K-theory of the integers and the Kummer-Vandiver Conjecture

GAUS AG winter term 2022/23

Though algebraic K-theory is a vital research area, the algebraic K-theory of the integers $\mathrm{K}_*(\mathbf{Z})$ is still not fully known. Letting B_k be the k-th Bernoulli number and setting $\frac{\mathrm{B}_k}{4k} = \frac{c_k}{w_{2k}}$ with $(c_k, w_{2k}) = 1$ we know the following:

n	0	1	2	3	4	5	6	7
$\mathrm{K}_{8k+n}(\mathbf{Z})$	?	$\mathbf{Z} \oplus \mathbf{Z}/2$	$\# = 2c_{2k+1}$	$\mathbf{Z}/2w_{4k+2}$	?	$\mathbf{Z}$	$\# = c_{2k+2}$	$\mathbf{Z}/w_{4k+4}$

As a consequence we obtain a link with the Riemann zeta function (for $k \geq 1$), namely

$$\frac{|\,\mathrm{K}_{4k-2}(\mathbf{Z})|}{|\,\mathrm{K}_{4k-1}(\mathbf{Z})|} = \frac{\mathrm{B}_k}{4k} = \frac{(-1)^k}{2}\zeta(1-2k).$$

In this seminar we want to better understand the higher algebraic K-theory of the integers as well as its relation to arithmetic. We will study Quillen's classical finiteness result for the K-theory of integers of number fields and connections between K-theory and étale cohomology. The latter will be used to link the conjectured vanishing of the groups $\mathrm{K}_{4n}(\mathbf{Z})$ (for $n \geq 1$) with the Kummer–Vandiver conjecture about cyclotomic extensions of the rational numbers [Kur92]. In the second half of the seminar we will study the proof of the known case that $\mathrm{K}_4(\mathbf{Z}) = 0$ whose constributions can be summed up as follows:

- Lee-Szczarba showed that $\mathrm{K}_4(\mathbf{Z})$ (and $\mathrm{K}_5(\mathbf{Z})$) contains no p-torsion for prime numbers $p > 5$ [LS78]. Soulé extended their arguments and showed that $\mathrm{K}_4(\mathbf{Z})$ (and $\mathrm{K}_5(\mathbf{Z})$) does not contain 5-torsion [Sou78].
- Weibel completely determined the 2-torsion of $\mathrm{K}_*(\mathbf{Z})$ [Wei97]. This resuls depends on the work of Voevodsky [Voe03][1], Suslin-Voevodsky [SV00], and Bloch-Lichtenbaum [BL94] on the Milnor conjecture, the Bloch-Kato conjecture, and the Quillen-Lichtenbaum conjecture. A gap was bridged by Rognes-Weibel [RW00, §5].
- Rognes showed that $\mathrm{K}_4(\mathbf{Z})$ does not have 3-torsion and concluded that $\mathrm{K}_4(\mathbf{Z}) = 0$. This uses previous work of Rognes [Rog92] as well as a homology computation by Soulé [Sou00].

Furthermore, it was shown recently that $\mathrm{K}_8(\mathbf{Z}) = 0$ [DSEVKM19, Kup19], but this will not be subject of the seminar.

Apéritif: One might want to read Soulé's overview article "Algebraic K-theory of the integers"
`https://link.springer.com/chapter/10.1007/BFb0088880`

Time and Place: Tuesday: 11:15–12:45 in Heidelberg, Mathematikon, SR 8, and online.
Last session on 31st January: Talk 9 (11:15–12:45, SR 8) and Talk 10 (14:15–15:45, SR 12). Afterwards we will have a cosy walk along Philosophenweg and dinner in Heidelberg Altstadt.

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[1] Depending on the eventually unpublished preprint [Voe94].

Talks

Talk 0: Introduction and overview (18.10. Christian Dahlhausen)
Introduction of the main players in the seminar, some historial background, and an overview of the seminar's content. Assignment of possibly free talks. Introduction of algebraic K-theory of rings: Define the Grothendieck group $\mathrm{K}_0(A)$ of a ring A and compute it for Dedekind domains. Define the first K-group $\mathrm{K}_1(A)$ and compute it for euclidean rings. Define the K-theory space $\mathrm{K}(A) = \mathrm{BGL}(A)^+$ using (without proof) the universal property of the +-construction. State that Quillen constructed a K-theory functor K^{Q} from the category of exact categories (with exact functors) to the category of topological spaces and that $\mathrm{K}(A) \simeq \mathrm{K}^{\mathrm{Q}}(\mathrm{Proj}^{\mathrm{fg}}(A))$ for the exact category $\mathrm{Proj}^{\mathrm{fg}}(A)$ of finitely generated projective A-modules. The foundational paper is Quillen's article [Qui73b] and a good textbook is [Wei13].

Talk 1: The Solomon-Tits theorem (08.11. Marlon Kocher)
This talk is the prequel for the subsequent talk about Quillen's finiteness result [Qui73a]. Introduce the building of a vector space and explain the Solomon-Tits theorem that these buildings have the homotopy type of a bouquet of spheres [Qui73a, Thm. 2].

Talk 2: Finiteness of the K-theory of the integers (15.11. Katharina Hübner)
Explain Quillen's proof that the K-theory of rings of integers of number fields is finitely generated [Qui73a]. Define the rank filtration on the category of finitely generated projective modules and on the induced Q-construction. Later in the seminar, we will see an analogous filtration on the spectrum level. State the exact sequence relating different levels of this filtration [Thm. 3] and deduce [Thm 1] from it. Afterwards, explain the proof of [Thm. 3] as detailed as time permits.

Talk 3: K-theory and étale cohomology (29.11. Tim Holzschuh)
Relate the K-theory of integers of number fields with their étale cohomology via the Chern character [Kur92, §2] so that we are prepared for the subsequent talk. In fact, we do not necessarily need the full result [Prop. 2.1]; it suffices to show that for a number field F with ring of integers $\mathcal{O}_F$ the Chern character

$$\mathrm{ch}\colon \mathrm{K}_{2r-2}(\mathcal{O}_F) \longrightarrow \mathrm{H}^2(\mathcal{O}_F[\tfrac{1}{p}], \mathbf{Z}_p(r))$$

is surjective. This is based on Soulé's paper [Sou79] and Dwyer-Friedlander's paper [DF85].

Talk 4: K-theory of the integers and cyclotomic fields (06.12. Max Witzelsperger)
This talk covers Kurihara's article relating K-theory with cyclotomic field extensions $\mathbf{Q}(\mu_p)|\mathbf{Q}$ for prime numbers $p \neq 2$ [Kur92, §1, §3]. Introduce the conjectures of Kummer-Vandiver (and Iwasawa) about the vanishing (resp. cyclicity) of certain eigenspaces of p-Sylow subgroups of class groups of cyclotomic extensions of the rational numbers [§0] and relate them to the vanishing (resp. cyclicity) of $\mathrm{H}^2_{\mathrm{et}}(\mathbf{Z}[\frac{1}{p}], \mathbf{Z}(r))$ [Cor. 1.5]. Show that the conjectured vanishing of $\mathrm{K}_{4n}(\mathbf{Z})$ for $n \geq 1$ implies the Kummer-Vandiver conjecture [Prop. 3.7], see also [Wei13, VI.10.9].

Talk 5: The p-torsion of $\mathrm{K}_*(\mathbf{Z})$ for $p > 5$ (13.12. Amine Koubaa)
Sketch the proof that $\mathrm{K}_4(\mathbf{Z})$ does not contain any p-torsion for $p \geq 5$ which was shown by Lee-Szczarba [LS78] ($p > 5$) and Soulé [Sou78] ($p = 5$).

Talk 6: A homology computation (24.01. Alexander Schmidt)
Explain Soulé's computation that the homology group $\mathrm{H}_1(\mathrm{SL}_4(\mathbf{Z}), \mathrm{St}_4)$ is a finite 2-group [Sou00].

Talk 7: The 2-torsion of $\mathrm{K}_*(\mathbf{Z})$ (17.01. Rustam Steingart)
Explain Weibel's computation of the K-theory with coefficients in $\mathbf{Z}/2$ [Wei97, Thm. 7]. Here we use without proof Voevodsky's result that $\mathrm{K}^{\mathrm{M}}_*(F)/2 \cong \mathrm{H}^*_{\mathrm{et}}(F; \mathbf{Z}/2)$ for fields F of characteristic $\neq 2$ together with the work of Suslin-Voevodsky [SV00] and Bloch-Lichtenbaum [BL94][2] in order

[2] A better reference for the Bloch-Lichtenbaum spectral sequence might be [Lev06].

to have a spectral sequene

$$E_2^{p,q} = \mathrm{H}^{p-q}_{\mathrm{et}}(F; \mathbf{Z}/2) \Rightarrow \mathrm{K}_{-p-q}(F; \mathbf{Z}/2).$$

Explain the consequences of this spectral sequence for $F = \mathbf{Q}_\ell$ and $F = \mathbf{R}$. Note that the proof for $F = \mathbf{R}$ [Prop. 4] relies on an unjustified assumption, but that this gap is bridged in a subsequent article of Rognes-Weibel [RW00, §5]; this subtlety shall be briefly mentioned and then ignored. As a consequence, deduce the descriptions of $\mathrm{K}_*(\mathbf{Q}; \mathbf{Z}/2)$ and of $\mathrm{K}_*(\mathbf{Z}; \mathbf{Z}/2)$.

Talk 8: The spectrum level rank filration (31.01. Christian Dahlhausen)
This talk covers the results [Rog00, §1–§4] whose proofs rely on [Rog92, §1–9]. First recall some relevant notions from topology which are needed. Introduce the rank filtration $F_\bullet\,\mathrm{K}(R)$ on the algebraic K-theory spectrum $\mathrm{K}(R)$ of a ring R and relate its subquotients $\bar{F}_\bullet\,\mathrm{K}(R)$ with the relative smash product of the universal bundle $\mathrm{EGL}_\bullet(R)$ with the stable building $D(R^\bullet)$ over $\mathrm{GL}_\bullet(R)$ R [Prop. 2.2]. In order to understand better the second component $D(R^\bullet)$ introduce the poset filtration on it [Def. 3.1] and then relate the subquotients of this filtration with the relative smash product of $\mathrm{GL}_\bullet(R)/P_\omega$ with the subquotients on the poset filtration on stable apartments [Prop. 4.2] which will be analysed in the subsequent talk.

Talk 9: Suspended Tits buildings (07.02. Lorenzo Mantovani)
This talk covers the results of [Rog00, §5, §6]. Explain the explicit identifications of the poset rank filtration and its subquotients for stable apartments [Prop. 5.1, Prop. 5.4]. Introduce Tits buildings and explain the relation between the Tits buildings of a PID and its fraction field [Lem. 6.1]. Maybe arrange with the subsequent talk's speaker to cover some material from [§7] in order to alleviate their job.

Talk 10: $\mathrm{K}_4(\mathbf{Z})$ is the trivial group (07.02. Georg Tamme)
This talks covers the results of [Rog00, §7, §8]. Introduce the component filtration of stable buldings [Def. 7.1] and explain (as much as time permits) the associated spectral sequence

$$E^1_{s,t} = \mathrm{H}_t(\mathrm{GL}_k(R); Z_s) \Rightarrow \mathrm{H}_{s+t}(\bar{F}_k\,\mathrm{K}(R)).$$

Finally, explain what we can conclude about the rank filtration spectral sequence for $\mathrm{K}(\mathbf{Z})$ modulo the Serre subcategory of finite 2-groups [(8.4)]. Compute the low degrees of the spectrum homology $\mathrm{H}_*(\mathrm{K}(\mathbf{Z}))$ of the spectrum $\mathrm{K}(\mathbf{Z})$ [Thm. 8.5] and subsequently the vanishing of $\mathrm{K}_4(\mathbf{Z})$ [Thm. 8.6].

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