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Wiles' Proof of Iwasawa's Main Conjecture

Hauptseminar (Summer Semester) 2011

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Short description: Iwasawa's Main Conjecture (MC) predicts the equality of two ideals in the so-called Iwasawa algebra, one "analytic", defined through the p-adic zeta function, and one "arithmetic" that encodes information about the ideal class groups of p-cyclotomic extensions of $\mathbb{Q}$.

Historically, this MC was first proved by Mazur and Wiles [9]. Wiles, in [15], went further and proved a generalization (from the rational numbers to totally real fields) of the MC. In that work Wiles made use of the theory of Hida families and in this way obtained also a "shorter" proof of the MC even over $\mathbb{Q}$. It is exactly this last proof that we will study in this seminar.

Both the joint work of Mazur and Wiles and the later work of Wiles are vast generalizations of an ingenious idea of Riber in his work [11], where he proved the converse of the so-called Herbrand's theorem. We believe it is better for the purposes of this seminar to take the time first to study this work of Ribet and then go further and see how this idea was generalized by Wiles.

We should mention that the Main Conjecture was generalized by Mazur and Greenberg for any motive (the classical Iwasawa's MC being for the Tate motive). It is a feature of the Tate motive that thanks to the analytic class number formula one needs to show only one of the inclusion of the ideals in order to conclude the MC. Actually the philosophy nowadays goes that one hopes to prove the one inclusion by constructing so-called Euler Systems and the other by studying Eisenstein ideals. Actually, in the case of the Tate motive (over $\mathbb{Q}$ only!), cyclotomic units can be used to construct an Euler System and hence provide yet another proof of Iwasawa's Main Conjecture over $\mathbb{Q}$ (but now proving the converse inclusion from that in Wiles' proof). Concerning the Eisenstein ideals approach we note that it is nothing else than the hope to generalize to other motives the work of Wiles that we study in this seminar. As a successful example of this philosophy we mention the recent work of Skinner and Urban where they prove the missing inclusion (the other has long been provided by Kato using Euler Systems) and establish the MC for particular elliptic curves over $\mathbb{Q}$.

Finally we mention that recently there is a great interest in yet a further generalization of the MC (of any motive) to a non-commutative setting as for example was initiated in [5], where one studies Galois extensions with Galois group isomorphic to a p-adic Lie group, which may not be abelian.

Talks: The proof of Wiles was the topic of a workshop held in Indian Institute of Technology, Guwahati in September 2008 organized by Coates, Dalawat, Saikia and Sujatha.

We will follow the program of this workshop quite closely and for all the talks in this seminar one can find a corresponding paper in the proceedings of this workshop (we provide below in more details the references for each talk).

1. (1 meeting) The goal of this talk is to provide an introduction to the classical Iwasawa theory (i.e. for the Tate motive) over totally real fields and then state the Main Conjecture of Iwasawa. It should also be explained that, thanks to the analytic class number formula, one has to prove only one divisibility in order to get the Main Conjecture. The main references for this talk are the first sections of [4] or [14].

2. (1,5 meetings) The aim of this and the next talk is to understand Ribet's proof [11] of the converse of the theorem of Herbrand. In this talk a weight 2 cusp eigenform with suitable congruence properties for its eigenvalues will be constructed. The first step is to construct a modular form that is congruent (modulo a suitable prime ideal) to an Eisenstein series (Proposition 4.1 in [12]). Then using the lifting lemma of Deligne-Serre one can obtain a modular form f with the same congruent properties but which is also an eigenform for all Hecke operators T_n with $(n,p) = 1$ (theorem 4.3 in [12]). Finally one shows that f is a cusp newform [12, Proposition 5.1 and 5.2].

3. (1,5 meetings) In this talk we conclude Ribet's proof. The first step is to reduce the problem to the existence of a suitable two dimensional (over a finite field) Galois representation as described in [6, theorem 2.3]. Next it has to be shown that this Galois representation can be obtained form the reduction (modulo a suitable prime) of the Galois representation associated to the cusp eigenform contructed in the previous talk (section 5 in [6]). The key step for this is a fundamental theorem of Ribet [6, theorem 3.1].

4. (1,5 meetings) In this talk and the next we develop the needed theory of Hida families (or Λ-adic modular forms) and the theory of pseudo-representations that we need for Wiles' proof of the main conjecture. The main reference for these two talks are the book of Hida [8] (chapter 7) and the introductory paper [1]. In this first talk the definition of a Λ-adic form is given and particular constructions are explained (Eisenstein families) (see for example [8, section 7.1]). Then we define Hida's ordinary operator and study the space of the ordinary Λ-adic forms. Finally this talk concludes with explaining the main ingredients of Hida's control theorem (see for example [1, theorem 5.7 (i) and (ii)].

5. (1,5 meeting) In this talk we study the theory of big Galois representations. The goal is to prove, using the theory of pseudo-representations of Wiles, that to an ordinary normalized Λ-adic eigenform we can associate, in a canonical way, a big Galois representation (for an explicit statement see [8, theorem 1 in section 7.5] or [1, theorem 6.1]. Also in this talk we are going to prove a theorem of Wiles that describes the form of this associated big Galois representation when it is restricted at the decomposition group corresponding to the prime p, the prime fixed in the Λ-adic theory. (see [1, theorem 6.2].

6. (1 meeting) In this talk we study the Λ-adic Eisenstein ideal, the generalization to the Λ-adic theory of Mazur's classical Eisenstein ideal. The main goal of this talk is to

prove a fundamental theorem of Wiles [15, thm 4.1] on the properties of this ideal. Since we restrict ourselves to the main conjecture over $\mathbb{Q}$ the proof of this theorem is also explained in details in [1, section 8, theorem 8.6].

7. (1 meeting) In this talk an introduction to the theory of Fitting ideals is given. We will follow the article of F. Nuccio [10]. In particular, for our purposes it is important to understand the relation of the characteristic elements of Iwasawa modules and Fitting ideals as for example it is explained in the second section of [10].

8. (1,5 meetings) Now we are ready to tackle the main conjecture. As we have seen in the first talk we have to prove only one divisibility. The one that we will prove can be "factored" through the Λ-adic Eisenstein ideal introduced before. We will follow the article of Coates and Sujatha [4] and brake the proof into two parts. These two parts should be seen as a generalization of Ribets's idea to the Λ-adic world. In this talk the first part (section 8 of [4]) should be covered.

9. (1 meeting) In this talk the proof of the main conjecture is concluded. This is the second part of the proof as explained in the paper of Coates and Sujatha [4] in section 9.

10. If the time allows we may also see the construction of the p-adic L-function for the Tate motive over totally real fields as was done by Deligne and Ribet using p-adic Hilbert modular forms [7] and in a different way by Cassou-Nogues [3] and Barsky [2] using the Shintani decomposition. For the Deligne-Ribet construction see also the introductory article [13].

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