On Lerman's Cut Construction

Overview:

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III. Topological Cuts
IV. Symplectic Cuts

I. Motivation

- Guillemin, Sternberg and McDuff noticed a connection between symplectic blow-ups (of the complex plane in the origin) and Marsden-Weinstein reduced spaces.
- Lerman generalized this concept and introduced "Symplectic Cuts".
- Later he gave the contact analog: "Contact Cuts".
- Various applications of these cuts in the context of symplectic and contact actions on manifolds.

II. Preliminaries

Requirements: Basic knowledge of symplectic and contact geometry

Quotient Manifold Theorem (QMT):

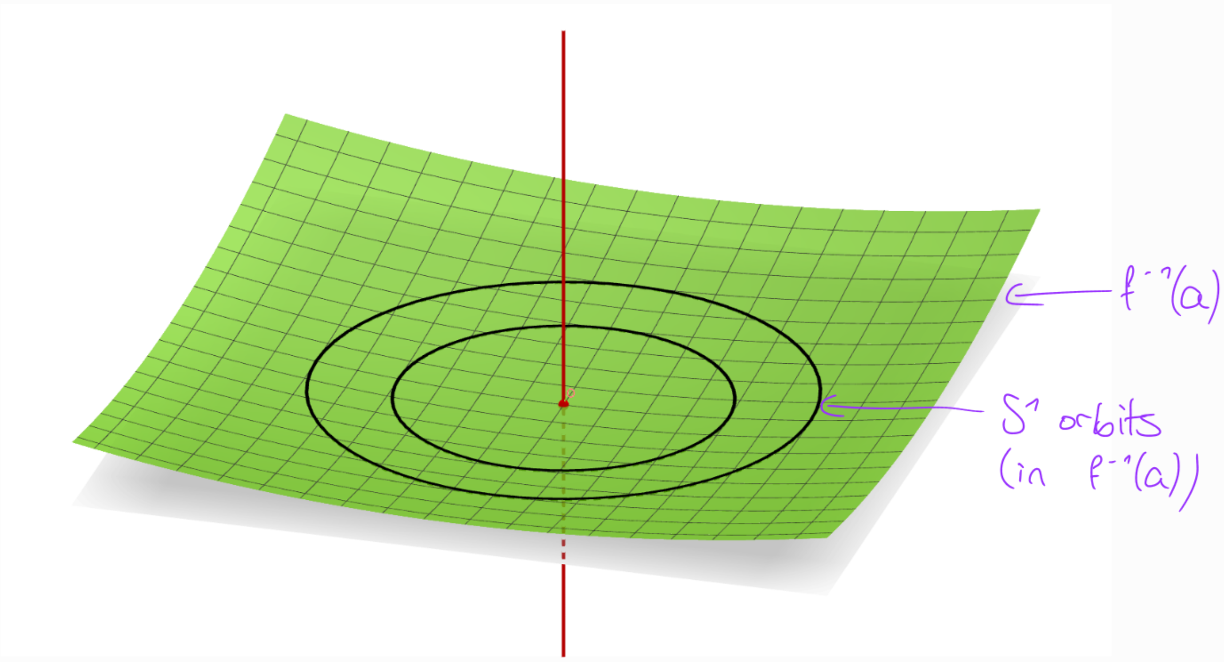
Given a smooth and free action of a compact Lie group G on the manifold M,
then the orbit space M/G (endowed with the quotient topology)
is a topological manifold of dimension $\dim(M) - \dim(G)$ and carries a unique smooth structure such that the projection $\pi : M \twoheadrightarrow M/G$ is a (smooth) submersion.

III. Topological Cuts

Setting: Given a smooth S^1-action on the manifold M.
Let $f: M \longrightarrow \mathbb{R}$ be smooth and S^1-invariant and $a \in \mathbb{R}$ a regular value of f. Suppose S^1 acts freely on $f^{-1}(a)$.

Consider the following equivalence relation $\sim$ on $f^{-1}([a,\infty))$
For $m \neq m'$: $m \sim m' :\Longleftrightarrow$ i) $m, m' \in f^{-1}(a)$ and
ii) $S^1 \cdot m = S^1 \cdot m'$

We call the quotient space (equipped with the quotient topology)
$f^{-1}([a,\infty))/_{\sim} =: M_{[a,\infty)}$ the cut with respect to the ray $[a,\infty)$.



<u>Example:</u> $M = \mathbb{R}^3 \setminus \mathbb{R} \cdot \begin{pmatrix}0\\0\\1\end{pmatrix}$, $a = 0$ and
$f(x,y,z) = \gamma \cdot (x^2 + y^2) - z$ for some real parameter $\gamma > 0$
S^1 acts on $M = \mathbb{C} \setminus \{0\} \times \mathbb{R}$ via $\underbrace{\lambda}_{\in S^1} \cdot (\underbrace{\omega}_{\in \mathbb{C}\setminus\{0\}}, \underbrace{z}_{\in \mathbb{R}}) = (\lambda \cdot \omega, z)$

Main Theorem: (Topological Cuts)

$M_{[a,\infty)}$ is a topological manifold of dimension $\dim(M)$ and carries a canonical smooth structure. Let $\pi: f^{-1}([a,\infty)) \twoheadrightarrow M_{[a,\infty)}$ denote the projection.
Then $\pi|_{f^{-1}((a,\infty))}$ is a diffeomorphism onto its open, dense image and $M_{[a,\infty)} \setminus \pi(f^{-1}((a,\infty))) \cong f^{-1}(a)/S^1$.

proof sketch: Consider $M \times \mathbb{C}$ with the S^1-action $\lambda \cdot (m,z) := (\lambda \cdot m, \lambda^{-1} z)$ and $\Psi: M \times \mathbb{C} \to \mathbb{R}$, $(m,z) \longmapsto f(m) - |z|^2$

Then Ψ is S^1-invariant with regular value a and the restricted S^1-action on $\Psi^{-1}(a)$ is free, so $\Psi^{-1}(a)/S^1$ is a smooth manifold.

Let $\sigma: f^{-1}([a,\infty)) \longrightarrow \Psi^{-1}(a)$, $\sigma(m) := (m, \sqrt{f(m)-a})$

Then σ descends to a homeomorphism (nontrivial) $\bar{\sigma}$:

$$\begin{tikzcd} f^{-1}([a,\infty)) \arrow[r, "\sigma"] \arrow[d, "pr"'] \arrow[dr, phantom, "\circlearrowleft"] & \Psi^{-1}(a) \arrow[d, "pr"] \\ M_{[a,\infty)} \arrow[r, "\bar{\sigma}"] & \Psi^{-1}(a)/S^1 \end{tikzcd}$$

Check: Well-defined: If $m \sim m'$, $m \neq m'$, then $m' = \lambda \cdot m$ for some $\lambda \in S^1$ and $f(m) = f(m') = a$

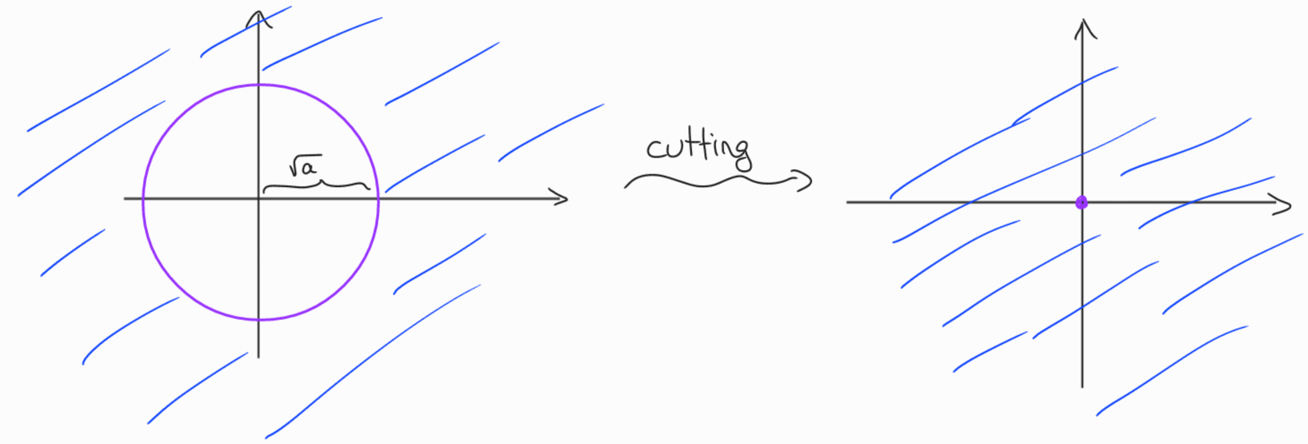
$\Longrightarrow pr(\sigma(m)) = S^1 \cdot (m, 0) = S^1 \cdot (m', 0) = pr(\sigma(m'))$

Bijective: $\bar{\sigma}^{-1}(S^1 \cdot (m,z)) := \begin{cases} \left[\frac{z}{|z|} \cdot m\right] & , \text{ if } z \neq 0 \\ [m] & , \text{ else} \end{cases}$

where $[m] := pr(m) \in M_{[a,\infty)}$

Give $M_{[a,\infty)}$ the smooth structure induced by $\overline{\sigma}$.
Since $(f^{-1}(a) \times \{0\})/_{S^1}$ is a submanifold of $\Psi^{-1}(a)/_{S^1}$,
$f^{-1}(a)/_{S^1}$ is a submanifold of $M_{[a,\infty)}$. $\square$

<u>Example:</u> $M = \mathbb{C}$, $f: M \to \mathbb{R}$, $f(w) := |w|^2$, $a > 0$ arbitrary
S^1 acts on $\mathbb{C}$ via multiplication



$+ \; = f^{-1}([a,\infty))$ $\qquad\qquad$ $M_{[a,\infty)}$

So $M_{[a,\infty)} \underset{\text{homeo}}{\cong} \mathbb{C}$, in fact $M_{[a,\infty)} \underset{\text{diffeo}}{\cong} \mathbb{C}$

#Example

IV. Symplectic Cuts

<u>Def.:</u> Let S^1 act smoothly on the symplectic manifold (M,ω).
A moment map $\mu: M \to \mathbb{R}$ is a smooth map with
$d\mu = -\iota_{\hat{\partial}_t}\,\omega$, where $\hat{\partial}_t$ is the infinitesimal generator of the S^1-action on M.
We call (M, ω, S^1, μ) a hamiltonian S^1-space.

<u>Remark:</u> If a moment map exists, it is determined up to a constant.
A moment map is S^1-invariant.

__Marsden-Weinstein:__ Let (M, ω, S^1, μ) be a hamiltonian S^1-space and $c \in \mathbb{R}$. Let $\mathrm{incl}: N := \mu^{-1}(c) \hookrightarrow M$ denote the inclusion.

If S^1 acts freely on N, then c is a regular value of μ. (Thus, N is a closed submanifold of codimension 1 in M.) The quotient manifold $M_{red} := N/_{S^1}$ carries a unique symplectic form ω_{red} such that $pr^*\omega_{red} = \mathrm{incl}^*\omega$ with $pr: N \twoheadrightarrow N/_{S^1}$ the projection.

__Symplectic Cuts:__ Let f be a moment map for a hamiltonian S^1-action on the symplectic manifold (M, ω). Suppose the action is free on the level set $f^{-1}(a)$ for $a \in \mathbb{R}$ fixed.

Then the cut $M_{[a,\infty)}$ is well-defined and carries a natural symplectic form ω_{cut} such that $pr|_{f^{-1}((a,\infty))} : f^{-1}((a,\infty)) \longrightarrow M_{[a,\infty)}$ is a symplectomorphism onto its open image and

$$\left(f^{-1}(a)/_{S^1}, \omega_{red} \right) \hookrightarrow \left(M_{[a,\infty)}, \omega_{cut} \right) \text{ is symplectic.}$$

__proof sketch:__ As above: $\Psi: M \times \mathbb{C} \longrightarrow \mathbb{R}$, $\Psi(m,z) = f(m) - |z|^2$ and $\sigma: f^{-1}([a,\infty)) \longrightarrow \Psi^{-1}(a)$, $\sigma(m) = (m, \sqrt{f(m)-a})$ descends to the diffeomorphism

$$\bar{\sigma}: M_{[a,\infty)} \xrightarrow{\cong} \Psi^{-1}(a)/_{S^1}$$

Recall that S^1 acts on $M \times \mathbb{C}$ via $\lambda \cdot (m,z) = (\lambda \cdot m, \lambda^{-1} z)$. Consider the symplectic product $(M \times \mathbb{C}, \omega + 2\,dx \wedge dy)$. Then Ψ is a moment map for the S^1 action on $M \times \mathbb{C}$.

Check: $d\Psi = df - 2x\,dx - 2y\,dy$

$$= \omega(-, \widehat{\partial_t}^M) - 2\,dx \wedge dy(-, iz)$$

$$= (\omega + 2\,dx \wedge dy)(-, \widehat{\partial_t}^M - iz)$$

$$= (\omega + 2\,dx \wedge dy)\,(-, \hat{\partial}_t^{M\times\mathbb{C}})$$ ✓

By Marsden-Weinstein we have a reduced symplectic form β_{red} on $\Psi^{-1}(a)/_{S^1}$ for $\beta = \omega + 2\,dx \wedge dy$ on $M \times \mathbb{C}$.

Then $\omega_{cut} := \bar{\sigma}^* \beta_{red}$.

We will omit the rest of the proposition. $\square$

<u>Example:</u> (continuation)

$\omega = 2\,dx \wedge dy$ on $M = \mathbb{C}$, $f(z) = |z|^2$, $a > 0$

and S^1 acts on $\mathbb{C}$ via multiplication.

Then f is a moment map since

$$\omega(-, \hat{\partial}_t) = 2\,dx \wedge dy\,(-, iz) = 2x\,dx + 2y\,dy = df$$

We have $(M_{[a,\infty)}, \omega_{cut}) \cong (\mathbb{C}, \omega)$

<u>Example:</u> $M = T^*\mathbb{T}^2$ with the standard symplectic form ω, i.e. if we denote coordinates with $(\theta_1, \theta_2, p_1, p_2)$ then

$$\omega = d\theta_1 \wedge dp_1 + d\theta_2 \wedge dp_2$$

Let S^1 act on M via $\lambda \cdot (\theta_1, \theta_2, p_1, p_2) := (\theta_1, \theta_2 + \lambda, p_1, p_2)$.

A moment map is given by $f(\theta_1, \theta_2, p_1, p_2) = p_2$

Check: $\omega(-, \hat{\partial}_t) = \omega(-, \partial_{\theta_2}) = dp_2 = df$

Then $M_{[0,\infty)} = \{(\theta_1, \theta_2, p_1, p_2) \mid p_2 \geq 0\}/_{\sim} \underset{\text{sympl.}}{\cong} T^*S^1 \times \mathbb{C}$

because $(\theta_1, \theta_2, p_1, p_2) \xmapsto{\cong} (\theta_1, p_1, \underbrace{p_2 \cdot e^{i\theta_2}}_{\text{polar coordinates}})$