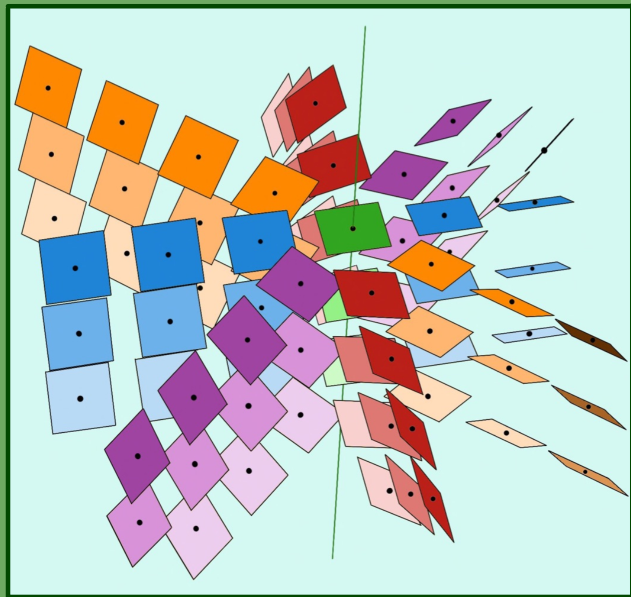
Contact structures induced by line fibrations of $\mathbb{R}^3$



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Content

1. Contact structures

Def. 1: (Distributions)

A d-dimensional Distribution ξ on a manifold M is a selection of d-dimensional linear subspaces $\xi_p \subset T_pM$ for all $p \in M$, so that $p \longmapsto \xi_p$ is smooth.

If ξ has codimension 1, we call it a hyperplane field.

Def. 2: (coorientable; integrable)

Let ξ be a hyperplane field.

- ξ is called **coorientable** if there exists a smooth 1-form α on M so that $\boxed{\xi = \ker \alpha}$

  (i.e. $\xi_p = \ker \alpha_p \ \forall p \in M$)

- ξ is then called **integrable** if α satifies the Frobenius integrability condition $\boxed{\alpha \wedge d\alpha \equiv 0}$.

Def. 3: (contact structures)

A contact structure on a 3-dim. manifold M is a (maximally) non-integrable, coorientable hyperplane field $\boxed{\xi = \ker \alpha} \subset TM$.

→ That is, α has to satisfy the contact condition:

$$\boxed{\alpha \wedge d\alpha \neq 0}$$

(i.e. $\alpha_p \wedge d\alpha_p \neq 0 \ \forall p \in M$)

→ α is called a contact form

→ (M, ξ) is called a contact manifold

Remark: In this case we have $M = \mathbb{R}^3$

$\longrightarrow$ A contact structure then is a 2-dim. plane field $\xi = \ker \alpha$ with $\alpha \wedge d\alpha \neq 0$

We always identify $T_p \mathbb{R}^3 = \mathbb{R}^3$ so that $\partial_x = e_1 = (1,0,0)$, $\partial_y = e_2 = (0,1,0)$, $\partial_z = e_3 = (0,0,1)$

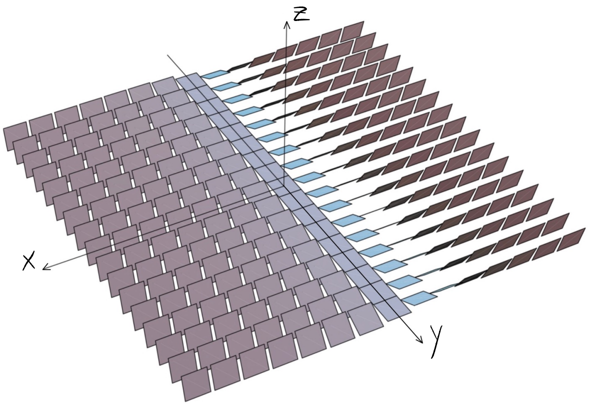
Example 1: (standard contact structure on $\mathbb{R}^3$)

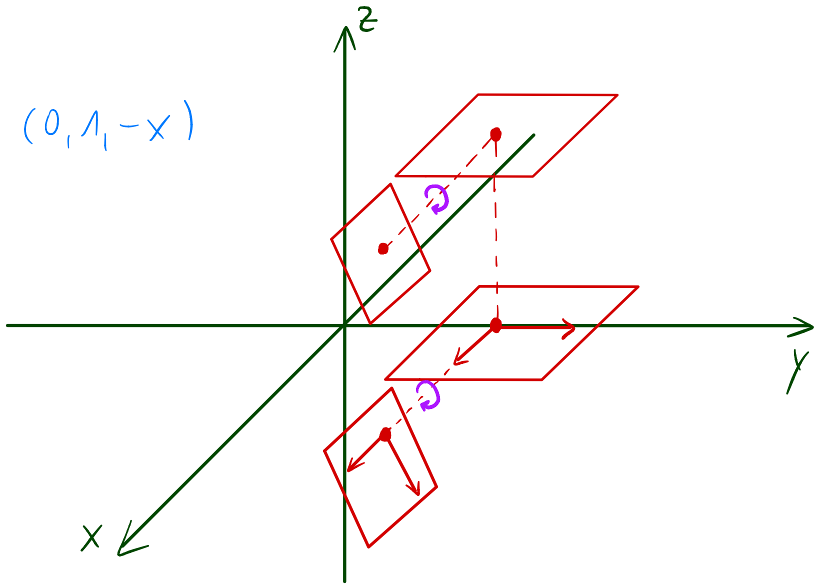
contact form: $\alpha_1 := dz + x\,dy \Rightarrow \xi_1 = \ker \alpha_1$

(check: $\alpha_1 \wedge d\alpha_1 = (dz + x\,dy) \wedge (\cancel{ddz} + dx \wedge dy) = dz \wedge dx \wedge dy \neq 0$)

$\xi_{1,(x,y,z)} = \mathrm{Lin}(\partial_x, \partial_y - x\partial_z)$

with $\partial_x = (1,0,0)$ and $\partial_y - x\partial_z = (0,1,-x)$

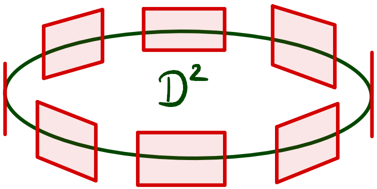




Def. 4: (tight & overtwisted)

A contact structure ξ on $\mathbb{R}^3$ is called overtwisted if there exists an <u>embedded</u> disk D^2 in $\mathbb{R}^3$ so that ∂D^2 is tangent to ξ and D^2 is transverse to ξ along ∂D^2.

Otherwise ξ is called tight.



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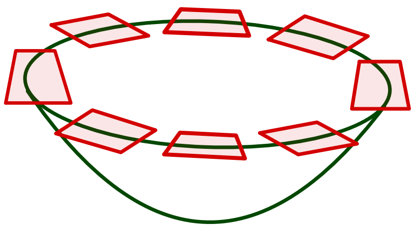

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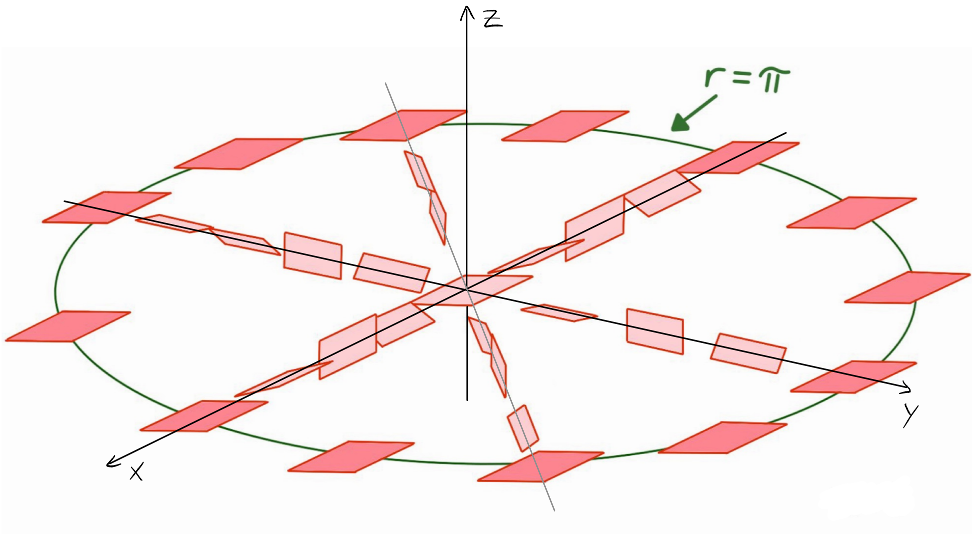
Example 2: (standard overtwisted contact structure)

contact form: $\alpha_{ot} = \cos(r)\,dz + r\sin(r)\,d\theta$

(cylindrical coordinates)

$\Rightarrow \xi_{ot} = \ker \alpha_{ot}$ is an overtwisted contact structure:



$$\xi_{ot,(r,\theta,z)} = \mathrm{Lin}\left(\partial_r,\ \cos(r)\,\partial_\theta - r\sin(r)\,\partial_z\right)$$

Def. 5: Let $\xi_1 = \ker \alpha_1$, $\xi_2 = \ker \alpha_2$ be two contact structures on the manifold M.

ξ_1 is **diffeomorphic** to ξ_2 if there exists a diffeomorphism $\phi : M \longrightarrow M$ so that α_1 and $\phi^* \alpha_2$ determine the same hyperplane field.

ϕ is then called a **contactomorphism**.

Remark: (Eliashberg, 1992)

- The standard contact structure ξ_1 on $\mathbb{R}^3$ is tight.
- Up to isotopy there is only one tight contact structure.

$\Rightarrow$ All tight contact structures on $\mathbb{R}^3$ are diffeomorphic to ξ_1

Example 3: (ξ_1 „again")

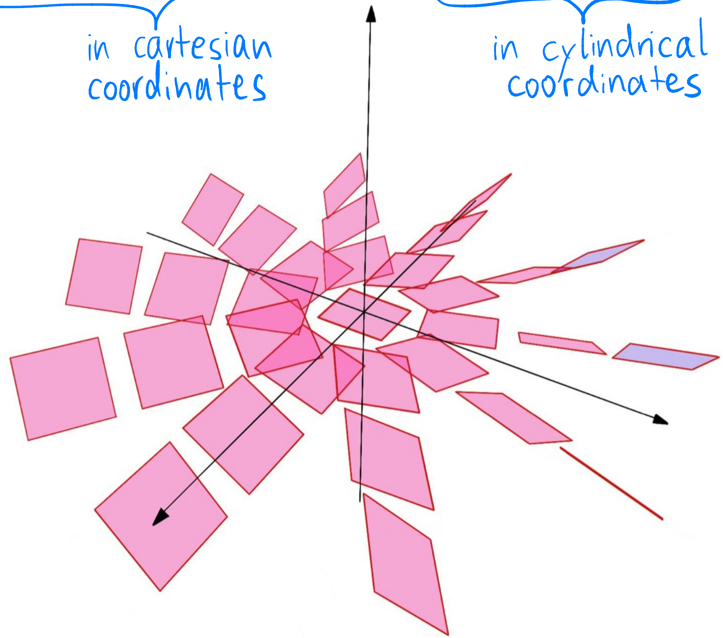
contact form: $\alpha_2 := \underbrace{dz + x\,dy - y\,dx}_{\text{in cartesian coordinates}} = \underbrace{dz + r^2\,d\theta}_{\text{in cylindrical coordinates}}$

$\Rightarrow \xi_2 = \ker \alpha_2$

is a contact structure diffeomorphic to ξ_1 and therefore tight

$\Rightarrow (\mathbb{R}^3, \xi_2)$ and $(\mathbb{R}^3, \xi_1)$ are contactomorphic

($(\mathbb{R}^3, \xi_2)$ and $(\mathbb{R}^3, \xi_{ot})$ not!)

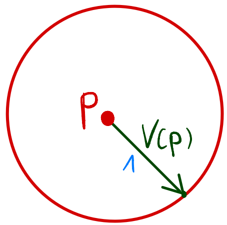


$$\xi_{2,(r,\theta,z)} = \mathrm{Lin}(\partial_r, \partial_\theta - r^2 \partial_z)$$

2. Line Fibrations

We start with a smooth vector field on $\mathbb{R}^3$ of unit length:

$$\boxed{V : \mathbb{R}^3 \longrightarrow S^2 \subset \mathbb{R}^3}$$



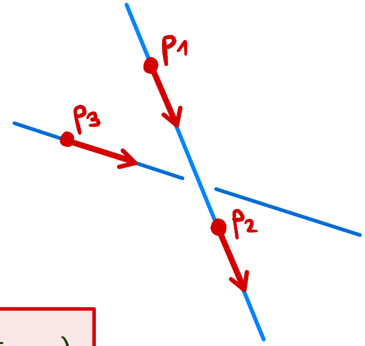
whose integral curves are straight, oriented lines

$\Rightarrow$ V induces a well-defined line fibration of $\mathbb{R}^3$ called $\{V\}$. The fibers are the oriented lines $\boxed{\ell_p = \{p + t \cdot V(p) : t \in \mathbb{R}\}} \in \{V\}$.

Def. 6: (skew)

$\{V\}$ is called skew if no two fibers are parallel,

i.e. $\boxed{V(p_1) = V(p_2) \Rightarrow \ell(p_1) = \ell(p_2)}$



Def. 7: (nondegenerate)

$\{V\}$ is called nondegenerate if the Differential DV vanishes only in the direction of V

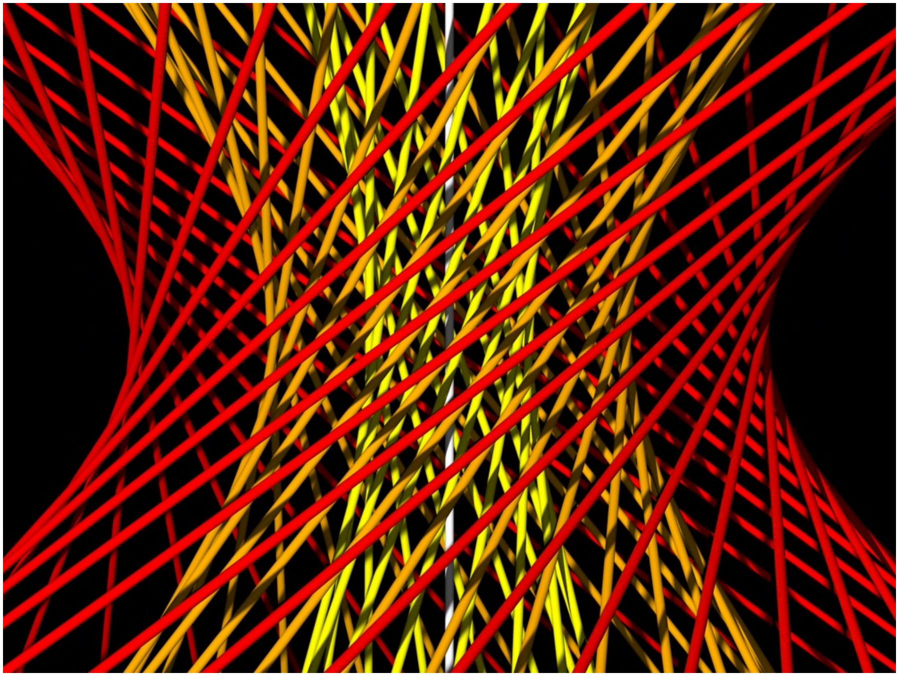
$D_pV(V(p)) = 0$
$\forall p \in \mathbb{R}^3$

$\Leftrightarrow \boxed{\forall p \in \mathbb{R}^3 : \ rk(D_pV) = 2}$

Remark: $\{V\}$ nondegenerate $\underset{\not\Leftarrow}{\Rightarrow}$ $\{V\}$ skew

Example 4: (standard Hopf fibration)

ℓ



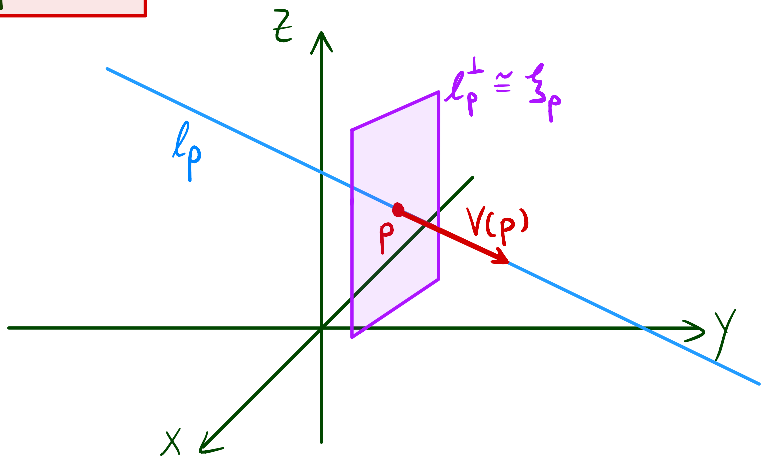
Choose an oriented line ℓ in $\mathbb{R}^3$ (white) and surround it with a family of hyperboloids (yellow, orange, red) which foliate $\mathbb{R}^3 \setminus \ell$.
Each hyperboloid is ruled. What we get is a skew line fibration.

Remark: The space of all great circle fibrations of $S^3 \subset \mathbb{R}^4$ sits naturally inside the space of skew line fibrations of $\mathbb{R}^3$ by a central projection from S^3 to any tangent $\mathbb{R}^3$.

This is the reason we can consider the Hopf fibration of S^3 as a line fibration as well.

$\Rightarrow$ Any line fibration $\{V\}$ induces a 2-plane distribution ξ on $\mathbb{R}^3$ by

$$\boxed{\xi_p = \langle V(p) \rangle^{\perp}}$$



ξ is cooorientable : $\xi = \ker \alpha$ with the contact form $\boxed{\alpha = V_1\, dx + V_2\, dy + V_3\, dz}$

with $\alpha \wedge d\alpha \neq 0 \iff \langle V, \operatorname{curl} V \rangle \neq 0$.

Question:

What conditions on $\{V\}$ do we need, so that the induced plane distribution ξ is a contact structure?

3. Main theorems

Theorem 1: (Becker & Geiges, 2020)

If a line fibration $\{V\}$ of $\mathbb{R}^3$ induces a contact structure ξ, then ξ is diffeomorphic to ξ_1.

(and therefore tight)

Theorem 2: (Harrison, 2019)

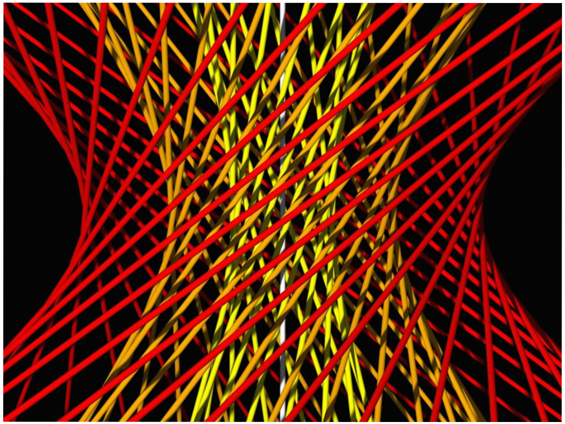
The plane distribution ξ induced by a nondegenerate line fibration $\{V\}$ of $\mathbb{R}^3$ is a tight contact structure.

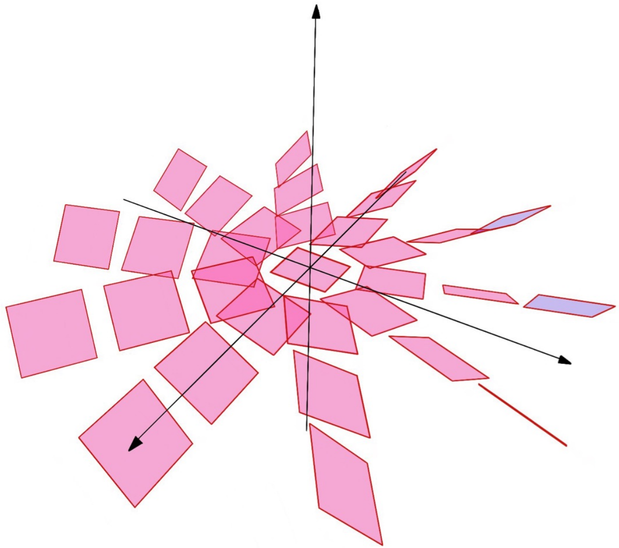
Remark: Because of Theorem 1, it only remains to show that ξ is a contact structure.

4. Illustrative examples

1) (nondegenerate & tight)

The Hopf fibration from above (which is nondegenerate) induces the familiar (tight) contact structure $\xi_2 = \ker(dz + r^2 d\theta)$





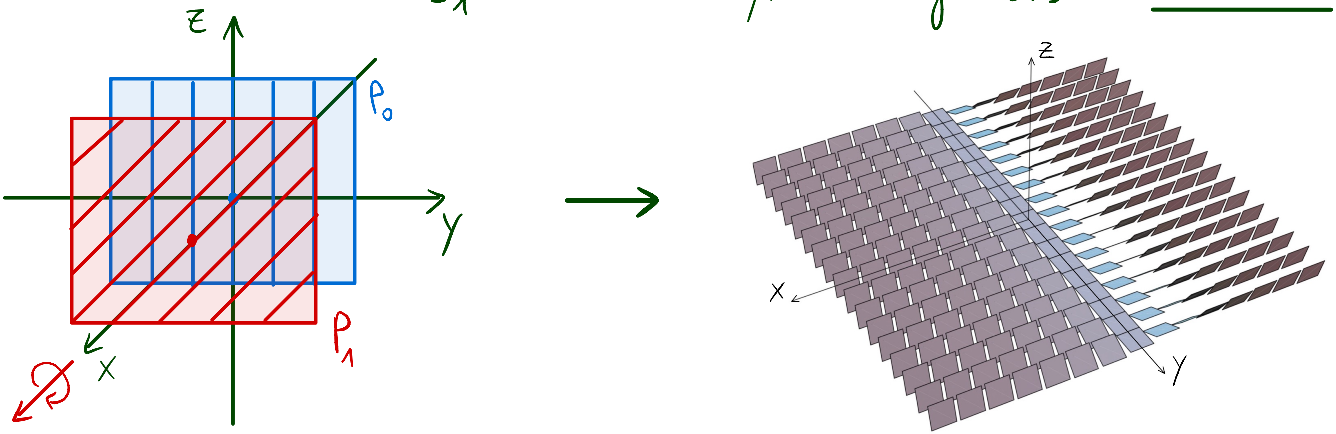
2) (degenerate & tight)

We construct a line fibration $\{V\}$ as follows:

$\boxed{1.}$ Foliate $\mathbb{R}^3$ by planes P_x parallel to the yz-plane.

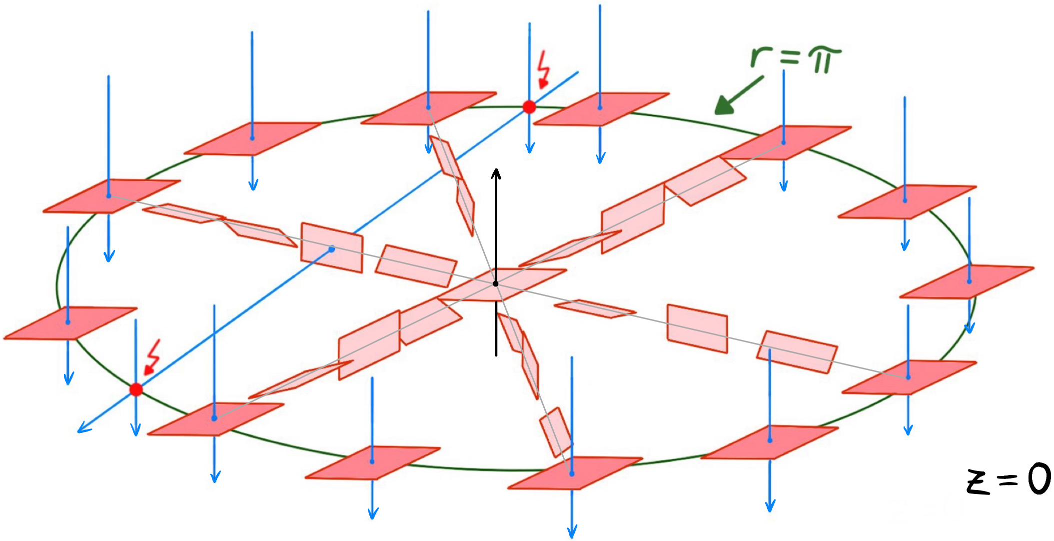
$\boxed{2.}$ Fiber each plane by parallel lines with direction $(0, x, 1)$ for P_x

$\Rightarrow$ $\{V\}$ induces the standard (tight) contact structure $\xi_1 = \ker(dz + x\,dy)$ although $\{V\}$ is <u>not skew</u>



3) (overtwisted & not a line fibration)

Vice versa: The straight lines that are orthogonal to the planes of the contact structure ξ_{ot} do not form a line fibration as the lines must not intersect each other.



$$\xi_{ot} = \ker(\cos(r)\,dz + r\sin(r)\,d\theta)$$

5. Outlook

- There are examples of skew, degenerate line fibrations that do not induce contact structures, and other interesting examples (more tools needed).

- One could also look at higher dimensions and ask:
  Does a nondegenerate line fibration of an odd-dimensional Euclidean space always induce a contact structure?

Thank you for your attention!

References:

1) M. Harrison. Contact structures induced by skew fibrations of $\mathbb{R}^3$, 2019.
2) T. Becker & H. Geiges. The contact structure induced by a line fibration of $\mathbb{R}^3$ is standard, 2020.
3) H. Geiges. An introduction to contact topology, 2008.
4) Y. Eliashberg. Contact 3-manifolds twenty years since J. Martinet's work, 1992.

Images by David Eppstein (Hopf line fibration)
and Matias Dahl (standard contact structure)