Seminar: Hyperbolic geometry, symmetry groups, and more

Prof. Dr. Anna Wienhard and Dr. Gye-Seon Lee

Wintersemester 2013/2014

1. Time and Location

To be determined

2. Description

In this seminar we will explore various topics related to symmetry groups, tilings of Euclidean or hyperbolic space, hyperbolic geometry and number theory.

The format of the seminar is a bit unusual. At the organizational meeting we will propose several projects. Students will get together in teams to carry out one of the projects. The project consists of a theoretical part, which can/should be complemented by an "experimental part". The experimental part could consist for example of some numerical computations or visualization on the computer. Upon completion of the project each team will present the project in the seminar.

The seminar is open for master and bachelor students. Some of the projects can also lead into a bachelor or master thesis.

3. Prerequisites

There are several topics for students to choose from, and hence any interested students are welcome to attend.

The seminar will be taught in English. If this seems to be inconvenient for you, keep in mind that nowadays English is the prevalent language in mathematical literature, mathematical conference and within the international mathematical community, and thus expressing yourself in English will be an important skill you will have to develop anyway.

4. Organization

Students who would like to sign up for the seminar should come to the organization meeting or send an email to Gye-Seon Lee (lee@mathi.uni-heidelberg.de) and let him know what project they would like to carry out.

Once you have agreed on the topic of your project with him, you should contact him to discuss the content of your project in more detail. Please let him help you during the preparation of your presentation.

Please read "Wie halte ich einen Seminarvortrag?" written by Prof. Dr. Manfred Lehn before you prepare your presentation:

http://www.mathematik.uni-mainz.de/Members/lehn/le/seminarvortrag

5. Topics

The schedule of presentations will be updated on the web page:

http://www.mathi.uni-heidelberg.de/~lee/seminarWS1314.html

5.1. **Apollonian Circle packings.** In this project, we will study Apollonian's theorem, Descartes' theorem and Apollonian groups.

- *Apllonius's theorem*: Given three mutually tangent circles, there are exactly two circles tangent to all three.
- *Descartes' theorem*: Given four mutually tangent circles whose curvatures are a_1, a_2, a_3, a_4 (with sign convention), then

$$2a_1^2 + 2a_2^2 + 2a_3^2 + 2a_4^2 = (a_1 + a_2 + a_3 + a_4)^2.$$

Moreover, we try to visualize apollonian circle packings.

Main references: [Sarnak] and [Graham, Lagarias, Mallows, Wilks, Yan]

Supplement: [Oh]

5.2. **Crystallographic groups.** Let $\mathbb{E}^n$ be the n-dimensional Euclidean space and $\mathrm{Isom}(\mathbb{E}^n)$ be the group of isometries of $\mathbb{E}^n$. An n-dimensional *crystallographic group* is a discrete group Γ of $\mathrm{Isom}(\mathbb{E}^n)$ such that $\mathbb{E}^n/\Gamma$ is compact. In this project, we study the theory of crystallographic groups.

5.2.1. *Bieberbach's theorem.* We will understand the proof of Bieberbach's (three) theorems:

(1) Every n-dimensional crystallographic group contains n linearly independent translations.
(2) For each fixed n, there are only finitely many isomorphism classes of n-dimensional crystallographic groups.
(3) Two crystallographic groups are isomorphic if and only if they are conjugate by an affine transformation.

Main references: [Ratcliffe §7.5] and [Buser]

Supplement: [Auslander]

5.2.2. *Wallpaper groups.* The exact numbers of isomorphic classes of n-dimensional crystallographic groups for $n = 1, 2, 3, 4$ are respectively $2, 17, 219, 4783$. In particular, we will understand 2-dimensional crystallographic groups, called the *wallpaper groups*. We will classify the seventeen wallpaper groups and visualize them.

Main References: [Martin §11] and [Conway, Burgiel, Goodman-Strauss §3]

Supplement: [Coxeter and Moser §4.5]

5.3. **Reflection groups.** Let X be the n-sphere, Euclidean n-space, or hyperbolic n-space. A *Coxeter polyhedron* in X is a convex polyhedron whose dihedral angles are all integer submultiples of π. A subgroup Γ of $\mathrm{Isom}(X)$ is a *reflection group* if Γ is generated by reflections with respect to the sides of a Coxeter polyhedron. In fact, reflection groups are discrete subgroups of $\mathrm{Isom}(X)$. In this project, we study the reflection groups.

5.3.1. *Simplex reflection groups.* A group is an *n-simplex reflection group* if it is generated by reflections with respect to the sides of Coxeter n-simplex in X. We will give the classification of simplex reflections groups, and for $n = 2$, we draw the corresponding figures on the computer.

Main references: [Ratcliffe §7.1] and [Thurston §13.5]

5.4. **Aperiodic tilings.** Let X be either $\mathbb{E}^n$ or $\mathbb{H}^2$. An X-tiling system is a finite collection of subsets of X, called "tiles", each being homeomorphic to a compact ball in X. A *tessellation* by these tiles is a decomposition of X, or more generally of a quotient of X, into a union of isometric copies of the various tiles intersecting only at their boundaries. A tiling system is called *aperiodic* or *non-periodic* if no compact quotient of X (by a discrete group of $\mathrm{Isom}(X)$) may be tessellated by the tiles of the system. In this project, we study the non-periodic tilings.

5.4.1. *Penrose's tilings.* We will understand an aperiodic $\mathbb{E}^2$-tiling system given by Penrose and its algebraic theory. Moreover we visualize Penrose's tilings on the computer.

Main references: [Penrose], [Grünbaum and Shephard §10.3], [de Bruijin]

5.4.2. *Aperiodic tilings of the hyperbolic plane.* We will show for each $n \geq 3$ there exists an aperiodic $\mathbb{H}^2$-tiling system whose set of tiles consists of a single convex hyperbolic n-gon, and visualize them. Note that it is an open question whether there exists an aperiodic $\mathbb{E}^2$-tiling system consisting of a single tile.

Main references: [Margulis and Mozes]

5.5. **Space of representations.** Let Γ_0 be a finitely generated group, $\mathbb{S}^n$ be the n-dimensional real projective sphere and $G = \mathrm{SL}^{\pm}(n+1, \mathbb{R})$. In this project, we study the space $\mathcal{D}(\Gamma_0)$ of the discrete faithful representations $\rho : \Gamma_0 \to G$ with the image $\Gamma = \rho(\Gamma_0)$ *dividing* a properly convex open subset Ω of $\mathbb{S}^n$, i.e. the discrete subgroup Γ of G acts on Ω such that the quotient Ω/Γ is compact.

Main references: [Benoist]

5.5.1. *Hyperbolic 3-manifold groups.* Let M be a closed oriented hyperbolic 3-manifold and $\pi_1(M)$ be the fundamental group of M. A hyperbolic structure on M corresponds to a discrete faithful representation ρ of $\pi_1(M)$ into $\mathrm{Isom}^+(\mathbb{H}^3) \subset \mathrm{SL}(4, \mathbb{R})$, i.e. $\rho \in \mathcal{D}(\pi_1(M))$. We will study the method for computing the space of representations near ρ.

Main references: [Cooper, Long, Thistlethwaite]

5.5.2. *Triangle reflection groups.* Assume that Γ is a triangle reflection group in $\mathbb{H}^2$. Let Γ_0 be a (abstract) group which is isomorphic to Γ. Then there is a discrete faithful representation ρ of Γ_0 into $\mathrm{Isom}(\mathbb{H}^2) \subset \mathrm{SL}^{\pm}(3, \mathbb{R})$, i.e. $\rho \in \mathcal{D}(\Gamma_0)$. We will study the space $\mathcal{D}(\Gamma_0)$ and visualize triangle reflection groups in $\mathbb{S}^2$.

Main references: [Lukyanenko] and [Vinberg and Kac]

Supplement: [Nie]

5.5.3. *Triangle groups.* We will understand the computational method to show that the family of representations of the triangle group

$$\Delta(3,3,4) = \langle a, b \,|\, a^3 = b^3 = (a.b)^4 = 1 \rangle$$

given by

$$a \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad b \mapsto \begin{pmatrix} 1 & 2 - t + t^2 & 3 + t^2 \\ 0 & -2 + 2t - t^2 & -1 + t - t^2 \\ 0 & 3 - 3t + t^2 & (-1+t)^2 \end{pmatrix}$$

are discrete and faithful for every $t \in \mathbb{R}$.

Main references: [Long, Reid, Thistlethwaite]

6. References

- P. Sarnak: *Integral Apollonian packings*, Amer. Math. Monthly 118 (2011) 291–306
- R.L. Graham, J.C. Lagarias, C.L. Mallows, A.R. Wilks, C.H. Yan: *Apollonian circle packings: geometry and group theory. I. The Apollonian group*, Discrete Comput. Geom. 34 (2005) 547–585
- H. Oh: *Apllonian circle packings, fractal geometry and dynamics on hyperbolic manifolds*, Slides, Joint mathematics meetings (2012), Available at http://www.math.brown.edu/~heeoh/
- J.G. Ratcliffe: *Foundations of hyperbolic manifolds*, Second edition, Springer (2006)
- P. Buser: *A Geometric proof of Bieberbach's theorems on crystallographic groups*, Enseign. Math. 31 (1985) 137–145
- L. Auslander: *An account of the theory of crystallographic groups*, Proc. Amer. Math. Soc. 16 (1965) 1230–1236
- G.E. Martin, *Transformation geometry: An introduction to symmetry*, Springer-Verlag (1982)
- J.H. Conway, H. Burgiel, C. Goodman-Strauss: *The symmetries of things*, A K Peters (2008)
- H.S.M. Coxeter, W.O.J. Moser, *Generators and relations for discrete groups*, Fourth edition, Springer-Verlag (1980)
- W.P. Thurston: *The geometry and topology of three-manifolds*, Lecture notes, Princeton University (1980) Available at http://library.msri.org/books/gt3m/
- R. Penrose: *Pentaplexity: a class of non-periodic tilings of the plane*, Math. Intelligencer 2 (1979/80) 32–37
- B. Grübaum, G.C. Shephard: *Tilings and patterns*, W.H. Freeman and Company (1987)
- N.G. de Bruijin: *Algebraic theory of Penrose's non-periodic tilings of the plane, I, II*, Nederl. Akad. Wetensch. Indag. Math. 43 (1981) 39–52
- G.A. Margulis, S. Mozes: *Aperiodic tilings of the hyperbolic plane by convex polygons*, Israel J. Math. 107 (1998) 319–325
- Y. Benoist: *A survey on divisible convex sets*, from "Geometry, analysis and topology of discrete groups"(2008) 1–18
- D. Cooper, D. Long, M. Thistlethwaite: *Computing varieties of representations of hyperbolic 3-manifolds into* $\mathrm{SL}(4,\mathbb{R})$, Experiment. Math. 15 (2006) 291–305
- A.V. Lukyanenko: *Projective deformations of triangle tilings*, Master Thesis, University of Maryland (2008)
- E.B. Vinberg, V.G. Kac: *Quasi-homogeneous cones*, Mat. Zametki 1 (1967) 347–354
- X. Nie, *On the Hilbert geometry of simplicial Tits sets*, arXiv:1111.1288
- D.D. Long, A. Reid, M. Thistlethwaite: *Zariski dense surface subgroups in* $\mathrm{SL}(3,\mathbb{Z})$, Geom. Topol. 15 (2011) 1–9