Seminar on Curves and Surfaces – Seminar über Kurven und Flächen

Wintersemester 2012/2013

Prof. Dr. Anna Wienhard, Laura Schaposnik and Dr. Gye-Seon Lee

Time and Location - Zeit und Ort

Thursday - Donnerstags 11 - 13 Uhr
INF 288, HS 5
First meeting: October 18, 2012

Topic – Inhalt

Curves and surfaces appear in many aspects of daily life, from knots, DNA, and soap bubbles to the surface of you coffee cup. This seminar will discuss geometric properties of curves and surfaces within three-dimensional space.

A standard theme will be the interplay between local geometric quantities (as e.g. curvature) and global properties.

In the first part of the seminar we will focus on curves, introduce their basic local invariants and discuss some global properties of closed curves.

In the second part of the seminar we will study surfaces, again first by introducing basic geometric invariants. We will see that some invariants depend crucially on the way the surface lies in three-dimensional space. Other invariants are determined by the inner geometry of the surface. This means that they could be detected and determined by someone who lives on the two-dimensional surface without seeing the ambient space. We will prove a theorem of Gauss, which he called Theorema egregious (distinguished theorem). This theorem states, that the Gauss curvature of a surface, which a priori is defined using information about how the surfaces lies in three-dimensional space, is in fact completely determined by the inner geometry of the surface. The Theorema egregium implies that it is impossible to make a correct map of the world. As second important theorem of Gauss, which we will prove is the Gauss-Bonnet Theorem, which states that a topological invariant, the Euler characteristic of the surface, is determined by the curvature.

In the second part we will discuss model geometries of surfaces of constant curvature, in particular spherical and hyperbolic geometry, as well as surfaces which carry these geometries. This will give us a small glimpse of other themes as e.g. homogeneous space, geometric structures and their moduli spaces.

Prerequisites – Vorkenntnisse

This seminar is aimed at students interested in differential geometry, but does not require any previous knowledge of differential geometry. Students are expected to have a solid background in analysis (e.g. by having taken the entire Analysis sequences). The seminar complements the topics of the lecture course on differential geometry. Interested students from physics are also welcome to attend.

The seminar will be taught in english. If this seems to be inconvenient for you, keep in mind that nowadays english is the prevalent language in mathematical literature, mathematical conference and within the international mathematical community, and thus expressing yourself in english will be an important skill you will have to develop anyway. The main references for the topics of the seminar are available in german as well as english.

Organization – Organisation

Students who would like to sign up for the seminar should send an email to Laura Schaposnik (lauraschaposnik@gmail.com) and let her know what talk you would like to give. If you are interested in one of the first talks, please be sure to contact her before the start of the semester. Once you picked a talk, please contact Laura or Gye-Seon (whoever is listed as advisor for your talk) to discuss the content of your talk in more detail.

You should start preparing for your talk several weeks before you are scheduled to give the talk. Please let Laura and Gye-Seon help you during the preparation.

Please read "Wie halte ich einen Seminarvortrag?" written by Prof. Dr. Manfred Lehn before you prepare your talk:
http://www.mathematik.uni-mainz.de/Members/lehn/le/seminarvortrag

1. Schedule – Vortragsplan

The schedule of talks is updated on www.????

1.1. **Local properties of curves.** This talk introduces regular curves and their basic local invariants (curvature and torsion). Some examples of interesting curves should be discussed (e.g. cycloid, tractix, logarithmic spiral), and the fundamental theorem of the local theory of curves should be formulated.

References: do Carmo 1.2- 1.5., Baer 2.1./2.3

(Advisor: Laura Schaposnik)

1.2. **Global properties of curves.** This talk discusses several results concerning global properties of curves: the isoperimetric inequality, Umlaufsatz and Four-Vertex Theorem. If time permits, one could formulate the converse to the Four Vertex Theorem.

References: do Carmo 1.7, Baer 2.2, perhaps also Gluck, and Turck/Gluck and others

(Advisor: Laura Schaposnik)

1.3. **Regular Surfaces.** This talk introduces regular surfaces in $\mathbb{R}^3$ and smooth maps between surfaces. Please discuss some examples, e.g. spheres, ellipsoids, zero sets of smooth functions (and counter-examples).

References: do Carmo 2.2/2.3, Baer 3.1

(Advisor: Gye-Seon Lee)

1.4. **Tangent space and the first fundamental form.** In this talk the tangent space to a regular surface is defined, then the first fundamental form is introduced. Application: surface area.

References: do Carmo 2.4./2.5. Baer 3.2/3.3

(Advisor: Gye-Seon Lee)

1.5. **Gauss map, second fundamental form and curvature.** This talk will introduce the Gauss map and and the central notion of curvature.

References: Do Carmo 2.6, 3.2 (until Definition 4, and Definition 6), 3.3., Baer 3.4/3.5/3.6

(Advisor: Gye-Seon Lee)

1.6. **Inner geometry - Theorema egregium.** This talk will introduce isometries, which are maps preserving the first fundamental form of the surface. The geometry which is governed by the first fundamental form is called the inner geometry of the surface. The Theorema egregium is proved in this talk: the Gauss curvature, which was defined using extrinsic quantities is actually a quantity of the inner geometry of the surface. A consequence of this is that there is no accurate map of the earth. If time permits discuss different projections of the earth.

References: Do Carmo 4.2/4.3, Baer 4.1/ 4.3

(Advisor: Gye-Seon Lee)

1.7. **Theorem of Gauss-Bonnet I.** This is the first of two talks which will introduce and prove the Gauss-Bonnet theorem.

References: do Carmo 4.5, Baer 6.1/6.2/6.3

(Advisor: Gye-Seon Lee)

1.8. **Theorem of Gauss-Bonnet II.** This is the second of two talks which will introduce and prove the Gauss-Bonnet theorem.

References: do Carmo 4.5, Baer 6.1/6.2/6.3

(Advisor: Gye-Seon Lee)

1.9. **Standard models of surfaces with constant Gauss curvature I.** This is the first of two talks that will discuss the standard models of surfaces of constant Gauss curvature, their geodesics, their isometries. The trigonometric formulas for hyperbolic and spherical triangles should be discussed, as well as other models for the hyperbolic plane.

References: Baer 4.9

(Advisor: Laura Schaposnik)

1.10. **Standard models of surfaces with constant Gauss curvature II.** This is the second of two talks that will discuss the standard models of surfaces of constant Gauss curvature, their geodesics, their isometries. The trigonometric formulas for hyperbolic and spherical triangles should be discussed, as well as other models for the hyperbolic plane.

References: Baer 4.9/4.11

(Advisor: Laura Schaposnik)

1.11. **Rigidity of the sphere.** The sphere is the only closed surface locally isometric to a sphere.

References: do Carmo 5.2 (only available in the english copy of the book)

(Advisor: Laura Schaposnik)

1.12. **Classification of flat tori.** A torus admits several different flat structures. This talk discusses this moduli space.

References: Gallot Hulin Lafontaine pp. 59-63

(Advisor: Laura Schaposnik)

1.13. **Metrics of negative curvature on surfaces.** In this talk we prove that surfaces of negative Euler characteristic admit metrics of constant negative curvature. The study of these metric leads to the study of Teichmüller space and the moduli space of Riemann surfaces.

References: Buser, Klingenberg Theorem 4.3.13.

(Advisor: Laura Schaposnik)

References

The main references for this seminar are the books by Manfredo doCarmo and by Christian Baer. Both books are available in english and german.

(1) Manfredo DoCarmo: Differential Geometry of Curves and Surfaces (available in english and german)
(2) Christian Bär: Elementare Differentialgeometrie (available in english and german)
(3) Gluck: L'Einseignement Mathematique 17(1971), 295– 309.
(4) Dennis DeTurck (University of Pennsylvania), Herman Gluck (University of Pennsylvania), Daniel Pomerleano (University of California, Berkeley) and Shea Vela-Vick (Columbia University) : the Four Vertex Theorem and its Converse, Notices Amer. Math. Soc, 54(2007), no. 2, 192-207
(5) Sylvestre Gallot, Dominique Hulin, Jacques LaFontaine: Riemannian geometry.
(6) Buser: Geometry and Spectra of Compact Riemann Surfaces.
(7) Wilhelm Klingenberg. Klassische Differentialgeometrie. Eagle, Leipzig, 2004.