Seminar: Geometry of Lie Groups

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Sommersemester 2016

1. Time and Location

Tuesdays 4 - 6 p.m. (First meeting: 19 April 2016) and INF 205, SR 2.

2. Description

The goal of this seminar is to familiarize students with the basic language of Lie groups and to introduce some important theorems about this topic, e.g. the Lie correspondence, the closed subgroup theorem and the quotient manifold theorem, and thereafter we begin the study of homogeneous spaces and, in particular, symmetric spaces. Moreover, we also briefly touch on lattices in Lie groups.

At the organizational meeting we will propose several topics (see Section 5). Among them, you can choose one topic you are interested in. During the first half of the semester, students will work on their project, which they will present during the second half of the semester. If the topic is difficult, it is also possible for students to get together in teams to carry out one of the topics. Every student is expected to hand in a short report as well.

3. Prerequisites

This seminar is aimed at students who are interested in differential geometry. Students are expected to have a good knowledge of calculus and a certain familiarity with the differential geometry of manifolds. There are several topics for students to choose from, and hence any interested students are welcome to attend.

The seminar is open for bachelor and master students. The seminar will be taught in English.

4. Organization

Students who would like to sign up for the seminar should come to the organization meeting, or send an email to Daniele (alessandrini@mathi.uni-heidelberg.de) or Gye-Seon (lee@mathi.uni-heidelberg.de). Please register at MÜSLI and let us know what project you would like to carry out. Once you have agreed on the topic of your project with us, you should contact us to discuss the content of your project in more detail. Please let us help you during the preparation of your presentation.

5. Topics

The schedule of presentations will be updated on the web page:
http://www.mathi.uni-heidelberg.de/~lee/seminarSS16.html

5.1. **Lie groups.**

We begin with the definition of Lie groups and some of the basic structures associated with them, and then present a number of examples. Next we study Lie group homomorphisms, and also introduce Lie subgroups and actions of Lie groups on manifolds, which are the primary purpose of Lie groups.

Main references: [Lee §7]

5.2. **The Lie algebra of a Lie group.** We introduce the Lie bracket operation and the Lie algebra of the Lie group. We describe a few basic properties of Lie algebras, and compute the Lie algebras of some familiar groups. Then we show how Lie group homomorphisms induce homomorphisms of their Lie algebras.

Main references: [Lee §8]

Supplement: [Duistermaat and Kolk §1.1–1.2]

5.3. **The Lie correspondence.** We prove that there is a one-to-one correspondence between finite-dimensional Lie algebras and simply connected Lie groups; and then we show that for any Lie group G, connected normal subgroups of G correspond to ideals in the Lie algebra of G.

Main references: [Lee §20]

5.4. **The closed subgroup theorem.** We introduce a canonical smooth map from the Lie algebra into the Lie group called the exponential map. As an application of the exponential map, we prove the closed subgroup theorem, which says that every topologically closed subgroup of a Lie group is actually an embedded Lie subgroup.

Main references: [Lee §20]

Supplement: [Duistermaat and Kolk §1.3–1.4]

5.5. **Classical groups and their real form.** Classical groups are the most important examples of Lie groups. They are presented as matrix groups, so they are very explicit, on the other hand they are also very general, since "almost" all simple Lie groups are classical groups. Classical groups are usually defined as the set of invertible matrices preserving a bilinear form. Their corresponding Lie algebra can also be described explicitly. Their topology is very interesting, for example we can compute their fundamental group.

Main references: [Rossmann §3, Goodman and Wallach §1]

5.6. **The quotient manifold theorem.** We prove the quotient manifold theorem, which asserts that a Lie group G acting smoothly, freely, and properly on a smooth manifold M yields a quotient space with a natural smooth manifold structure. Moreover, M become a principal fiber bundle with structure group G. After the proof, we explore one special classes of Lie group actions, which are actions by discrete groups and which yield covering maps.

Main references: [Lee §21]

Supplement: [Duistermaat and Kolk §1.11]

5.7. **Homogeneous spaces.** We study homogeneous spaces, which are smooth manifolds endowed with smooth transitive Lie group actions. We show that they are equivalent to Lie groups modulo closed subgroups. Then we describe a number of applications of homogeneous space theory to the Lie theory, and also give some examples of homogeneous spaces.

Main references: [Lee §21] and [Warner §3]

Supplement: [Arvanitoyeorgos §4]

5.8. **Symmetric spaces.** We introduce symmetric spaces, which are among the most important examples of Riemannian manifolds. They are homogeneous spaces together with involutive isometries. We explain some general properties of symmetric spaces, and also give some examples of them.

Main references: [Boothby §7.8 – 7.9]

Supplement: [Arvanitoyeorgos §6]

5.9. **Bieberbach's theorems.** Let $\mathbb{E}^n$ be the n-dimensional Euclidean space and $\mathrm{Isom}(\mathbb{E}^n)$ be the group of isometries of $\mathbb{E}^n$. An n-dimensional crystallographic group is a discrete group Γ of $\mathrm{Isom}(\mathbb{E}^n)$ such that $\mathbb{E}^n/\Gamma$ is compact. In this talk, we study the theory of crystallographic groups. In particular, we will understand the proof of Bieberbach's (three) theorems:

(1) Every n-dimensional crystallographic group contains n linearly independent translations.
(2) For each fixed n, there are only finitely many isomorphism classes of n-dimensional crystallographic groups.
(3) Two crystallographic groups are isomorphic if and only if they are conjugate by an affine transformation.

Main references: [Ratcliffe §7.5] and [Buser]
Supplement: [Auslander]

5.10. **A geometric construction of lattices.** We construct a few lattices in $SO(p,1)$ by the geometric method, when $p \leq 9$. This geometric method of construction of lattices has been initiated by Poincaré in 1880. We present a proof, due to Vinberg, of an extension of Poincaré theorem and to describe some of these explicit lattices. In particular, these are reflection groups.
Main references: [Benoist §1]
Supplement: [Vinberg]

5.11. **A example of arithmetic groups.** The aim of this talk is to give explicit constructions of lattices in the real Lie group $\mathrm{SL}(d,\mathbb{R})$, which are examples of a general arithmetic construction of lattices in any semisimple Lie group, due to Borel and Harish-Chandra. In particular, we will show that $\mathrm{SL}(d,\mathbb{R})$ is a non-compact lattice in $\mathrm{SL}(d,\mathbb{R})$.
Main references: [Benoist §2] and [Morris §7]
Supplement: [Abbaspour and Moskowitz §8]

6. References

- J.M. Lee: *Introduction to smooth manifolds*, Second edition, Springer (2013)
- J.J.Duistermaat, J.A.C.Kolk: *Lie groups*, Springer (2000)
- F.W. Warner: *Foundations of differentiable manifolds and Lie groups*, Springer (1983)
- A. Arvanitoyeorgos: *An introduction to Lie groups and the geometry of homogeneous spaces*, American Mathematical Society (2003)
- W.M. Boothby: *An introduction to differentiable manifolds and Riemannian geometry*, Revised second edition, Academic Press (2003)
- J.G. Ratcliffe: *Foundations of hyperbolic manifolds*, Second edition, Springer (2006)
- P. Buser: *A Geometric proof of Bieberbach's theorems on crystallographic groups*, Enseign. Math. 31 (1985) 137–145
- L. Auslander: *An account of the theory of crystallographic groups*, Proc. Amer. Math. Soc. 16 (1965) 1230–1236

- Y. Benoist: *Five lectures on lattices in semisimple Lie groups* in the book *Géométries à courbure négative ou nulle, groupes discrets et rigidités*, Sémin. Congr. 18 (2009) 117–176
- È.B. Vinberg: *Discrete linear groups that are generated by reflections*, Izv. Akad. Nauk SSSR Ser. Mat. 35 (1971) 1072–1112
- E.W. Morris: *Introduction to arithmetic groups*, Deductive Press (2015)
- W. Rossmann: *Lie groups An introduction through Linear Groups*, Oxford GTM (2002).
- R. Goodman, N. Wallach: *Symmetry, Representations and Invariants*, Springer (2009).