Seminar on Characteristic Classes

Sommersemester 2013

Prof. Dr. Anna Wienhard, Dr. Laura Schaposnik and Dr. Gye-Seon Lee

1. Time and Location

Thursdays 2 - 4 p.m. (First meeting: 18 April 2013) and INF 288, HS 1

2. Topic

Characteristic classes are invariants of fibre bundles or principal bundles on manifolds. They are important for the classification of manifolds. They also play an important role in differential geometry, gauge theory and related topics in physics.

In this seminar we will introduce Stiefel-Whitney classes, the Euler class, Chern classes and Pontrjagin classes and study some of their properties. We will then discuss Chern-Weil theory, which describes how characteristic classes can be calculated using the curvature of the principal bundle. In the end we discuss an example of characteristic classes of flat bundles, due to Milnor, which shows us the limitations of Chern-Weil theory.

3. Prerequisites

Students should ideally have some background in algebraic topology and differential geometry. Interested students from physics are also welcome to attend.

The seminar will be taught in english. If this seems to be inconvenient for you, keep in mind that nowadays english is the prevalent language in mathematical literature, mathematical conference and within the international mathematical community, and thus expressing yourself in english will be an important skill you will have to develop anyway.

4. Organization

Students who would like to sign up for the seminar should send an email to Laura Schaposnik (schaposnik@mathi.uni-heidelberg.de) and let her know what talk they would like to give. If you are interested in one of the first talks, please be sure to contact her before the start of the semester.

Once you have agreed on the date of your talk with Laura, you should contact the person who appears as advisor for your talk, Laura or Gye-Seon (lee@mathi.uni-heidelberg.de), to discuss the content of your talk in more detail. You should start preparing for your talk several weeks before you are scheduled to give speak. Please let Laura and Gye-Seon help you during the preparation.

Please read "Wie halte ich einen Seminarvortrag?"written by Prof. Dr. Manfred Lehn before you prepare your talk:

http://www.mathematik.uni-mainz.de/Members/lehn/le/seminarvortrag

5. Schedule

The schedule of talks is updated on the web page:

http://www.mathi.uni-heidelberg.de/~lee/seminarSS13.html

5.1. **Vector bundles.** This talk introduces vector bundles and their basic constructions. Some examples of vector bundles should be discussed (e.g. tangent bundle, normal bundle), and it should be described how to construct new vector bundles out of old ones. Moreover, complex vector bundles and Hermitian metrics should be covered.

References: Milnor and Stasheff §2/§3/§13/§14 p.156–p.157, Hatcher(2) §1.1, Morita §5.1

(Advisor: Laura)

5.2. **Homology and Cohomology.** This talk reviews homology and cohomology theory including the relationship between them. The cross product, cup product and Poincaré duality theorem should be discussed. Moreover, the Gysin sequence of a vector bundle should be covered. For this, oriented bundles, Euler classes and the Thom isomorphism theorem might be briefly introduced.

References: Milnor and Stasheff §9 p.95–p.98/§10/§12 p.143–p.145/Appendix A, Hatcher(1) §2/§3/§4.D

(Speaker: Laura)

5.3. **Axioms of Stiefel-Whitney and Chern classes.** This talk introduces Stiefel-Whitney classes and Chern classes, and discusses the properties of Euler classes. Axioms which characterize Stiefel-Whitney classes (resp. Chern classes) shall be given, and their consequences and examples should be discussed.

References: Milnor and Stasheff §4/§9 p.98–p.102, Hatcher(2) §3.1/§3.2, Hirzebruch §4.2

(Advisor: Laura)

5.4. **Constructions of Stiefel-Whitney and Chern classes.** This talk discusses the constructions of Stiefel-Whitney classes and Chern classes. The universal bundles, the cell structure and the cohomology for Grassmann manifolds should be discussed.

References: Milnor and Stasheff §5/§6/§7/§8/§14, Hatcher(2) §1.2/§3.1

(Speaker: Gye-Seon)

5.5. **Pontrajagin classes and Pontrjagin numbergs.** This talk introduces Pontrjagin classes. The cohomology of the oriented Grassmannian manifold and its relation with Pontrjagin classes should be discussed. Moreover, Chern numbers of a compact complex manifold should be discussed, as well as Pontrjagin numbers of a compact oriented manifold.

References: Milnor and Stasheff §15/§16, Hatcher(2) §3.2

(Advisor: Gye-Seon)

5.6. **Chern-Weil theory I.** This is the first of two talks that discuss Chern-Weil theory. Connections and curvatures on vector bundles should be introduced, then it should be discussed how to get a characteristic cohomology class from the curvature.

References: Milnor and Stasheff Appendix C p.289–p.300, Morita §5.3/§5.4/§5.5

(Advisor: Gye-Seon)

5.7. **Chern-Weil theory II.** This is the second of two talks that discuss Chern-Weil theory. Chern classes and Pontrjagin classes should be described in terms of curvature, and the generalized Gauss-Bonnet theorem should be proven.

References: Milnor and Stasheff Appendix C p.300–p.312, Morita §5.5/§5.6/§5.7, Milnor's article

(Advisor: Gye-Seon)

5.8. **Characteristic classes of flat bundles.** In this talk flat bundles, an example of vector bundles such that the Euler class with real coefficients is non-zero, should be discussed.

References: Milnor and Stasheff Appendix C p.312–p.314, Milnor's article

(Advisor: Laura)

6. References

The main references for this seminar is the book by John W. Milnor and James D. Stasheff.

- John W. Milnor and James D. Stasheff: Characteristic classes
- Allen Hatcher: (1) Algebraic topology, (2) Vector bundles and K-theory
- Shigeyuki Morita: Geometry of differential forms
- John Milnor: On the existence of a connection with curvature zero, Comment. Math. Helv. 32 (1958) 215–223
- Norman Steenrod: The topology of fibre bundles
- Dale Husemöller: Fibre bundles
- Friedrich Hirzebruch: Topological methods in algebraic geometry