Seminar: "Differntial forms and their use"
Various operations on differential forms

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February 5, 2015

Let M be an n-dimensional C^∞ manifold. We denote all k-forms on M by $\mathcal{A}^k(M)$ and consider their direct sum

$$\mathcal{A}^*(M) = \bigoplus_{k=0}^{n} \mathcal{A}^k(M)$$

with respect to k, that is, the set of all differential forms on M. We will define various operations on $\mathcal{A}^*(M)$. And denote the set of all the vector fields on M by $\mathfrak{X}(M)$.

1 Preparation

Proposition 1.1. *The map $\iota : \Lambda^* V^* \to \mathcal{A}^*(V)$ is an isomorphism. That is, the exterior algebra $\Lambda^* V^*$ of V^* and the vector space $\mathcal{A}^*(V)$ of all alternating forms on V can be identified by ι. Using this, a product is defined on $\mathcal{A}^*(V)$ which is described as follows. If for $\omega \in \Lambda^k V^*$, $\eta \in \Lambda^l V^*$, we consider their exterior product $\omega \wedge \eta$ as an element of $\mathcal{A}^{k+l}(V)$ by the identification ι, we have*

$$\begin{aligned}
&\omega \wedge \eta(X_1, \cdots, X_{k+l}) \\
&= \frac{1}{(k+l)!} \sum_{\sigma} \operatorname{sgn}\sigma \, \omega(X_{\sigma(1)}, \cdots, X_{\sigma(k)}) \eta(X_{\sigma(k+1)}, \cdots, X_{\sigma(k+l)}) \\
&(X_i \in V)
\end{aligned} \tag{1.1}$$

Here σ runs over the set $\mathfrak{G}_{k+l}$ of all permutations of $k+l$ letters $1, 2, \cdots, k+l$.

Theorem 1.1. *Let M be a C^∞ manifold. Then the set $\mathcal{A}^k(M)$ of all k-forms on M can be naturally identified with that of all multilinear and alternating maps, as $C^\infty(M)$ modules, from k-fold direct product of $\mathfrak{X}(M)$ to $C^\infty(M)$.*

1.1 The bracket of vector fields

Definition 1.1. *Let M be a C^∞ manifold and p a point of M. If a map $v : C^\infty(M) \mapsto \mathbb{R}$ satisfies the conditions*

(i) $v(f+g) = v(f) + v(g)$, $v(af) = av(f)$,

(ii) $v(fg) = v(g)g(p) + f(p)v(g)$,

for arbitrary functions $f, g \in C^\infty(M)$ and $a \in \mathbb{R}$, then v is said to be a ***tangent vector*** *to M at p.*

Proposition 1.2. *Let M and N be C^∞ manifolds and $f : M \to N$ a C^∞ map. Then, for an arbitrary tangent vector $v \in T_pM$ at the point p on M and an arbitrary function $h \in C^\infty(N)$ on N,*

$$v(h \circ f) = f_*(v)h.$$

Proposition 1.3. *Let M be a C^∞ manifold and X, Y vector field on M. If $Xf = Yf$ for an arbitrary C^∞ function f on M, then $X = Y$*

Proof. see page 38 in the reference book $\square$

Let $X, Y \in X(M)$ be two vector fields on a C^∞ manifold M. Then, both X, Y act on $C^\infty(M)$ as derivations. Consider a map

$$C^\infty(M) \ni f \mapsto X(Yf) - Y(Xf) \in C^\infty(M). \tag{1.2}$$

By an easy calculation, we can check that this map also has two properties of the derivation. If we rewrite it as $X(Yf) - Y(Xf) = (XY - YX)f$, it indicates that $XY - YX$ expresses a vector field on M. Actually, using the symbol $[X, Y]$ instead of $XY - YX$, we consider the correspondence

$$C^\infty(M) \ni f \mapsto [X, Y]_p f = X_p(Yf) - Y_p(Xf) \in \mathbb{R} \tag{1.3}$$

at each point $p \in M$. From the fact that (1.2) satisfies the properties of a derivation, we immediately see that the correspondence (1.3) satisfies the condition (see Definition 1.1) of tangent vectors at the point p. That is, we can consider $[X, Y]_p$ as a tangent vector to M at p. If it is shown that $[X, Y]_p$ is of class C^∞ with respect to p, we can conclude that $[X, Y]$ is a vector field on M. In order to check that $[X, Y]$ is a vector field on M, we give X, Y the local expressions

$$X = \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i}, Y = \sum_{i=1}^{n} b_i \frac{\partial}{\partial x_i}.$$

By an easy calculation, we have

$$[X, Y]_p f = \sum_{i,j=1}^{n} (a_i(p) \frac{\partial b_i}{\partial x_i}(p) - b_i(p) \frac{\partial a_j}{\partial x_i}(p)) \frac{\partial f}{\partial x_j}(p).$$

From this, we see that $[X, Y]$ is a vector field on M, and simultaneously its local experssion is given by

$$[X, Y] = \sum_{i,j=1}^{n} (a_i \frac{\partial b_j}{\partial x_i} - b_j \frac{\partial a_j}{\partial x_i}) \frac{\partial}{\partial x_j} \tag{1.4}$$

Proposition 1.4. *The bracket of vector fields has the following properties.*

(i) $[aX + bX', Y] = a[X, Y] + b[X', Y]$ *($a, b \in \mathbb{R}$), and the same for Y.*

(ii) $[Y, X] = -[X, Y]$

(iii) *(Jacobi identity)* $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$

(iv) $[fX, gY] = fg[X, Y] + f(Xg)Y - g(Yf)X$ *($f, g \in C^\infty(M)$).*

1.2 Transformations of vector fields by diffeomorphism

Let M, N be C^∞ manifolds and $f : M \to N$ a diffeomorphism from M to N. Then, for an arbitrary vector field X on M, a vector field f_*X on N is defined by

$$(f_*X)_q = f_*(X_{f^{-1}(q)})(q \in N).$$

Or equivalently we can write $f_*(X_p) = (f_*X)_{f(p)}(p \in M)$. Then, for an arbitrary function $h \in C^\infty(N)$ on N, we have

$$(f_*X)h = X(h \circ f) \circ f^{-1}. \tag{1.5}$$

This holds because, for a point $q \in N$, we have

$$((f_*X)h)(q) = (f_*X)_q h = f_*(X_{f^{-1}(q)})h,$$

and on the other hand, by Proposition 1.2, we have

$$f_*(X_{f^{-1}(q)})h = X_{f^{-1}(q)}(h \circ f) = (X(h \circ f) \circ f^{-1})(q).$$

2 Exterior product

Definition 2.1. *The **exterior product** $\omega \wedge \eta \in \mathcal{A}^{k+l}(M)$ of a k-form $\omega \in \mathcal{A}^k(M)$ and an l-form $\eta \in \mathcal{A}^l(M)$ on M is defined as follows. Since at each point $p \in M$ we have $\omega_p \in \Lambda^k T_p^* M$, $\eta_p \in \Lambda^l T_p^* M$, their product $\omega_p \wedge \eta_p \in \Lambda^{k+l} T_p^* M$ is defined. Then, we put*

$$(\omega \wedge \eta)_p = \omega_p \wedge \eta_p.$$

By definition, the exterior product is obviously associative. That is, if $\tau \in \mathcal{A}^m(M)$, we have $(\omega \wedge \eta) \wedge \tau = \omega \wedge (\eta \wedge \tau)$. Therefore we do not need the parentheses. If they are locally expressed as $\omega = f dx_{i_1} \wedge \cdots \wedge dx_{i_k}$, $\eta = g dx_{j_1} \wedge \cdots \wedge dx_{j_l}$, we have

$$\omega \wedge \eta = fg dx_{i_1} \wedge \cdots dx_{i_k} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_l}.$$

Proposition 2.1. *The exterior product induces a bilinear map*

$$\mathcal{A}^k(M) \times \mathcal{A}^l(M) \ni (\omega, \eta) \mapsto \omega \wedge \eta \in \mathcal{A}^{k+l}(M)$$

and it has the following properties.

(i) $\eta \wedge \omega = (-1)^{kl} \omega\eta$.

(ii) For arbitrary vector fields $X_1, \cdots, X_{k+l} \in \mathfrak{X}(M)$,

$$\begin{aligned} &\omega \wedge \eta(X_1, \cdots, X_{k+l}) \\ &= \frac{1}{(k+l)!} \sum_{\sigma \in \mathfrak{S}_{k+l}} \mathrm{sgn}\sigma \omega(X_{\sigma(1)}, \cdots, X_{\sigma(k)}) \eta(X_{\sigma(k+1)}, \cdots, X_{\sigma(k+l)}) \end{aligned} \tag{2.1}$$

Property (i) is obvious from the description above, and (ii) follows from 1.1.

3 Exterior differentiation

Definition 3.1. *For a k-form $\omega \in \mathcal{A}^k(M)$ on M, its **exterior differentiation** $d\omega \in \mathcal{A}^{k+1}(M)$ is the operation defined by*

$$d\omega = \sum_j \frac{\partial f}{\partial x_j} dx_j \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}; \tag{3.1}$$

here ω is locally expressed as $\omega = f dx_{i_1} \wedge \cdots \wedge dx_{i_k}$.

In view of the fact that for the isomorphism $\varphi^* : \mathcal{A}^*(U') \to \mathcal{A}^*(U)$ induced by an arbitrary diffeomorphism $\varphi : U \to U'$ between two open sets U, U' of $\mathbb{R}^n$, the equation $d \circ \varphi^* = \varphi^* \circ d$ holds (see the description following (2.4) in the reference book), we see that the above d does not depend on the local expression. Therefore, the operation of taking the exterior differentiation defines a degree 1 (that is, increasing the degree by 1) linear map

$$d : \mathcal{A}^k(M) \to \mathcal{A}^{k+1}(M),$$

andwe can see that it has the following properties.

Proposition 3.1. *(i)* $d \circ d = 0$.

(ii) For $\omega \in \mathcal{A}^k(M)$, $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$.

These can be easily proved by previous results.

Next, we shall characterize the exterior differentiation without using the local expression. Namely, we have the following theorem.

Theorem 3.1. *Let M be a C^∞ manifold and $\omega \in \mathcal{A}^k(M)$ an arbitrary k-form on M. Then for arbitrary vector fields $X_1, \cdots, X_{k+1} \in \mathfrak{X}(M)$, we have*

$$\begin{aligned} & d\omega(X_1, \cdots, X_{k+1}) \\ & = \frac{1}{k+1}\{\sum_{i=1}^{k+1}(-1)^{i+1} X_i(\omega(X_1, \cdots, \hat{X}_i, \cdots, X_{k+1})) \\ & + \sum_{i<j}(-1)^{i+j}\omega([X_i, X_j], X_1, \cdots, \hat{X}_i, \cdots, \hat{X}_j, \cdots, X_{k+1})\}. \end{aligned} \tag{3.2}$$

Here the symbol $\hat{X}_i$ means X_i is omitted. In particular, the often-used case of $k = 1$ is

$$d\omega(X, Y) = \frac{1}{2}\{X\omega(Y) - Y\omega(X) - \omega([X, Y])\} \quad (\omega \in \mathcal{A}^1(M)).$$

Proof. If we consider the right-hand side of the formula to be proved, as a map from the $(k+1)$-fold direct product of $\mathfrak{X}(M)$ to $C^\infty(M)$, we see that it satisfies the conditions of degree $k+1$ alternating from as a map between modules over $C^\infty(M)$. Since it is easy to verify this fact by using Proposition 1.4 (iv), we leave it to reader. Therefore, by Theorem 1.1, we see that the right-hand side is a $(k+1)$-form on M.

If two differential forms coincide in some neighborhood of an arbitrary point, they coincide on the whole. Then, consider a local coordinate system $(U; x_1, \cdots, x_n)$ around an arbitrary point $p \in M$. Let the local expression of ω with respect to this local coordinate system be $\omega = \sum_{i_1<\cdots<i_k} f_{i_1\cdots i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k}$. Then we have

$$d\omega = \sum_{i_1<\cdots<i_k} df_{i_1\cdots i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k} \tag{3.3}$$

From the linearity of differential forms with respect to the functions on M, it is enough to consider only vector fields X_i such that $X_i = \frac{\partial}{\partial x_{j_i}}(i = 1, \cdots, k+1)$ in a neighborhood of p. Then $[X_i, X_j] = 0$ near p. Moreover, by the alternating property of differential forms, we may assume that $j_1 < \cdots < j_{k+1}$. Then, if we apply (3.3) to $(X_1, \cdots, X_{k+1})$, we have

$$d\omega(X_1, \cdots, X_{k+1}) = \frac{1}{(k+1)!}\{\sum_{s=1}^{k+1}(-1)^{s-1}\frac{\partial}{\partial x_{j_s}} f_{j_1\cdots\hat{j}_s\cdots j_{k+1}}\}.$$

On the other hand, when we calculate the right hand side of the formula using $[X_i, X_j] = 0$, we obtain the same value. This finishes the proof. $\square$

We consider Theorem 3.2 as a definition of the exterior differentiation that is independent of the local coordinates.

4 Pullback by a map

We shall study the relationship between differential forms and C^∞ maps. Let

$$f : M \to N$$

be a C^∞ map from a C^∞ manifold M to N. Consider the differential $f_* : T_pM \to T_{f(p)}N$ of f at each point $p \in M$. f_* induces its dual map $f^* : T^*_{f(p)}N \to T^*_pM$, that is, the map defined by $f^*(\alpha)(X) = \alpha(f_*(X))$ for $\alpha \in T^*_{f(p)}N$, $X \in T_pM$. Furthermore, f^* defines a linear map $f^* : \Lambda^k T^*_{f(p)}N \to \Lambda^k T^*_p M$ for an arbitrary k, and they induce an algebra homomorphism

$$f^* : \mathcal{A}^*(N) \to \mathcal{A}^*(M).$$

For a differential form $\omega \in \mathcal{A}^k(N)$ on N, $f^*\omega \in \mathcal{A}^k(M)$ is called the **pullback** by f. Explicitly, for $X_1, \cdots, X_k \in T_pM$,

$$f^*\omega(X_1, \cdots, X_k) = \omega(f_*X_1, \cdots, f_*X_k).$$

Proposition 4.1. *Let M, N, be C^∞ manifolds. Let $f : M \to N$ be a C^∞ map and $f^* : \mathcal{A}^*(N) \to \mathcal{A}^*(M)$ the map induced by f. Then f^* is linear and has the following properties.*

(i) $f^*(\omega \wedge \eta) = f^*\omega \wedge f^*\eta \quad (\omega \in \mathcal{A}^k(N), \eta \in \mathcal{A}^l(N)).$

(ii) $d(f^*\omega) = f^*(d\omega)$

The proof can be given easily by using the previous results.

5 Interior product and Lie derivative

Definition 5.1. *Let M be a C^∞ manifold and $X \in \mathfrak{X}(M)$ a vector field on M. Then a linear map*

$$i(X) : \mathcal{A}^k(M) \to \mathcal{A}^{k-1}(M)$$

is defined by

$$(i(X)\omega)(X_1, \cdots, X_{k-1}) = k\omega(X, X_1, \cdots, X_{k-1})$$

*for $\omega \in \mathcal{A}^k(M)$, $X_1, \cdots, X_{k-1} \in \mathfrak{X}(M)$. Note that if $k = 0$, we define $i(X) = 0$. We call $i(X)\omega$ the **interior product** of ω by X.*

By definition, $i(X)$ is obviously linear with respect to functions. That is, $i(X)(f\omega) = fi(X)\omega$. Using Proposition 1.1, we see that $i(X)$ is an **anti-derivation** of degree -1, that is,

$$\begin{aligned} &i(X)(\omega \wedge \eta) \\ &= i(X)\omega \wedge \eta + (-1)^k \omega \wedge i(X)\eta \quad (\omega \in \mathcal{A}^k(M), \eta \in \mathcal{A}^l(M)). \end{aligned} \tag{5.1}$$

Definition 5.2. *Define a linear operator*

$$L_X : \mathcal{A}^k(M) \to \mathcal{A}^k(M),$$

*called **Lie derivative**, also concerning the vector field $X \in \mathfrak{X}(M)$. This is defined by*

$$\begin{aligned} &(L_X\omega)(X_1, \cdots, X_k) \\ &= X\omega(X_1, \cdots, X_k) - \sum_{i=1}^k \omega(X_1, \cdots, [X, X_i], \cdots, X_k). \end{aligned} \tag{5.2}$$

It is easy to see that the right-hand side of this formula 5.2 satisfies the condition of Theorem 1.1, so that $L_X\omega$ is definitely a differential form. Obviously L_X is linear. This definition 5.2 is extremely algebraic. Although we may say that the formula is neat and beautiful, it is not clear what it means geometrically. We shall give a definition that makes the meaning clearer in the next section.

Similar things can be said also for the exterior product and the exterior differentiation. We first introduced both exterior product and the exterior differentiation with geometric definitions int terms of local expressions. However, leaving them aside, we can use formula (2.1) for exterior product and Theorem 3.2 for exterior differentiation as algebraic definitions.

As for the Lie derivative, we use (5.2) as its definition for the moment, and proceed.

6 The Cartan formula and properties of Lie derivatives

The following theorem represents the relationship between two operators concerning a vector field X, namely, the interior product $i(X)$ and the Lie derivative L_X, and is sometimes called the Cartan formula.

Theorem 6.1. *(Cartan formula)*

(i) $L_X i(Y) - i(Y)L_X = i([X,Y])$.

(ii) $L_X = i(X)d + di(X)$.

Proof. First, we prove (i). It is obvious for $k=0$, so let ω be an arbitrary k-form with $k>0$. Then, for any $X_1, \cdots, X_{k-1} \in \mathfrak{X}(M)$,

$$\begin{aligned}
&(L_X i(Y)\omega)(X_1,\cdots,X_{k-1})\\
&= X((i(Y)\omega)(X_1,\cdots,X_{k-1})) - \sum_{i=1}^{k-1}(i(Y)\omega)(X_1,\cdots,[X,X_i],\cdots,X_{k-1})\\
&= k\{X(\omega(Y,X_1,\cdots,X_{k-1})) - \sum_{i=1}^{k-1}\omega(Y,X_1,\cdots,[X,X_i],\cdots,X_{k-1})\}.
\end{aligned} \tag{6.1}$$

On the other hand,

$$\begin{aligned}
&(i(Y)L_X\omega)(X_1,\cdots,X_{k-1})\\
&= kL_X\omega(Y,X_1,\cdots,X_{k-1})\\
&= k\{X(\omega(Y,X_1,\cdots,X_{k-1})) - \omega([X,Y],X_1,\cdots,X_{k-1})\\
&\quad - \sum_{i=1}^{k-1}\omega(Y,X_1,\cdots,[X,X_i],\cdots,X_{k-1})\}.
\end{aligned} \tag{6.2}$$

Subtracting (6.2) from (6.1), we have

$$L_X i(Y)\omega - i(Y)L_X\omega = i([X,Y])\omega,$$

and (i) is proved.

Next we shall prove (ii). When $k=0$, since $L_X f = Xf$ for a function f and on the other hand $i(X)f = 0$ and $i(X)df = df(X) = Xf$, (ii) holds. Thus, let $k>0$, and let ω be a k-form and

$X_1, \cdots, X_k$ vector fields. Then, we have

$$
\begin{aligned}
&(i(X)d\omega)(X_1, \cdots, X_k) \\
&= (k+1)d\omega(X, X_1, \cdots, X_k) \\
&= X(\omega(X_1, \cdots, X_k)) + \sum_{i=1}^{k}(-1)^i X_i(\omega(X, X_1, \cdots, \hat{X}_i, \cdots, X_k)) \\
&\quad + \sum_{j=1}^{k}(-1)^j \omega([X, X_j], X_1, \cdots, \hat{X}_j, \cdots, X_k) \\
&\quad + \sum_{i<j}(-1)^{i+j}\omega([X_i, X_j], X, \cdots, \hat{X}_i, \cdots, \hat{X}_j \cdots, X_k),
\end{aligned} \tag{6.3}
$$

and on the other hand we have

$$
\begin{aligned}
&(di(X)\omega)(X_1, \cdots, X_k) \\
&= \sum_{i=1}^{k}(-1)^{i+1} X_i(\omega(X, X_1, \cdots, \hat{X}_i, \cdots, X_k)) \\
&\quad + \sum_{i<j}(-1)^{i+j}\omega(X, [X, X_j], X_1, \cdots, \hat{X}_i, \cdots, \hat{X}_j, \cdots, X_k).
\end{aligned} \tag{6.4}
$$

Summing up (6.3) and (6.4), we have

$$
\begin{aligned}
&(i(X)d + di(X))\omega(X_1, \cdots, X_k) \\
&= X(\omega(X_1, \cdots, X_k)) + \sum_{i=1}^{k}(-1)^j \omega([X, X_j], X_1, \cdots, \hat{X}_j, X_k) \\
&= (L_X\omega)(X_1, \cdots, X_k),
\end{aligned} \tag{6.5}
$$

and (ii) is proved. $\square$

Using the Cartan formula (Theorem 6.1), we can prove some properties of the Lie derivative L_X.

Proposition 6.1. *(i)* $L_X(\omega \wedge \eta) = L_X\omega \wedge \eta + \omega \wedge L_X\eta \quad (\omega \in \mathcal{A}^k(M), \eta \in \mathcal{A}^l(M))$.

(ii) $L_X d\omega = dL_X\omega \quad (\omega \in \mathcal{A}^k(M))$.

(iii) $L_X L_Y - L_Y L_X = L_{[X,Y]} \quad (X, Y \in \mathfrak{X}(M))$.

Proof. According to Cartan formula (ii) and

$$i(X)(\omega \wedge \eta) = i(X)\omega \wedge \eta + (-1)^k \omega \wedge i(X)\eta,$$

we can get

$$
\begin{aligned}
&L_X(\omega \wedge \eta) \\
&= i(X)d(\omega \wedge \eta) + di(X)(\omega \wedge \eta) \\
&= i(X)(d\omega \wedge \eta + (-1)^k \omega \wedge d\eta) + d(i(X)\omega \wedge \eta + (-1)^k \omega \wedge i(X)\eta) \\
&= \{[i(X)d\omega \wedge \eta + (-1)^{k+1} d\omega \wedge i(X)\eta] + (-1)^k[i(X)\omega \wedge d\eta + (-1)^k \omega \wedge di(X)\eta]\} \\
&\quad + \{[di(X)\omega \wedge \eta + (-1)^{k-1} i(X)\omega \wedge d\eta] + (-1)^k[d\omega \wedge i(X)\eta + (-1)^k \omega \wedge di(X)\eta]\} \\
&= i(X)d\omega \wedge \eta + \omega \wedge di(X)\eta + di(X)\omega \wedge \eta + \omega \wedge di(X)\eta \\
&= L_X\omega \wedge \eta + \omega \wedge L_X\eta.
\end{aligned}
$$

For the property (ii),

$$L_X d\omega = i(X)dd\omega + di(X)d\omega = di(X)d\omega,$$

and right side is

$$dL_X\omega = di(X)d\omega + ddi(X)\omega = di(X)d\omega.$$

Property (i) and (ii) are proved.

Now we prove (iii), we use induction on k. First, if $k = 0$, since $L_{[X,Y]}f = [X,Y]f = (L_X L_Y - L_Y L_X)f$ for a function f, it certainly hold. Next assume that it is true up to $k(\geq 0)$, and we shall prove the case of $k+1$. Let ω be an arbitrary $(k+1)$-form. Then since, for an arbitrary vector field Z, $i(Z)\omega$ is k-form, by the assumption of induction we have

$$L_{[X,Y]}i(Z)\omega = (L_X L_Y - L_Y L_X)i(Z)\omega. \tag{6.6}$$

On the other hand, by Cartan formula (i), we have

$$L_{[X,Y]}i(Z)\omega = i(Z)L_{[X,Y]} + i([[X,Y],Z]), \tag{6.7}$$

and, again using (i),

$$\begin{aligned} &L_X L_Y i(Z) \\ &\quad = L_X(i(Z)L_Y + i([Y,Z])) \\ &\quad = i(Z)L_X L_Y + i([X,Z])L_Y + i([Y,Z])L_X + i([X,[Y,Z]]), \end{aligned} \tag{6.8}$$

and similarly,

$$\begin{aligned} &L_Y L_X i(Z) \\ &\quad = i(Z)L_Y L_X + i([Y,Z])L_X + i([X,Z])L_Y + i([Y,[X,Z]]). \end{aligned} \tag{6.9}$$

Subtracting (6.9) from (6.8), we have

$$\begin{aligned} &L_X L_Y i(Z) - L_Y L_X i(Z) \\ &\quad = i(Z)(L_X L_Y - L_Y L_X) + i([X,[Y,Z]]) - i([Y,[X,Z]]). \end{aligned} \tag{6.10}$$

Also subtracting (6.10) from (6.7), we have

$$(L_{[X,Y]} - L_X L_Y + L_Y L_X)i(Z) = i(Z)(L_{[X,Y]} - L_X L_Y + L_Y L_X). \tag{6.11}$$

Here we used the Jacobi identity $[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0$. If we substitute (6.11) in (6.6), we have

$$i(Z)(L_{[X,Y]} - L_X L_Y + L_Y L_X)\omega = 0.$$

Here, since Z was an arbitrary vector field, we obtain

$$(L_{[X,Y]} - L_X L_Y + L_Y L_X)\omega = 0,$$

and the proof is finished. □

7 Lie derivative and one-parameter group of local transformations

Here, as we promised in previous section, we shall give a more geometric definition of the Lie derivative.

Suppose a vector field X is given on a C^∞ manifold M. We can consider X as an assignment of a direction $X_p \in T_pM$ at each point p on M. Therefore, for instance, if a C^∞ function $f \in C^\infty(M)$ on M is given, we can "differentiate f in the direction of X". This is nothing but Xf. Now what would happen if a differential form is given on M instead of a function? Since a function is a special case (the case of degree 0) of differential forms, it will be natural to try to "differentiate" also a general differential form ω in the direction of X. Actually, such a natural operation is defined, and furthermore it operates not only on differential forms but also on so-called tensors, which is a notion including vector fields on a manifold. We call this a (general) **Lie derivative**. The geometric definition of a Lie derivative is given using the one-parameter group φ_t of local transformations on M generated by X (see §1.4(c) in reference book) rather than the vector field X itself.

At first, we shall study the relationship between the differential Xf of a function $f \in C^\infty(M)$ by X and the one-parameter group of local transformations. The result is

$$(Xf)(p) = \lim_{t\to 0} \frac{(\varphi_t^* f)(p) - f(p)}{t} \quad (p \in M) \tag{7.1}$$

(here $\varphi_t^* f$ stands for $f \circ \varphi_t$). This follows because, by the notation of §1.4(c) in the reference book, $\varphi_t(p) = c(p)(t)$ and $\dot{c}(p)(0) = X_p$, we have

$$\lim_{t\to 0} \frac{(\varphi_t^* f)(p) - f(p)}{t} = \lim_{t\to 0} \frac{f(\varphi_t(p)) - f(p)}{t} = X_p f.$$

Though φ_t is not always defined on the whole of M, for each point $p \in M$, φ_t is defined in a neighborhood of p for sufficiently small t, and there is no problem in the above calculation.

Next we shall see that the bracket $[X, Y]$ of vector fields can be considered as the Lie derivative of Y by X (the symbol $L_X Y$ is used). That is,

$$[X, Y] = \lim_{t\to 0} \frac{(\varphi_{-t})_* Y - Y}{t}. \tag{7.2}$$

Here, the equation (7.2) means that the values of both sides are equal at each point p on M, and then the limit of the right-hand side is taken with respect to the usual topology of T_pM as a vector space. We shall prove (7.2). By Proposition (1.3), it is enough to show that the operations of both sides on an arbitrary C^∞ function $f \in C^\infty(M)$ on M are equal. We shall calculate the operation of the right-hand side of f. Since by (1.5),

$$((\varphi_{-t})_* Y) f = Y(f \circ \varphi_{-t}) \circ \varphi_t = \varphi_t^*(Y(f \circ \varphi_{-t})),$$

we have

$$\begin{aligned}
&\lim_{t\to 0} \frac{(\varphi_{-t})_* Y - Y}{t} f \\
&= \lim_{t\to 0} \frac{\varphi_t^*(Y(f \circ \varphi_{-t})) - \varphi_t^*(Yf) + \varphi_t^*(Yf) - Yf}{t} \\
&= \lim_{t\to 0} \varphi_t^* \{Y(\frac{f \circ \varphi_{-t} - f}{t})\} + \lim_{t\to 0} \frac{\varphi_t^*(Yf) - Yf}{t} \\
&= \lim_{t\to 0} \varphi_t^* Y\{(\frac{\varphi_{-t}^* f - f}{t})\} + \lim_{t\to 0} \frac{\varphi_t^*(Yf) - Yf}{t} \\
&= Y(-Xf) + X(Yf) = [X, Y]f.
\end{aligned}$$

Here we have used (7.1), the fact that the functions which appear in the calculation are all of class C^∞ so that we can change the order of differentiation, and also the fact that $\{\varphi_{-t}\}$ is the one-parameter group of local transformations generated by $-X$. Thus (7.2) is proved.

As for the Lie derivative of differential forms, the following proposition holds.

Proposition 7.1. *Let X be a vector field on C^∞ manifold M, and $\{\varphi_t\}$ the one-parameter group of local transformations generated by X. Then for an arbitrary k-form $\omega \in \mathcal{A}^k(M)$, we have*

$$L_X\omega = \lim_{t\to 0}\frac{\varphi_t^*\omega - \omega}{t}.$$

Proof. First, we shall show that if $\varphi : M \to M$ is an arbitrary diffeomorphism, we have

$$(\varphi^*\omega)_p(X_1,\cdots,X_k) = \varphi^*(\omega(\varphi_*X_1,\cdots,\varphi_*X_k)) \tag{7.3}$$

for vector fields $X_1,\cdots,X_k$ on M. By the definition of pullback of differential forms, we have

$$(\varphi^*\omega)_p(X_1,\cdots,X_k) = \omega_{\varphi(p)}(\varphi_*X_1,\cdots,\varphi_*X_k)$$

for an arbitrary point $p \in M$. (7.3) immediately follows from this. If we calculate the right-hand side on $X_1,\cdots,X_k$ using (7.3), we obtain

$$\begin{aligned}
&\lim_{t\to 0}\frac{(\varphi_t^*\omega)(X_1,\cdots,X_k) - \omega(X_1,\cdots,X_k)}{t}\\
&= \lim_{t\to 0}\frac{\varphi_t^*(\omega((\varphi_t)_*X_1,\cdots,(\varphi_t)_*X_k)) - \omega(X_1,\cdots,X_k)}{t}\\
&= \lim_{t\to 0}\frac{\varphi_t^*(\omega((\varphi_t)_*X_1,\cdots,(\varphi_t)_*X_k)) - \varphi_t^*(\omega(X_1,\cdots,X_k))}{t}\\
&\quad + \lim_{t\to 0}\frac{\varphi_t^*(\omega(X_1,\cdots,X_k)) - \omega(X_1,\cdots,X_k)}{t}.
\end{aligned}$$

Let A be the first term and B the second term in this last formula. Then by (7.1) we have

$$B = X\omega(X_1,\cdots,X_k). \tag{7.4}$$

On the other hand, we have

$$\begin{aligned}
A &= \lim_{t\to 0}\varphi_t^*\Big(\frac{\omega((\varphi_t)_*X_1,\cdots,(\varphi_t)_*X_k) - \omega(X_1,\cdots,X_k)}{t}\Big)\\
&= \lim_{t\to 0}\varphi_t^*\Big(\frac{\omega((\varphi_t)_*X_1,\cdots,(\varphi_t)_*X_k)}{t} - \frac{\omega(X_1,(\varphi_t)_*X_2,\cdots,(\varphi_t)_*X_k)}{t}\Big)\\
&\quad + \lim_{t\to 0}\varphi_t^*\Big(\frac{\omega(X_1,(\varphi_t)_*X_2,\cdots,(\varphi_t)_*X_k)}{t} - \frac{\omega(X_1,X_2,(\varphi_t)_*X_3,\cdots,(\varphi_t)_*X_k)}{t}\Big)\\
&\quad + \cdots + \lim_{t\to 0}\varphi_t^*\Big(\frac{\omega(X_1,X_2,\cdots,X_{k-1},(\varphi_t)_*X_k) - \omega(X_1,\cdots,X_k)}{t}\Big)\\
&= \sum_{i=1}^k \omega(X_1,\cdots,[-X,X_i],\cdots,X_k).
\end{aligned}$$

Therefore we have

$$\begin{aligned}
A + B &= X\omega(X_1,\cdots,X_k) - \sum_{i=1}^k \omega(X_1,\cdots,[X,X_i],\cdots,X_k)\\
&= (L_X\omega)(X_1,\cdots,X_k)
\end{aligned}$$

and the proof is completed. □

Reference

Shigeyuki Morita. Geometry of differential forms, American Mathematical Society, volume 201, 2001