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**University of Heidelberg**

**Differential Geometry of Curves and Surfaces**

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**Introduction**

There are two aspects regarding the differential geometry of curves and surfaces:

* The classical differential geometry
* The global differential geometry

The classical differential geometry analyzes the local properties of the curves and surfaces by using methods based on differential calculus. Thus, curves and surfaces are defined by functions that can be differentiated a certain number of times. As classical differential geometry represents mostly the study of surfaces, the local properties of curves are an important part of this, since they appear naturally while studying surfaces.

The global differential geometry on the other hand, studies the influence of the local properties on the behavior on the entire curve or surface, on contrary to the classical geometry that studies the local properties that depend only on the behavior of the curve or surface in the neighborhood of a point.

*1.2 Parameterized Curves*

Differentiable functions are used to define the certain subsets in , subsets which are one dimensional and to which the methods of differential calculus can be applied. A real function of a real variable is *differentiable (smooth)* if it has, at all points, derivatives of all orders (which are automatically continuous).

**DEFINITON:**  A parameterized differentiable curve is a differentiable map α: I -> of an open interval I=(a,b) of the real line R into .

A parameterized continuous curve, for which the map α: I → is differentiable up to all orders, is called a parameterized smooth (differentiable) curve.

* t - the parameter of the curve
* Interval - used to emphasis that a=−∞ and b=+∞ are not excluded
* Tangent vector - if we denote by x’(t), y’(t) and z’(t) the first derivatives of x, y, z at the point t, then the vector (x’(t), y’(t),z’(t))=α’(t) ∈ is called the tangent vector/velocity vector.
* Trace of α- the image set α(*I)* *⊂R*

Difference between curve map and trace

Physically, a curve describes the motion of a particle in n-space, and the trace is the trajectory of the particle. If the particle follows the same trajectory, but with different speed or direction, the curve is considered to be different**. [1]**

*1.3 Properties of product vectors in*

|u|=

where u=(,) and |u|-length (norm)

u\*v=|u||v| cos

Properties:

1. If u and v are nonzero vectors, then u \* v =0 if and only if u is orthogonal to v.
2. u \* v =v \* u
3. λ(u\*v)= λu\*v=u\*λv
4. u\*(v+w)=u\*v+u\*w

Proof that u(t)\*u(v) is a differentiable function:

Let e1=(1,0,0), e2=(0,1,0), e3=(0,0,1). Since ei\*ej=1 if i=j and ei\*ej=0 if i, where i, j=1,2,3, then:

u=u1\*e1+u2\*e2+u3\*e3, v=v1\*e1+v2\*e2+v3\*e3

Considering (3) and (4):

u\*v=u1\*v1+u2\*v2+u3\*v3

From this expression it follows that if u(t) and v(t), t I, are differentiable curves, then u(t)\*v(t) is a differentiable function and:

(u(t)\*v(t))=u’(t)\*v(t)+u(t)\*v’(t)

*1.4 Regular Curves: Arc Length*

**Definition:** *A parameterized differentiable curve α: I-> is said to be regular if α’ (t) ≠0 for all t* ∈ I.

Given t ∈ *I* the arc length of a regular parameterized curve α: I ->, from the point , by definition:

s(t)=

where |α’(t)|=*is the length of the vector α’(t).*

There might be case when the parameter t is already the arc length measured from some point, so ds/dt=1=|α’(t)|. So, the velocity vector has a constant length of 1. Conversely, if |α’(t)|1, then

s=

Another convention important to mention is that two curves differ by a change of orientation if given a curve α parameterized by arc length s (a,b), we consider the curve β(-s)=α(s) that has the same trace but it is described in the opposite direction.

*1-5 The vector (cross) product in*

Elementary properties of determinants

1. e ~ e
2. If f~e then f~e
3. If e~f, f~g then e~g

If the two ordered bases e={ei}, f={fi}, i=1,…n have a positive determinant for the matrix of change of basis, then we say that these two bases have the same orientation. The change of the determinant can be either positive or negative, and as such we say that V has two orientations-positive or negative. In case the matrix that changes the basis has determinant -1, we say that ordered basis is negative.

Let u and v ∈ The vector product of u and v is the unique vector u X v ∈ characterized by:

(u X v)\*w=det(u,v,w) for all w ∈

Here det(u,v,w) means that if we express u, v, w in natural basis {ei},

u=

v=

w= , i=1, 2, 3

det(u,v,w)=

uXv=

Properties of determinants:

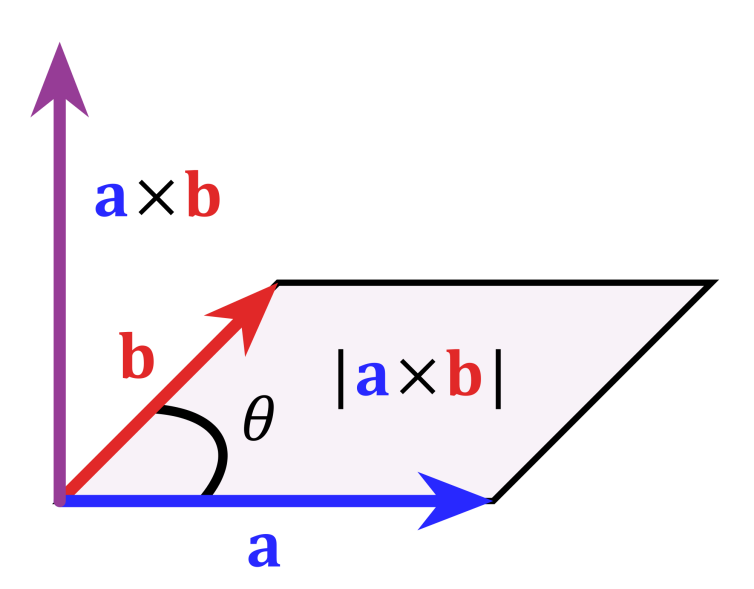
1. u X v =-v X u
2. (au+bw) X v =au X v +bw X v for any real numbers a,b
3. u X v =0 if and only if u and v are linearly dependent
4. (u X v)\*u=0 and (u X v) \*v=0

Based on property 4, it can be seen that u X v≠0 is a vector normal to a plane generated by u and v.

= = \* (1-)=, where is the angle of u and v and A is the

area of the parallelogram generated by u and v. (Fig.1)

#Remark: at the figure 1 “a” refers to u, “b” refers to v.



*Fig.1 Vector product example*

The vector product is not associative. The following identity holds for the vector product:

(u X v) X w=(u\*w)v-(v\*w)u

*1-6 The Local Theory of Curves Parametrized by Arc Length*

Let α: *I=(a,b) ->* be a curve parameterized by arc length s. |α’’(s)|, as the second derivative of the tangent vector measures how rapidly the curve pulls away from the tangent line at s, in a neighborhood of s. This suggests the following definition:

**Definition:** Let α: *I-> be a curve parametrized by arc length s* ∈ I.  *The number* |α’’(s)|=k(s) is called the curvature of α at s.

* If α is a straight line, then k0, and conversely if k0, then α=0, so the curve is a straight line.
* If the tangent vector changes the orientation, α’’(s) and |α’’(s)| remain invariant under this change.

If k(s)≠0:

* n(s)= α’’(s)/ |α’’(s)| (α’’(s)=k(s)\*n(s)) where n(s) is normal to α’(s) and it is known as the normal vector at s. n(s) is in the direction of α’’(s) and defined by : n(s)= α’’(s)/| α’’(s) |
  + n(s) is normal to α’(s) because:
    - * (α’(s)\* α’(s)=1) ’
      * α’’(s)\* α’(s)+ α’(s)\* α’’(s)=0=>
      * α’’(s)\* α’(s)=0 so, α’’(s) is normal to α’(s).
* *The osculating plane* at s - the plane determined by the unit tangent and normal vectors, α’(s) and n(s).

If k(s)=0:

* n(s) and the osculating plane is not defined.

In order to work with the local analysis of the curves, the osculating plane is needed.

t’(s)=k(s)\*n(s) where t(s)= α’(s), the unit tangent vector of α at s.

b(s)=t(s) ˄ n(s) , where b(s) is the unit vector normal to the osculating plane and is called binormal vector at s. The length |b’(s)| measures the rate of change the neighboring osculating planes with the osculating plane at s. Thus, b’(s) measures how rapidly the curve pulls away from the osculating plane at s, in a neighborhood of s.

To compute b’(s), on one hand we have:

|b(s)|=1

(b(s)\*b(s))=1

b’(s)\*b(s)+b(s)\*b’(s)=0 => b’(s)\*b(s)=0 so b’(s) is normal to b(s).

On the other hand:

By definition: b(s)=t(s) ∧ n(s)

(b(s)=t(s) ∧ n(s))’

b’(s)=t’(s) ∧ n(s) +t(s) ∧ n’(s)

t’(s) ∧ n(s)=0 because t’(s)=k(s) n(s), and thus k(s) n(s) ∧ n(s)=0 since n(s) ∧ n(s) =0

Thus, b’(s)=t(s) ∧ n’(s)

🡺 b’ (s) ∧ t(s)

b’(s) ∧ n’(s)

b’(s) ∧ b(s)

It follows that b’(s) is parallel to n(s) and we may write b’(s)=(s) n(s), where (s) is some function.

**Definition:** *Let α: I->* be a curve parameterized by arc length s such that α’’(s)≠0, s I. The number (s) defined by b’(s)= (s)n(s) is called the of α at s.

If α is a plane curve, then the plane of the curve agrees with the osculating plane, thus (s)0. Conversely, if (s)0 and k≠0, we have that b(s)=constant and it follows that:

(

Thus, =constant, which means is contained in a plane normal to .

* Contrary to the curvature, the torsion can be negative or positive.
* b’(s) and the torsion remain invariant under the change of the orientation of the binormal vector.

*Frenet trihedron* at s refers to the three orthogonal unit vectors t(s), n(s) and b(s) to each value of the parameter s. t’(s) , b’(s) expressed as kn and n respectively give more information about the behavior of in a neighborhood of s.

Thus, so far we have: t’(s)=kn and b’(s)=

The derivative of n(s) on the other hand is n’(s)=-, since n(s)=b˄t.

Frenet formulas:

* t’=kn
* n’=-
* b’=

Planes notions:

*-Rectifying plane*-the tb plane

*-Normal plane*- the nb plane

*-Principal normal*-the lines that contain n(s) and pass through α(s)

*-Binormal*-the lines that contain b(s) and pass through α(s)

-Radius of curvature - r=1/k of the curvature

Difference between curvature and torsion:

* **Curvature** measures the failure of a curve to be a line. If  α(s) has zero curvature, it is a line. High curvature (positive or negative corresponding to right or left) means that the curve fails to be a line quite badly, owing to the existence of sharp turns.
* **Torsion** measures the failure of a curve to be planar. If α(s) has zero torsion, it lies in a plane. High torsion (positive or negative corresponding to up and down) means that the curve fails to be planar quite badly, owing to it curving in various directions and through many planes.

Examples:

* τ=0,κ=0: A line. Lines look very much like lines, and they are certainly planar.
* τ=0,κ=k>0: A circle. Circles don't look like lines, especially small ones. They have constant curvature. However, they do lie in a plane.
* τ=c>0,κ=k>0: A helix. Helixes curve like circles, failing to be lines. They also swirl upwards with constant torsion, failing to lie in a plane.
* τ>0,κ=k>0: A broken slinky. Slinkies curve like circles, failing to be lines. They generally have constant positive torsion, like helixes. But if you break them, the torsion remains positive (viewed from the bottom up), but how large the torsion is corresponds to how stretched the slinky is. A very stretched slinky has large torsion, compacted slinkies have small torsion. **[2]**

It can be said that it is the curvature and the torsion that describe completely the local behavior of the curve. This leads to the following theorem.

**1.7 FUNDAMENTAL THEOREM OF THE LOCAL THEORY OF CURVES**

*Given differentiable functions k(s)>0 and there exists a regular parametrized curve α: I-> , such that s is the arc length, k(s) is the curvature and (s) is the torsion of α. Any other curve α, satisfying the same condition differs from α by a rigid motion, that is there exists an orthogonal linear map of , with positive determinant and a vector c, such that =+c*

Proof of uniqueness part of the fundamental theorem:

1. The arc length, the curvature and the torsion are invariant under rigid motion.
2. Assuming that two curves α= α (s) and (s) satisfy the conditions:

* k(s)= (s)
* (s)= (s), s

Let , , and be the Frenet trihedrons at s= I of α and , respectively. There is a rigid motion that takes () into α( and into , , .

After performing the rigid motion on , so after taking ( in α( , in regards of Frenet equations we have:

with t= ), n()=), and b)=)

Following the expression:

=<>+<>+<>  
=

=0, for all s I. As the above expression is constant and since it is zero for s= it follows that it is identically zero. So for all constants =>t(s).

=>=>

Since,

d/ds( =>

*1-8 The Local Canonical Form*

In geometry one of the best ways to solve problems is finding a coordinate system which is adapted to the problem. The Frenet trihedron at s, is the natural coordinate system that is used for the analysis of the local properties of the curve, in the neighbourhoud of the point s.

Let be a curve parametrized by arc length without singular points of order 1. The equations of the curve, in a neghbourhood of , using the trihedron t( as a basis of and assuming that =0, using the finite Taylor expansion look as below:

Taylor expansion

where

because =kn; ’=(kn)’=k’n+kn’ (n’=-)

so =k’n-k-t

Now by replacing: , =kn and

where all terms are computed at s=0

In the coordinate system 0xyz, where the origin 0 agrees with and where t=(1,0,0), n=(0,1,0) and b=(0,0,1), is given by:

, where R=(Rx, Ry,Rz)

The above representation is known as the *local canonical form of ,* in a neighbourhood of s=0.

Geometrical applications of the local canonical form:

1. The sign of −*τ* is the sign of *z*′(*s*), so the torsion is positive if the curve pulls ‘down’ from the osculating plane, and negative if it pulls ‘up’;
2. *y*(*s*)≥0 and *y*(*s*)=0 only when *s*=0 in some neighborhood of *s*, so that the curve is entirely on one side of the rectifying plane;
3. The osculating plane is the limit of the planes spanned by the tangent line and the point *αs*+*h* as *h*→0. **[3]**

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