Simplex reflection groups

Seminar WS 13/14

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1 Introduction

Reflections groups are those generated by reflections on the sides of polyhedra. To study these you can not only use pure mathematical equations but also your geometric understanding.

Given "nice" enough polyhedra we get discrete groups which we can characterize by certain graphs called *coxeter graphs*.
For those who like to get a nice way to visualize the results we can show that for a nice polyhedron P the set $\{g_i P\};\ g_i \in \Gamma$ is a tesselation of e.g the $\mathbb{R}^n$.

This talk will give an introduction to reflection groups and coxeter graphs. Our main goal is to classify the simplex reflection groups and to compute some tesselations in 2 dimensions.

Enjoy!

2 Reflection groups

Theorem 2.1. Let G be the group generated by the reflections of X in the sides of a finite-sided, n-dimensional, convex polyhedron P in X of finite volume. Then

$$X = \bigcup \{gP \mid g \in G\}$$

Proof. Induction over n. Left for the reader as an exercise. Or see Ratcliff, Thm 7.1.1 □

One of our most important properties regarding the angles of the simplex will be the following:

Definition 2.2. An angle α is a submultiple of π if and only if $\exists k \in \mathbb{N} : \alpha = \pi/k$

With this we can conclude:

Theorem 2.3. Let Γ be a discrete reflection group with respect to the polyhedron P. Then all dihedral angles of P are submultiples of π and if g_S and g_T are reflections in adjacent sides S and T of P with $\theta(S,T) = \pi/k$, then $g_S g_T$ has order k in Γ.

Proof. We consider two cases:

$\underline{\theta = 0 :}$ then $g_S g_T$ is a translation.

$\underline{\theta > 0 :}$ we will sketch this proof . Considering words $g_i = g_{S_1} \cdot \cdots \cdot g_{S_i}$ one can show that $\{g_i P\}_{i \in \mathbb{N}}$ defines a cycle of Polyhedra (see Ratcliff, Page 256). This tells us that there are only finitely many distinct Polyhedra and thus there exsists $k \in \mathbb{N}$ such that $2\theta(S,T) = \frac{2\pi}{k}$ and $g_S g_T$ has order k.
For a more detailed proof consider the literatur.

□

The following theorem ensures that under the right prerequisites the group Γ which is generated by the reflection of X in the sides of P is indeed a discrete reflection group.

Theorem 2.4. Let P be as in theorem 1 such that all dihedral angles are submultiples of π. Them Γ is a discrete reflection group with respect to P.

Proof. We will just give the idea:
for $n = 1$ the theorem is obviously true. The rest will be shown by induction on n. Therefore you can construct a topological space $\tilde{X}$ for which the theorem holds by construction. Then using a covering space argument you can show that $\tilde{X}$ is homeomoprhic to X.
See Ratcliff, Thm 7.1.3 □

To see this we can take a look at some examples:

Example 2.5. Consider the sphere S^n and

$$P = \{x \in S^n \mid x_i \geq 0;\ i = 1, \ldots, n+1\}$$

P is a n-simplex in S^n with dihedral angles $\pi/2$. Thus Γ is a discrete reflection group.

Example 2.6. If we consider an n-cube in E^n then the dihedral angles are again $\pi/2$. Thus Γ is a discrete reflection group.

Example 2.7. To get an hyperbolic example we can take a look at the 2-simplex in H^2 whose dihedral angles are $30°, 45°, 90°$. If we keep the vertex at the $30°$ angle fixed we get the following picture:

3 Coxeter groups and graphs

As we will see Later we can classify our reflection groups through certain graphs. Thus we first need to take a look on special groups:

Definition 3.1 (Coxeter groups). A **Coxeter group** is a group G presented by $(S_i \mid (S_iS_j)^{k_{ij}})$ such that:

1. $i, j \in I$ for some countable set I
2. $k_{ij} \in \mathbb{N} \cup \{\infty\}$ for all i, j
3. $k_{ij} = k_{ji}$ for each pair i, j
4. $k_{ii} = 1\ \forall i$
5. $k_{ij} > 1$ if $i \neq j$
6. if $k_{ij} = \infty$ then $(S_iS_j)^{k_{ij}}$ is deleted

Note: Since $S_i^2 = S_j^2$ we can derive $(S_jS_i)^{k_{ji}}$ from $(S_iS_j)^{k_{ij}}$. Thus only one of the latter is required.

Definition 3.2 (Coxeter graphs). Let $G = (S_i \mid (S_iS_j)^{k_{ij}})$ be a coxeter group. The coxeter graph is a labeled graph with vertices I an edges

$$\{(i, j) \mid k_{ij} > 2\}$$

where each edge is labeled k_{ij}.
Note: for simplicity those edges with $k_{ij} = 3$ are not labeled.

Theorem 3.3. Let Γ be a discrete reflection group with respect to P. Let $\{S_i\}$ be the sides of P. If S_i and S_j are adjacent sides we set $k_{ij} = \pi/\theta(S_i, S_j)$. Then

$$\left(S_i \mid (S_i, S_j)^{k_{ij}}\right)$$

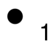
is a group representation for Γ under the mapping $S_i \mapsto g_{S_i}$

Proof. See Ratcliff 7.1.4.
Using the proof of Theorem 3 you can compute these representations. □

With this every discrete reflection group is a coxeter group. Just set $k_{ij} = \infty$ if S_i, S_j are not adjacent.

To understand these graphs we will take a look at some examples

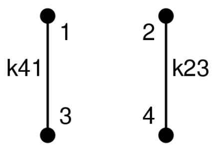
Example 3.4. Let $G = (S_1 \mid S_1^2)$. Then the graph looks like this:



Example 3.5. Let G be the group generated by the reflections in the sides of a rectangle P in E^2. G has the following representation:

$$\left(S_1, S_2, S_3, S_4 \mid S_i^2, (S_i S_{i+1})^2 \ i \ mod \ 4\right)$$

Thus the graph looks like this:



4 Gram matrices

Definition 4.1 (Gram matrix). Let Δ be a n-simplex in S^n, E^n or a generalized n-simplex in H^n with sides $S_1, \ldots, S_{n+1}$. Let v_i be a (nonzero) normal vector to S_i directed inwards. Then the *Gram matrix* w.r.t. $v_1, \ldots, v_{n+1}$ is

$$A = \begin{cases} (v_i \cdot v_j) & X = S^n, E^n \\ (v_i \circ v_j) & X = H^n \end{cases},$$

where "$\cdot$" is the Euclidean inner product and "$\circ$" is the Lorentzian inner product. If the v_i are unit vectors and it is $n > 1$, then $A = (-\cos(\Theta(S_i, S_j)))$ holds.

Our goal is to prove the following theorem:

Theorem 4.2. Let $A = (-\cos(\Theta_{ij})) \in M_{(n+1)\times(n+1)}(\mathbb{R})$ be symmetric, s.t. $0 < \Theta_{ij} \leq \frac{\pi}{2}$ for all $i \neq j$ and $\Theta_{ii} = \pi$ for all i holds. Then A is a Gram matrix of an n-simplex $X \in \{S^n, E^n, H^n\}$ if and only if A_{ii} is positive definite for all i.

Furthermore there holds:

- $X = S^n \quad \Leftrightarrow \quad \det A > 0$
- $X = E^n \quad \Leftrightarrow \quad \det A = 0$
- $X = H^n \quad \Leftrightarrow \quad \det A < 0$

We will prove this theorem for each case seperatly and start with the following lemma, which should be known from linear algebra classes.

Lemma 4.3. Let $A \in M_{n\times n}(\mathbb{R})$ be symmetric, s.t. A_{nn} is positive definite. Then A is

1. positive definite, if and only if $\det A > 0$,
2. of type $(n-1, 0)$, if and only if $\det A = 0$ or
3. of type $(n-1, 1)$, if and only if $\det A < 0$.

Theorem 4.4. Let A be a symmetric matrix. Then A is a Gram matrix of a simplex Δ in S^n if and only if A is positive definite.

Proof. "$\Rightarrow$" Let $v_1, \dots, v_{n+1}$ be the normalvectors of $S_1, \dots, S_{n+1}$ the sides of Δ. Furthermore let $V_i \subseteq \mathbb{R}^{n+1}$ be linear subspaces, s.t.

$$\langle S_i \rangle = V_i \cap S^n$$

holds. Define H_i to be the halfspace bounded by V_i containing Δ, i.e.

$$H_i = \{x \in \mathbb{R}^{n+1} \mid x \cdot v_i \geq 0\}.$$

So we can compute $\Delta = (\cap_{i=1}^{n+1} H_i) \cap S^n$. Define $B = (v_1, \dots, v_{n+1})$. Then we have

$$B^{\perp} = \cap_{i=1}^{n+1} = \{0\}.$$

Thus B is nondegenerate and $v_1, \dots, v_{n+1}$ is a basis of $\mathbb{R}^{n+1}$. Define an inner product by

$$\langle x, y \rangle := Bx \cdot By \qquad \forall x, y \in \mathbb{R}^{n+1}.$$

Then we can compute

$$\langle e_i, e_j \rangle = Be_i \cdot Be_j = v_i \cdot v_j,$$

yielding that A is the matrix of this inner product, so A is positive definite.

"$\Leftarrow$" Conversly suppose A is positive definite. Then there exists an orthonormal Basis $(u_1, \dots, u_{n+1})$ with regard to the inner product defined by A. Let $C = (u_1, \dots, u_{n+1})$ and we get $C^t AC = \mathrm{id}$. Setting $B := C^{-1}$ we see $A = B^t B$. We define v_j to be the jth columnvector of B. Then $(v_1, \dots, v_{n+1})$ forms a basis of $\mathbb{R}^{n+1}$ and $A = (v_i \cdot v_j)$. Next we define

$$Q := \{y \in \mathbb{R}^{n+1} \mid y_i \geq 0 \forall i = 1, \dots, n+1\}.$$

Thus Q is a $n+1$-dimensional convex polyhedron in E^{n+1} with $n+1$ sides and exactly one vertex at the origin. We set

$$H_i := \{x \in \mathbb{R}^{n+1} \mid v_i \cdot x \geq 0\}$$
$$V_i := \{x \in \mathbb{R}^{n+1} \mid v_i \cdot x = 0\}$$
$$K = \bigcap_{i=1}^{n+1} H_i$$

Take a $x \in K$ and we see $(B^t x)_i = v_i \cdot x \geq 0$. Thus we get $B^t K \subset Q$. On the other hand let y be arbitrary in Q. We set $x = C^t y$ and get $B^t x = y" \geq 0"$. So $v_i \cdot x \geq 0$ and we see that x lies in K. Thus

$$B^t K = Q.$$

We see that K is a $(n+1)$-dimensional convex polyhedron in E^{n+1} with $n+1$ sides $V_i \cap K$ and exactly one vertex at 0. Thus $\Delta = K \cap S^n$ is a n-dimensional convex polyhedron in S^n with sides $S_i = V_i \cap \Delta$. Now Δ lies in a open hemisphere because of [Rat05, Thm 6.3.16]. Thus Δ is a polytope because of [Rat05, Thm 6.5.1]. Now Δ has exactly $n+1$ sides thus we see that Δ is a n-simplex [Rat05, Thm 6.5.4] an its Gram matrix is$A$. □

Lemma 4.5. Let Δ be an n-simplex in E^n. Let v be a vertex of Δ, let S be the side of Δ opposite v and let h be the distance from v to $\langle S \rangle$. Then there holds

$$\mathrm{Vol}_n(\Delta) = \frac{1}{n} h \,\mathrm{Vol}_{n-1}(S).$$

Proof. This is a consequence of the theroem of Cavalieri. □

Theorem 4.6. Let $A \in M_{(n+1)\times(n+1)}(\mathbb{R})$ be symmetric and $n > 0$. A is a Gram matrix of Δ in E^n if and only if

1. A_{ii} is positive definite for all i,
2. $\det A = 0$ and
3. all entries of $\operatorname{adj} A$ are positive.

Proof. "$\Rightarrow$" Let A be a Gram matrix of an n-simplex in E^n w.r.t. the normalvectors $v_1, \dots, v_{n+1}$ of the sides $S_1, \dots, S_{n+1}$. Let H_i be the halfspace of E^n bounded by $\langle S_i \rangle$ containing Δ. Then we get

$$\Delta = \bigcap_{i=1}^{n+1} H_i.$$

By translation we can assume that the vertex opposite S_j is the origin. Then

$$\left(\bigcap_{\substack{i=1\\i\neq j}}^{n+1} H_i\right) \cap S^{n-1} =: \Delta_j$$

is a $(n-1)$-dimensional simplex in S^{n-1}. Thus $v_1, \dots, \hat{v}_j, \dots, v_{n+1}$ form a basis of $\mathbb{R}^n$ and A_{jj} is a Gram matrix of Δ_j. Therfore A_{jj} is positive definite.

We define $B := (v_1, \dots, v_{n+1}) \in M_{n\times(n+1)}(\mathbb{R})$. If we set $\langle x, y\rangle = Bx \cdot By$, we seet that A is the matrix of this inner product. Now there holds $\operatorname{rank} B = n$ and thus $\operatorname{rank} A = n$. Therfore we yield $\det A = 0$.

Now let u_i be the vertex opposite of S_i and let $h_i := \operatorname{dist}(u_i, \langle S_i\rangle)$, $s_i = h_i^{-1}$ and $F_i = \operatorname{Vol}_{n-1}(S_i)$. We see

$$\frac{F_i}{s_i} = n \operatorname{Vol}_n(\Delta)$$

Let x be in $\mathring{\Delta}$. Then Δ ist subdivided in $n+1$ simplices. We set $\bar{x}_i := \operatorname{dist}(x, \langle S_i\rangle)$. By the above lemma we see

$$\sum_{i=1}^{n+1} F_i \bar{x}_i = n \operatorname{Vol}_n(\Delta)$$

Using $F_i = n s_i \operatorname{Vol}_n(\Delta)$ we yield

$$\sum_{i=1}^{n+1} s_i \bar{x}_i = 1. \tag{1}$$

Now we translate Δ such that $u_{n+1} = 0$ holds and we define

$$\hat{v}_i = \frac{v_i}{|v_i|}.$$

Thus we see

$$\begin{aligned} \hat{v}_i \cdot x &= \bar{x}_i && i \leq n \\ \hat{v}_{n+1} \cdot x &= \bar{x}_{n+1} - h_{n+1} \end{aligned}$$

Plugging these equations into (1) we get

$$\left(\sum_{i=1}^{n+1} s_i \hat{v}_i\right) \cdot x = \sum_{i=1}^{n+1} s_i \bar{x}_i - \underbrace{s_{n+1} h_{n+1}}_{=1} = 0$$

Now $x \in \mathring{\Delta}$ is arbitrary and $\mathring{\Delta}$ contains a basis of $\mathbb{R}^{n+1}$. Hence we yield

$$\sum_{i=1}^{n+1} s_i \hat{v}_i = 0$$

and

$$v_i \sum_{j=1}^{n+1} s_j \hat{v}_j = \sum_{j=1}^{n+1} s_j a_{ij} |v_j|^{-1} = 0.$$

So $w := (s_1|v_1|^{-1}, \dots, s_{n+1}|v_{n+1}|^{-1})$ is a eigenvector of A ot the eigenvalue 0. We see that all the components of w are positive and since $\operatorname{rank} A = n$ the eigenspace of 0 is onedimensional. So we see that all components of all the elements of $\ker(A)$ have the same sign. From the equation

$$A \operatorname{adj} A = \det A \operatorname{id} = 0$$

we get that all the column vectors fo $\operatorname{adj} A$ lie in $\ker A$ and since $(\operatorname{adj} A)_{ii} = \det A_{ii} > 0$ we have that all entries of $\operatorname{adj} A$ are positive.

"$\Leftarrow$" We see that A is of type $(n, 0)$ by our first Lemma. Thus we have $\dim \ker(A) = 1$ and again we see by

$$A \operatorname{adj} A = \det A \operatorname{id} = 0$$

that all the columnvectors of $\operatorname{adj} A$ lie in $\ker A$. Thus all the components of elements in $\ker A$ have the same sign. Since A ist of type $(n, 0)$ we find a matrix $B \in \mathrm{GL}_{n+1}(\mathbb{R})$ with

$$A = B^t \operatorname{diag}(1, \dots, 1, 0) B$$

Let v_j be the jth columnvector of B and we define $\bar{v}_j \in \mathbb{R}^n$ by dropping the last coordinate of v_j. Then we see $A = (\bar{v}_i \cdot \bar{v}_j)$. We set $\bar{B} = (\bar{v}_1 \dots \bar{v}_n)$. Then we have $\bar{B}e_i \cdot \bar{B}e_j = \bar{v}_i \cdot \bar{v}_j$ and so the restriction of the bilinear form A to $\mathbb{R}^n$ ist given by

$$\langle x, y \rangle = \bar{B}x \cdot \bar{B}y.$$

Now $A_{n+1,n+1}$ is positive definite, therefore $\bar{B}$ is nonsingular. Tus $\bar{v}_1, \dots, \bar{v}_n$ is a basis vo $\mathbb{R}^n$. We define

$$H_i = \{x \in \mathbb{R}^n \mid \bar{v}_i \cdot x \geq 0\},$$
$$V_i = \{x \in \mathbb{R}^n \mid \bar{v}_i \cdot x = 0\}$$

and

$$C = \begin{pmatrix} \bar{v}_1^t \\ \vdots \\ \bar{v}_{n+1}^t \end{pmatrix}$$

Then $CC^t = A$. Thus the columnspace of C is the same as the columnspace of A. Suppose $x \in \bigcap_{i=1}^{n+1} H_i$. Then $\bar{v}_i x \geq 0$ for all $i = 1, \dots, n+1$. So each component vo Cx is nonnegative. We take a $0 \neq y \in \ker A$. Since A is symmetric y is orthogonal to the columnspace. Thus

$$(Cx) \cdot y = 0$$

holds. Now all components of y have the same sign and we get $Cx = 0$. Thus $x \in \bigcap_{i=1}^{n} V_i = \{0\}$. So we see $\bigcap_{i=1}^{n+1} H_i = \{0\}$. So we see thath $\bigcap_{i=1}^{n} H_i$ is a n-dimensional convex polyhedron in E^n with sides $V_i \cap \left(\bigcap_{j=1}^{n} H_j\right)$ and exactly one vertax at 0. $\bigcap_{i=1}^{n+1} H_i = \{0\}$ implies

$$\bigcap_{i=1}^{n} H_i \subset -\mathring{H}_{n+1} = \{x \in \mathbb{R}^n \mid \bar{v}_{n+1} \cdot x < 0\}.$$

Let

$$H_0 := \{x \in \mathbb{R}^n \mid \bar{v}_{n+1} \cdot x \geq -1\}$$

and V_0 its bounding hyperplane. Then

$$\Delta = \bigcap_{i=0}^{n} H_i$$

is a convex n-simplex in E^n with the normalvectors $\bar{v}_1, \ldots, \bar{v}_{n+1}$ and thus with Gram matrix A.

□

The next step is to consider the hyperbolic case.

Theorem 4.7. Let $A \in M_{(n+1)\times(n+1)}(\mathbb{R})$, $n > 0$. A is the Gram matrix of a n-simplex Δ in n if and only if

1. A_{ii} is positive definite for all i,
2. $\det A < 0$ and
3. all entries of adj A are positive

Proof. "$\Rightarrow$" Suppose A is a Gram matrix with regard to the (Lorentz orthogonal) normalvectors $v_1, \ldots, v_{n+1}$ of the sides $S_1, \ldots, S_{n+1}$. Let V_i be the n-dimensional subspace of $\mathbb{R}^{n+1}$ with $\langle S_i \rangle = V_i \cap H^n$. Furthermore we define, as above, H^n to be the halfspace of $\mathbb{R}^{n+1}$ bounded by V_i containing Δ. Thus we yield

$$H_i = \{x \in \mathbb{R}^{n,1} \mid x \circ v_i \geq 0\}$$

and

$$\Delta = \left(\bigcap_{i=1}^{n+1} H_I\right) \cap H^n$$

We define $B := (v_1 \ldots v_{n+1})$. Then we see

$$B^\perp = \{x \in \mathbb{R}^{n+1} \mid x \circ v_i = 0 \forall i\} = \bigcap_{i=1}^{n+1} V_i = \{0\}.$$

Thus B is nonsingular and we can define a bilinear form of type $(n,1)$ by setting

$$\langle x, y\rangle := Bx \circ By.$$

We can compute $\langle e_i, e_j\rangle = Be_i \circ Be_j = v_i \circ v_j$. Thus A is the matrix of the form $\langle\cdot,\cdot\rangle$. Thus A is of typ $(n,1)$, which leads to $\det A < 0$.

Next, by a translation we assume that the vertex opposite S_j is e_{n+1} and we set $r_j := \frac{1}{2}\operatorname{dist}(e_{n+1}, S_j)$. Now we can define

$$\Delta' := S(e_{n+1}, r_j) \cap \Delta.$$

So Δ' is a $(n-1)$-dimensional spherical simplex with sides $S_i' = S_i \cap S(e_{n+1}, r_j)$ and v_i is normal to the side S_i' for all $i \neq j$ in th horizontal hyperplane $P(e_{n+1}, \cosh r_j)$ of E^{n+1} containing S_i', since the laste coordinate of v_i is zero foreach $i \neq j$. Thus we get A_{jj} is positive definite.

We define v_j^* to be the jth rowvector of B^{-1} and set $w_j = Jv_i^*$, where we have $J = \operatorname{diag}(-1, 1, \ldots, 1)$. Thus we get

$$\begin{aligned} w_i \circ v_j^* &= Jv_i^* \circ v_j \\ &= v_i^* \cdot v_j \\ &= (B^{-1})^t e_i \cdot v_j \\ &= e_i \cdot B^{-1} v_j \\ &= e_i \cdot e_j \\ &= \delta_{ij} \end{aligned}$$

Since $A = B^t JB$ hold we get

$$A^{-1} = B^{-1}J(B^{-1})^t = (v_i^* \circ v_j^*) = (w_i \circ w_j).$$

The iith entry of A^{-1} is $\underbrace{\det A_{ii}}_{>0}\underbrace{(\det A)^{-1}}_{<0}$. Thus $\|w_i\|$ is imaginary, i.e. w_i is time-like.

Now $w_i \circ v_j = 0$ for all $i \neq j$, so w_i lies in a onedimensional time-like subspace spanned by the vertex of Δ opposite S_i. Furthermore, since $w_i \circ v_i = 1 > 0$, w_i is positive time-like. [Rat05, Thm 3.1.1] yields $w_i \circ w_j < 0$ for all i, j. Thus all entries of A^{-1} are negative and since $\det A < 0$ holds, we get that alle entries of adj A are positive.

"$\Leftarrow$" We know by our first lemma that A is of type $(n,1)$. So we find a $B \in GL_{n+1}(\mathbb{R})$, s.t. $A = B^t JB$. Let v_j be the jth columnvector of B. Then $V_1, \ldots, v_{n+1}$ is a basis of $\mathbb{R}^{n+1}$ and $A = (v_i \circ v_j)$. We set

$$Q := \{y \in \mathbb{R}^{n+1} \mid y_i \geq 0 \forall i\}$$

and see that Q is a $(n+1)$-dimensional convex polyhedron in E^{n+1} with $n+1$ sides, $n+1$ edges and exactly one vertex at 0. We define

$$H_i := \{x \in \mathbb{R}^{n,1} \mid v_i \circ x \geq 0\}$$
$$V_i := \{x \in \mathbb{R}^{n,1} \mid v_i \circ x = 0\}$$
$$K := \bigcap_{i=1}^{n+1} H_i$$

As in the proof of the euclidean case we have $B^t JK = Q$. Thus K is a $n+1$-dimensional convex polyhedron in E^{n+1} with $n+1$ sides $V_i \cap K$, $n+1$ edges and exactly one vertex at 0. Let v_j^* be the jth rowvector fo B^{-1} and $w_i = Jv_i^*$. Then we have, as before, $w_i \circ v_J = \delta_{ij}$ and thus $w_i \in K$ for all i. As $w_i \circ v_j = 0$ for all $i \neq j$, we have that w_i is on the edge of K opposite the side $V_i \cap K$ for all i. It is

$$A^{-1} = B^{-1}J(B^{-1})^t = (v_i^* \circ v_j^*) = (w_i \circ w_j).$$

On the other hand we have

$$A^{-1} = (\det A)^{-1} \operatorname{adj} A$$

So all entries of A^{-1} are negative and we get $w_i \circ w_j j < 0$ for all i, j. So $w_1, \ldots, w_{n+1}$ are time-like with some parity by [Rat05, Thm 3.1.1]. Without loss of generality we may assume that all w_i are positive time-like or we replace B by $-B$. Let $0 \neq x \in K$ and $y = B^t Jx$. Then y is in Q an so $y_i \geq 0$ holds fo rall i. A short computation shows $X = \sum_{i=1}^{n+1} y_i w_i$ and so x is positive time-like by [Rat05, Thm 3.1.2]. Therefore $\Delta = K \cap H^n$ is a n-dimensional convex polyhedron with sides $S_i = V_i \cap \Delta$. Now radial projection from the origion maps a link of the origion onto Δ. So Δ is compact and therefore a polyhedron. Since it has exactly $n+1$ sides it is an n-simplex by theorems [Rat05, Thms 6.5.1, 6.5.4] and A is its Gram matrix with regard to $v_1, \ldots v_{n+1}$.

□

Now the only thing left is to get rid of adj A. The first step is the following lemma.

Lemma 4.8. Let $A = (a_{ij}) \in M_{n\times n}(\mathbb{R})$ be symmetric with $a_{ij} \leq 0$ for all $i \neq j$. Suppose A_{ii} is positve definite for all i. Then adj A is nonnegative.

Proof. Let $x \in \mathbb{R}^n$ with $Ax \geq 0$. We claim that $x \geq 0$ or $x \leq 0$ holds, where we compare componentwise. On the contrary assume we find $x_i < 0$ and $x_j > 0$ for some $i \neq j$. Let x' be the vector obtained by x by deleting the nonnegative components. Furthermore, let A' be the diagonalminor of A obtained by deleting rows and columns assosciated to x'. Then we have $A'x' \geq 0$ since the terms omitted are of the form $a_{ij}x_j$ where $x_i < 0$ and $x_j \geq 0$. Thus $i \neq j$ and by assumption we have $a_{ij} \leq 0$ and $a_{ij}x_J \leq 0$. So we see $x' \cdot A'x' \leq 0$, since $x' < 0$ holds. However A' is positive definite, which is a contradiction. Hence we have $x \geq 0$ or $x \leq 0$.

Now suppose $A \in GL_n(\mathbb{R})$. Then $A \cdot A^{-1}e_i \geq 0$ and by the previous argument we see $A^{-1}e_i \geq 0$ or $A^{-1}e_i \leq 0$. There holds adj $A = \det AA^{-1}$, so again we have adj $Ae_i \geq 0$ or adj $Ae_j \leq 0$.

Suppose A is singular. Then we get $A \operatorname{adj} A e_i = \det A e_i = 0 \geq 0$. So $\operatorname{adj} A e_i \geq 0$ or $\operatorname{adj} A e_i \leq 0$.

So in general we have $\operatorname{adj} A e_i \leq 0$ or $\operatorname{adj} A e_i \geq 0$ and since $(\operatorname{adj} A)_{ii} = \det A_{ii} > 0$ we have $\operatorname{adj} A \geq 0$. □

proof of 4.2.

1. Let $X = S^n$. This case follows directly form 4.4.

2. Let $X = E^n$. If A is a Gram matrix, then A_{ii} is positive definite for all i and $\det A = 0$ by 4.6. Conversly we know that A is of type $(n, 0)$ and so $\ker(A)$ is onedimensional. Let $0 \neq x \in \ker A$. Then $x_i \neq 0$, since A_{ii} is positive definite. We have the equation

$$A \operatorname{adj} A = \det A \operatorname{id} = 0.$$

   So the column vectors of $\operatorname{adj} A$ are in the kernel of A. Since $(\operatorname{adj} A)_{ii} = \det A_{ii} > 0$ we have that $\operatorname{adj} A > 0$. Thus the theorem follows from 4.6

3. Let $X = H^n$. If A is a Gram matrix, then A_{ii} is positive definite for all i and $\det A < 0$ by 4.7. Conversly we consider the vectors $w_1, \ldots, w_{n+1}$ in the second half of the proof of 4.7. We have that $A^{-1} = (w_i \circ w_j)$. By our last lemma we have that $A^{-1} = (\det A)^{-1} \operatorname{adj} A$ is nonpositive. Hence we have $w_i \circ w_j \leq 0$ for all i, j. Since $(A^{-1})_{ii} < 0$ holds, all $w_1, \ldots, w_{n+1}$ are time-like. So [Rat05, Thm 3.1.1] gives aus $w_i \circ w_j < 0$ for all i, j and thus all entries of A^{-1} are negative and at last all entries of $\operatorname{adj} A$ are positive and we can use 4.7 to get the result.

□

References

[Rat05] John G. Ratcliffe. *Foundations of Hyperbolic Manifolds*. Springer, 2005.