Knot theory and Topology of 3-Spaces

A triangulation for the Figure-8 complement

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Seminar: Knot Theory and its Applications

1 Motivation

One aim of this seminar is to tabulate all the knots. We want to be able to tell from a projection of one knot if it is the same as another knot. To achieve this we introduced two invariants (or conecpts of invariants) so far: the Seifert surfaces and the knot polynomials.

We now want to work towards a third invariant: the volume of a knot. How can we assign a volume to a knot K? The process is to look at $M := S^3 \setminus K$, which is a 3-manifold and by a thm and a conjecture of W. Thurston (c.f. [Ada94]), we assume that most knots are of the form, that M has the structure of a hyperbolic 3-manifold with finite volume. By this process we see, that we get many more invariants for our knot. Since we get a unique manifold we can use all of its invariants as knot invariants.

However today we can only take a first small step towards assigning a manifold to our knot K. This first step consists in the construction of a triangulation of $S^3 \setminus K$. After that one could give these tetrahedra the structure of compatible hyperbolic tetrahedra and could then compute their (hyperbolic) volume. This is then the knot volume we were talking about earlier.

2 The Figure-8 knot

We now want to show the construction of the Figure-8 knot complement very explicitly. For our purposes we use the the projection given in figure 1 on page 2. Our aim will be to proof the following

Theorem 2.1. $M = S^3 \setminus K$, where K is the Figure-8 knot, is homeomorphic to the tesselation given by the two tetrahedra in figure 2 on page 2, where sides with the same edges and edges with the same arrows are identified and all vertices are removed.

After this we would have to show, that these two tetrahedra are in fact (ideal) hyperbolic tetrahedra and then we can compute their volume and there by compute the volume of the Figure-8 knot.

Before we can start with our proof, we need to define *CW-complexes*. The will give us a decomposition of $S^3 \setminus K$, which will easily be seen to be homeomorphic to our two tetrahedra.

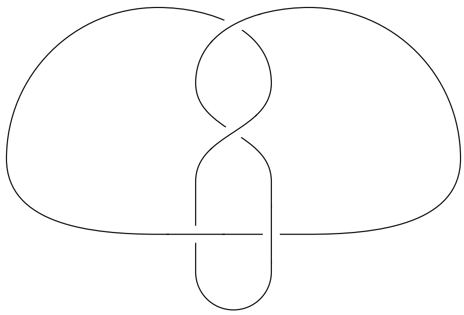

Figure 1: One projection of the Figure-8 knot

Definition 2.2 (CW-complex). A *k-cell* is a topological space that is homeomorphic to $B^k := [0,1]^k$. A *open k-cell* is a topological space that is homeomorphic to the interior of B^k.

A *CW-complex* is a topological Hausdorff space X, which decomposes into a family of open cells $(c_i)_{i \in I}$, s.t. the following properties hold:

1. For every k-cell $c_i \subset X$, there exists a continuous map $f_i : B^k \to X$, s.t. the interior of B^k is mapped homeomorphically to c_i and the boundary of B^k is mapped onto a finite union of cells of dimension (strictly) smaller than k.
2. $M \subset X$ is closed if and only if $M \cap f_i(B^k)$ is closed for all $i \in I$.

This definition is perhaps a little bit hard to grasp. One should think of CW-complexes in an inductive way. First one scatters some 0-cells (i.e. points) in space. Afterwards we attach some 1-cells (i.e. line segments) to these points. Then some discs, then some balls and so far and so forth.

Our aim is now to give $S^3 \setminus K$ the structure of a CW-complex and show that it is the same structure as the one given by our tesselation.

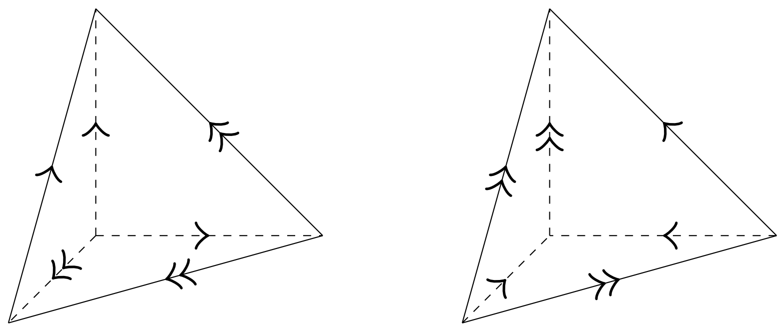

Figure 2: The labeling for the triangulation of the Figure-8 complement

Proof of Thm 2.1. We start by giving S^3 the structure of a CW-complex, which uses K as part of its 0- and 1-cells. For that we take the projection of K from figure 1 on page 2 and add two more linesegments (which will later on correspond to the two edges of the tetrahedra) and add an orientation and a labeling, too. The result is figure 3.

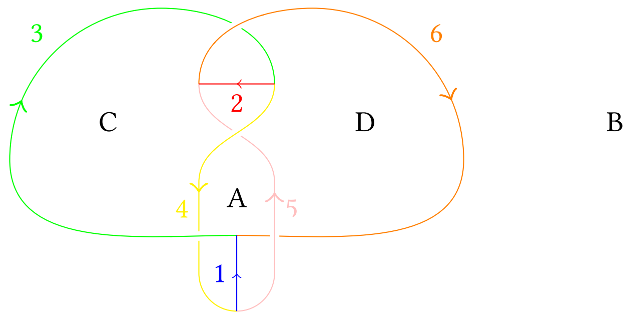

Figure 3: A labeling of the 1- and 2-cells introduced by the figure-8 knot. The 1-cells are labeled 1 to 6 and the 2-cells are labeled A to D. Remember that we have a point at infinity, so B really is a disc, which includes infinity. We call the union of all the 1-cells K^1 and the union of all the 1- and 2-cells K^2.

We can now think of it as being embedded in S^3 which we always identify with $\mathbb{R}^3 \cup \{\infty\}$. So now we have four points (the endpoints of the segment 1 and 2) and six lines embedded in S^3. The next step are the 2-cells, i.e. the discs. We will attach four of them (which is great, because after the identification our two tetrahedra have only four sides left, which is the same number...). The discs are labeled from A to D and their boundary is given by the 1-cells surrounding them. We have to keep in mind that we have a point at infinity, so B really is a disc with boundary 6, 1, 1, 3, 2.

So now we are almost finished with giving S^3 a CW-complex structure. The last thing we have to prove is, that we can attach two 3-cells (balls) in a way that is compatible with the interiors of our tetrahedra. The problem is, that it is not clear at all, that we can attach two balls to our cells in a meaningful way. To show that this is possible we use a trick. We will show that $S^3 \setminus K^1$, where K^1 is the union of alle the 1-cells, is homeomorphic to $S^3 \setminus K'^1$, where K'^1 is given in figure 4 on page 4.

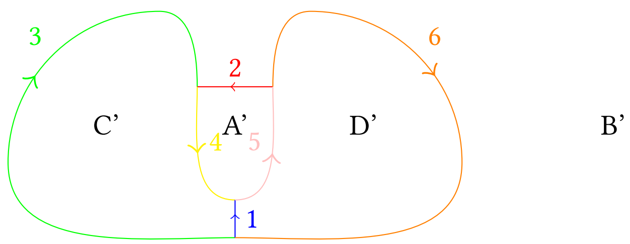

Figure 4: The definition of K'^1 and the labeling of the discs A to D are mapped to.

It is not that obvious that we have this homeomorphism. To get it, we first note, that our homeomorphism can be chosen as the identity outside a neighbourhood of segment 1 and a neighbourhood of segment 2. We now concentrate on the neighbourhood of segment 1.

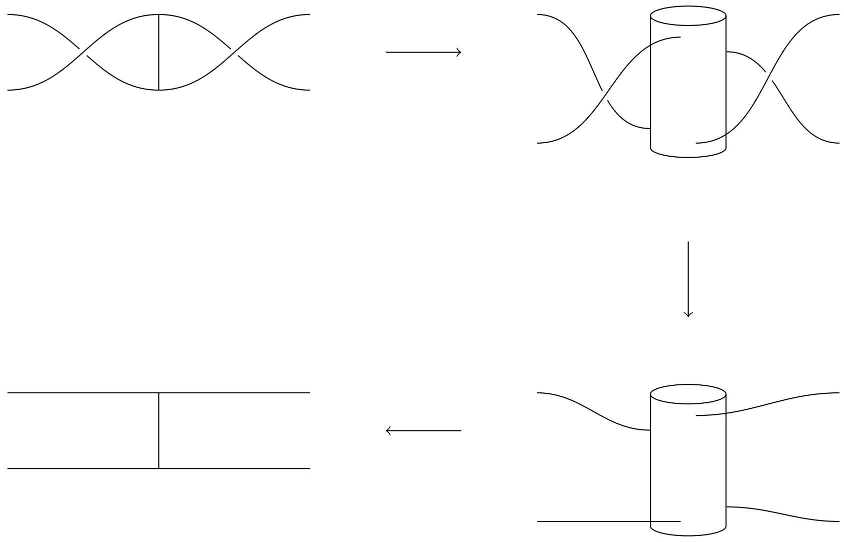

Figure 5: The homeomorphism in a neighbourhood of segment 1.

Figure 5 gives the motivation for the local homeomorphism. One can make this idea more concrete, if one looks at the map

$$\dot{B}^2 \times [0,1] \to (B^2 \setminus B_r^2) \times [0,1], \quad (x,t) \mapsto \left(\frac{x}{1-r} + r, t \right),$$

which one can easily check to be an homeomorphism, if $r < 1$ and B_r^2 is the disc of radius r and $\dot{B}^2 = B^2 \setminus \{0\}$ holds. With this map we get the first and the third arrow and the second one is induced by an isotopy, so all the arrows are given by homeomorphisms, which yields our assumption. In the case of segment 2, we do the same thing.

So now we have an homemorphism from $S^3 \setminus K^1$ to $S^3 \setminus K'^1$ and our discs A to D are mapped under this homeomorphism to A' to D'. However to the union of K'^1 with A' to D' it is very easy to attach two 3-cells, because the graph given by K'^1 ist planar and thus divides $\mathbb{R}^3$ (if we forget for one moment about the point at infinity) in two halfspaces. So one ball will become the upper halfspace and the other one the lower halfspace and by this we cover all of S^3, so we found a CW-complex structure on S^3. At last we use the homeomorphism we constructed to get to $S^3 \setminus K'^1$ to get back to $S^3 \setminus K^1$ and attach by it the 3-cells to K^2 und by this get again a CW-complex structure which incorporates our knot K. If we study what happens to the upper 3-cells under the homeomorphism, especially to which sides it gets attached to and then remember, that we are interested in $S^3 \setminus K$, i.e. we have to collapse the line segments 3 to 6 and remove the remaining endpoints, we can draw the left tetrahedra shown in figure 6 on page 5.

The right tetrahedron is just the first from figure 2 on page 2 and we see that the two are compatible if we identify segment 1 with the single arrow and segment 2 with te double arrow. We can do the same thing with the lower 3-cell and see that it is compatible with the other tetrahedron.

This completes our proof of the structure of the Figure-8 knot complement. □

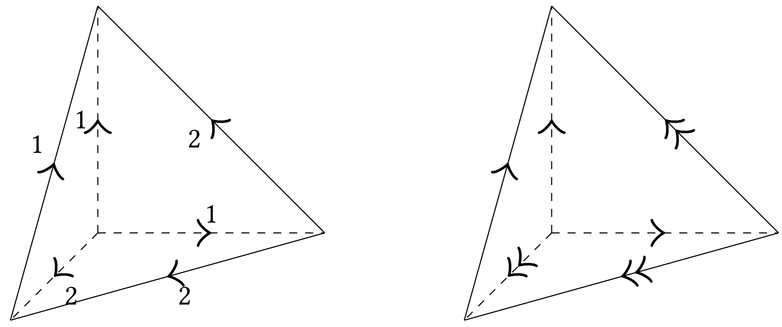

Figure 6: The left tetrahedron shows how the upper 3-cell is attached. The right tetrahedron shows the right tetrahedron from figure 2 on page 2. We can see, that both are compatible.

3 Outlook

As was stated in the motivation part, this is only the starting point. If one liked to stick to the example of the Figure-8 knot, we would now have to show that the two tetrahedra can be chosen as (ideal) hyperbolic tetrahedra. Thus $S^3 \setminus K$ would get the structure of a hyperbolic manifold. This is shown in [Rat06, §10]. There are even more examples on hyperbolic knots and links, if you want to look at some more examples.

However one should try to generalize the approach to general knots and links. This has been done by W. Thurston and J. Weeks. Weeks was even able to implement an algorithm, that can compute the triangulation of a knot, if it exists and afterwards tries to calculate its volume (and many more invariants). For further reading I would advice to have a look at [CDW].

References

[Ada94] Colin C. Adams. *The knot book*. An elementary introduction to the mathematical theory of knots. W. H. Freeman and Company, New York, 1994, pp. xiv+306. ISBN: 0-7167-2393-X.

[CDW] Marc Culler, Nathan M. Dunfield, and Jeffrey R. Weeks. *SnapPy, a computer program for studying the topology of 3-manifolds*. Available at `http://snappy.computop.org` (07/07/2014).

[Rat06] John G. Ratcliffe. *Foundations of hyperbolic manifolds*. Second. Vol. 149. Graduate Texts in Mathematics. Springer, New York, 2006, pp. xii+779. ISBN: 978-0387-33197-3; 0-387-33197-2.