Summersemester 2016

University of Heidelberg

Geometry of Lie Groups seminar

Classical groups and their real forms

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Introduction

Classical groups are among the most important lie groups there are. On the one hand they are very simple and on the other hand nearly all simple lie groups are in fact classical groups. As matrix groups they are easy to understand and many computations can be done explicitly.

For example their corresponding Lie Algebras are easy to compute and even some topological features can be derived with moderate or little effort.

classical groups

Throughout this talk let $E \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ unless otherwise stated. Also, every vector space considered has finite dimension.

First of all we want to get familiar with classical groups. Therefor we will define them and consider both coordinate and coordinate-free representations.

Definition 1. Let V be a vector space over E. A **classical group** is either $GL(V)$ or a subgroup that preserves a non-degenerate sesquilinear form being either symmetric or skew-symmetric.

Whenever we choose a basis $(v_i)_i$ for V this induces an isomorphism

$$GL(V) \longrightarrow GL(n, E)$$

where the latter is the group of all $n \times n$ matrices with non-zero determinant and koefficents in E. If the context is clear we will switch between notations as needed and omit the underlying field, thus writing $GL(n)$.

We assume that the reader has sufficent knowledge of those groups from linear algebra and is familiar with matrix multiplication and basic operations.

Now it is time to consider some examples in detail. Therefore consider an $E-$vector space V and a bilinear form

$$B: \ V \times V \longrightarrow E$$

Let $\text{Isom}(V, B)$ be the isometry group of B, i.e.

$$\text{Isom}(V, B) = \{M \in GL(V) \mid B(Mx, My) = B(x, y) \ \forall x, y \in V\}$$

If we choose coordinates in V and identifiy M and B with their matrices in $GL(n, E)$ we get

$$B(Mx, My) = B(x, y) \ \forall x, y \Leftrightarrow B = M^T B M \tag{1}$$

Since B is non-degenerate if follows $\det(B) \neq 0$ and thus (1) tells us that the condition $B(Tv, Tw) = B(v, w) \ \forall v, w$ directly implies $T \in \text{Isom}(V, B)$.

orthogonal groups

Let $E \neq \mathbb{H}$ for now.

Lets start with a coordinate representation. The first one is well known:

Definition 2. The **orthogonal group** is

$$O(n, E) := \left\{M \in GL(n, E) \mid M^T M = I\right\} = \left\{M \in GL(n, E) \mid M^T = M^{-1}\right\}$$

As we will see soon if $E = \mathbb{R}$ there is a slight generalisation we are interested in

Definition 3. The **indefinite orthogonal group** is

$$O(p, q) \coloneqq \left\{ M \in GL(n, E) \mid M^T I_{p,q} M = I_{p,q} \right\}, \quad I_{p,q} = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$$

Note that $O(p, q) \cong O(q, p)$ and thus $O(0, n) = O(n, 0) = O(n, \mathbb{R})$.

It would not make sense to consider the indefinite case for $E = \mathbb{C}$ which can be seen through the following lemma:

Lemma 4. *Let V be an $E-$vector space and B a symmetric non-degenerate bilinear form on V. Then*

a) if $E = \mathbb{C}$ there is a basis $(v_i)_i$ of V sucht that $B(v_i, v_j) = \delta_{ij}$

*b) if $E = \mathbb{R}$ there are $p, q \geq 0$ sucht that $p+q = n$ and a basis $(v_i)_i$ with $B(v_i, v_j) = \varepsilon_i \delta_{ij}$ where $\varepsilon_i = 1$ if $i \leq p$ and $\varepsilon_i = -1$ otherwise. Such a basis will be called **pseudo-orthonormal**.*

Furthermore p only depends on the choice of B.

Proof. Via the Gram-Schmidt algorithm we obtain an orthogonal basis $(v_i)_i$ of V.

Now if $E = \mathbb{C}$ we can choose a square root of $B(v_i, v_i)$ for all i and set

$$v_i' = \frac{1}{\sqrt{B(v_i, v_i)}} v_i$$

this gives us the desired basis. If $E = \mathbb{R}$ we do the same thing except we first order our basis such that $B(v_i, v_i) \geq B(v_{i+1}, v_{i+1})$ and set

$$v_i' = \frac{1}{\sqrt{\pm B(v_i, v_i)}} v_i$$

where the sign is determined by the sign of $B(v_i, v_i)$.

It remains to show that p is intrinsic to B. Define

$$V^+ = span \{v_1, \ldots, v_p\} \qquad V^- = span \{v_{p+1}, \ldots, v_n\}$$

Then $V = V^+ \oplus V^-$ and B is positive definite on $V^+ \times V^+$.

take any subspace W of V such that B is positive definite on W. Assume $w \in W$ and $\pi_{V^+}(w) = 0$. Then it follows that $w \in V^-$ and thus $w = \sum_{i>p} a_i v_i$. We get

$$B(w, w) = \sum_{i,j>p} a_i a_j B(v_i, v_j) = -\sum_{i>p} a_i^2 \leq q$$

Since B was assumed to be positive definite on W we get $w = 0$. Thus $\pi : W \longrightarrow V^+$ is injective and hence $\dim(W) \leq \dim(V^+) = p$. So p is uniquely number defined as the maximum dimension of a subspace on which B is positive definite. $\square$

Thus we now have a good understanding of these orthogonal matrices. Yet a coordinate-free version would be nice to have. Since we already have the identification

$$GL(V) \longrightarrow GL(n, E)$$

we should try if we can indetify these orthogonal matrices as isometrys of certain bilinear forms. If we consider the definition of $O(p, q)$ or $O(n, E)$ we know that those bilinear forms need to be at least symmetric. This leads us to

Theorem 1. *Let B be a non-degenerate symmetric bilinear form on V (E-vector space) and $(v_i)_i$ be a (pseudo-)orthonormal basis with respect to B. Then the map*

$$\mathrm{Isom}(V,B) \longrightarrow O(p,n-p)$$

sending M onto its matrix-representation is an isomorphism. Note that for $E=\mathbb{C}$ we get $p=n$.

Proof. After choosing a suitable basis we get the result from

$$B(Mx,My) = (Mx)^T (B(v_i,v_j))_{ij} My = x^T M^T (B(v_i,v_j))_{ij} My = x^T M^T I_{p,q} My$$

□

We now try to get similar results for different kinds of groups.

Symplectic groups

If we define $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ we consider the **symplectic group**

$$Sp(n,E) = \left\{ M \in M_{2n}(E) \mid M^T J M = J \right\} < GL(n,E)$$

As this is the coordinate representation we are now interested in a coordinate-free version using bilinear forms. This time we take a look at skew-symmetric bilinear forms, e.g $B(x,y) = -B(y,x)$. Note that there is no vector space with uneven dimension equipped with a skew-symmetric bilinear form because if we consider the matrix-representation of such a form we get

$$\det(A) = \det(A^T) = \det(-A) = (-1)^n \det(A)$$

Since the matrix B for a non-degenerate skew-symmetric bilinear form $B(x,y)$ is non-zero it vanishes in uneven dimensions.

So we have

Lemma 5. *Let V be a $2n-$dimensional vector space over E and B a non-degenerate, swek-symmetric bilinear form on V. Then there is a basis $(v_i)_i$ of V such that $(B(v_i,v_j))_{ij} = J$. Such a basis will be called $B-$symplectic.*

Proof. We only sketch the proof. Since B is non-degenerate we find v,w such that $B(v,w) \neq 0$. After rescaling we thus achieve $B(v,w)=1$ and $B(w,v)=-1$. Consider

$$W = \{x \in V \mid B(v,x) = 0 \;\&\; B(w,x) = 0\}$$

and show that $\dim(W) = 2n-2$. By Induction we get the desired basis. □

Like before we achieve

Theorem 2. *Let V be a $2n-$dimensional vector space over E and B a non-degenerate, swek-symmetric bilinear form on V. Fix a $B-$symplectic basis for V. Thenn this induces an isomorphism*

$$\mathrm{Isom}(V,B) \longrightarrow Sp(n,E)$$

unitary groups

If we consider complex matrices it makes sense to take a look at unitary matrices. Let $A^* = \bar{A}^T$. Then we can define the **unitary group**

$$U(n) = \{M \in M_n(\mathbb{C}) \mid M^* M = I\}$$

Since this is similar to the orthogonal case we also take a look at the indefinite case. For this let $I_{p,q}$ as before. Then

$$U(p,q) = \{M \in M_n(\mathbb{C}) \mid M^* I_{p,q} M = I_{p,q}\}$$

is the **indefinite unitary group** of signature (p,q). To get to the coordinate-free version we take a look at specific bilinear forms:

Definition 6. Let V be a $\mathbb{C}-$vector space. A bilinear form $B(x,y)$ is called **hermitian** iff

1. $B(av,w) = aB(v,w)$ for $a \in C$
2. $B(w,v) = \overline{B(v,w)}$

Since $B(v,v)$ is always real the notion of positve definitness is the same as in the real case.

Again we get the following result

Lemma 7. *Let V be a $\mathbb{C}-$vector space and B a non-degenerate hermitian form on V. Then there is an integer p and a basis $(v_i)_i$ of V such that $B(v_i,v_j) = \varepsilon_i\delta_{ij}$ where $\varepsilon_i = 1$ if $i \le p$ and $\varepsilon_i = -1$ otherwise. Moreover p only depends on the choice of B and not on the choice of the basis.*

Proof. Nearly the same as in Lemma 4 except we have complex numbers. □

Since we now have a nice basis we conclude

Theorem 3. *Let V be a $\mathbb{C}-$vector space and B a non-degenerate hermitian formwith signature (p,q) on V. If we fix a pseudo-orthonormal basis on V this induces an isomorphism*

$$\mathrm{Isom}(V,B) \longrightarrow U(p,q)$$

Quarternion groups

Up until know we only considered $E \in \{\mathbb{R}, \mathbb{C}\}$ so now we take a look at $E = \mathbb{H}$. Before we consider quarternionic groups we should review some basic fact about $\mathbb{H}$.

Definition 8. Let $\{1,i,j,k\}$ be a basis of $\mathbb{R}^4$. We define a multipliction by setting

$$i^2 = j^2 = k^2 = -1 \qquad ij = -ji = k \qquad ki = -ik = j \qquad jk = -kj = i$$

and extending this linear. This space is called the **quarternions**.

Like this we can define a conjugation

$$(a + ib + jc + kd)^* = a - ib - jc - kd$$

and an absolute value by

$$|w| = |a + ib + jc + kd| = w^* w = ww^* = \sqrt{a^2 + b^2 + c^2 + d^2}$$

We should note that multiplication in the quarternions is not commutative.

In arbitrary dimension we can define a right multiplication on $\mathbb{H}^n$ by

$$(u_1, \dots, u_n) \cdot a = (u_1 \cdot a, \dots, u_n \cdot a)$$

for $u_i, a \in \mathbb{H}$. Therefore we can think of $\mathbb{H}^n$ as a vector space over $\mathbb{H}$ (instead of an $4n$-dimensional real vector space).

It is often useful to see quarternions in a complex form. For this we write $z = x + jy \in \mathbb{H}^n$ with $x, y \in \mathbb{C}^n$ if we see $\mathbb{C}$ embedded into $\mathbb{H}$ via $\mathbb{C} = \mathbb{R} + i\mathbb{R} \subset \mathbb{H}$. Moreover if we write $C = A + jB \in M_n(\mathbb{H})$ for $A, B \in M_n(\mathbb{C})$ we get

Lemma 9. *Consider the maps*

$$z \mapsto \begin{pmatrix} x \\ y \end{pmatrix} \qquad\qquad C \mapsto \begin{pmatrix} A & -\bar{B} \\ B & \bar{A} \end{pmatrix}$$

They define an isomporphism $\mathbb{H}^n \longrightarrow \mathbb{C}^{2n}$ and

$$M_n(\mathbb{H}) \longrightarrow \left\{T \in M_{2n}(\mathbb{C}) \mid JT = \bar{T}J\right\}, \quad J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

This tells as that $\mathbb{H}-$linear transformations can be seen as complex-linear ones. Using this embedding it is possible to define the following:

Definition 10. The **general linear group** $GL(n, \mathbb{H})$ is defined as

$$GL(n, \mathbb{H}) = \left\{T \in GL(2n, \mathbb{C}) \mid JT = \bar{T}J\right\}$$

With these basic facts we can take a look at two more classical groups. Let Ip, q be as before and set $X^* = (x^*_{ji})_{ij}$ then

Definition 11. The **indefinite quaternionic unitary group** is

$$Sp(p, q) = \{M \in GL(n, \mathbb{H}) \mid M^* I_{p,q} M = I_{p,q}\}$$

In the same manner as with unitary groups we get

$$\mathrm{Isom}(V, B) \cong Sp(p, q)$$

for non-degenerate quaternionic hermitian forms $B(w, z) = w^* I_{p,q} z$, e.g. forms, such that

$$B(w, z) = B(z, w)^* \qquad\qquad B(w \cdot a, z \cdot b) = a^* B(w, z) b$$

Our last example is to take a look at symplectic groups with quarternions. Let J be the matrix from the symplectic groups. Since $J^2 = -I_{2n}$ we get an auotmorphism of $GL(2n, \mathbb{C})$ via

$$\theta(M) = -JMJ$$

such that $\theta^2 = Id$. This leads us to

Definition 12.

$$SO^*(2n) = \left\{M \in SO(2n, \mathbb{C}) \mid \theta(\bar{M}) = M\right\}$$

Since we can identify $\mathbb{C}^{2n}$ with $\mathbb{H}^n$ by $(a, b) \mapsto a + jb$. Then the group above is the isometry group of non-degenerate quarternionic skew-hermition forms

$$C(x, y) = x^* \cdot j \cdot y$$

e.g. forms that satisfy $C(x, y) = -C(y.x)^*$ and $C(xa, yb) = a^* C(x, y) b$

lie algebras of classical groups

Now that we have seen all classical groups we want to calculate their Lie algebras.

general lie algebra

First of all we want to take a look at the general and special linear lie algebras.

Definition 13. The **general linear lie algebra** is

$$Lie(End(V)) = \mathfrak{gl}(V) \cong \mathfrak{gl}(n, E) = Lie(M_n(E))$$

where $End(V)$ is equipped with $[X, Y] = XY - YX$

Another easy to calculate example is the special lie algebra. For this define (note: the trace is basis independend)

$$\mathfrak{sl}(V) = \{T \in End(V) \mid tr(T) = 0\}$$

Since $tr(AB) = tr(BA)$ we get $tr([A, B]) = 0$ for all $A, B \in End(V)$ and thus $\mathfrak{sl}(V)$ is a sub-lie-algebra of $\mathfrak{gl}(V)$. With this we get

Definition 14. The **special linear lie algebra** is

$$\mathfrak{sl}(n, E) = \{A \in \mathfrak{gl}(n, E) \mid tr(A) = 0\}$$

Lie algebras of bilinear forms

For now let us consider $E \in \{\mathbb{R}, \mathbb{C}\}$.

In an earlier talk we have seen that for a Lie group G we have the identification

$$Lie(G) = T_eG$$

Let us consider differentiable curves $\sigma : (-\varepsilon, \varepsilon) \longrightarrow GL(V)$ such that $\sigma(0) = I$ and $\sigma(t) \in G$, e.g. $\sigma(t)$ satisfies the conditions which defined G as a subgroup of $GL(V)$. Now we have shown that $\sigma'(0) \in Lie(G)$ and that every element in the lie algebra is obtained like this.

We can now consider $\mathrm{Isom}(V, B)$ for a bilinear form B. Then

$$B(\sigma(t)v, \sigma(t)w) = B(v, w)$$

for every t. Differentiating this equation yields

$$0 = B(\sigma'(0)v, \sigma(0)w) + B(\sigma(0)v, \sigma'(0)w)$$

because $\sigma(0) = I$ we can define

Definition 15. The lie algebra associated with a bilinear form is

$$\mathfrak{so}(V, B) = \{X \in End(V) \mid B(Xv, w) = -B(v, Xw)\}$$

This is indeed a subalgebra of $\mathfrak{gl}(V)$ because

$$B(XYv, w) = -B(Yv, Xw) = B(v, YXw)$$

implies that $B([X, Y]v, w) = -B(v, [X, Y]w)$.

If we fix a basis of V and let $\Gamma = (B(v_i, v_j))_{ij}$ then this implies

$$A^T\Gamma + \Gamma A = 0$$

for every matrix representation A of an element in $\mathfrak{so}(V, B)$. Since B is non degenerate in our cases we have

$$A^T = -\Gamma A \Gamma^{-1}$$

and thus $tr(T) = 0$ for all $T \in \mathfrak{so}(V, B)$.

orthogonal and symplectic lie algebras

If we apply the previous results it is easy to compute the orthogonal lie algebras.
First of all if $\Gamma = I$ we have

$$\mathfrak{so}(n, E) = \left\{X \in M_n(E) \mid X^T = -X\right\}$$

Now let B be a bilinear form with matrix $I_{p,q}$. Then

$$\mathfrak{so}(p, q) = \left\{X \in M_n(\mathbb{R}) \mid X^T I_{p,q} = -I_{p,q}X\right\}$$

Remember that because B is non degenerate they are sub lie algebras of $\mathfrak{sl}(n, F)$.
As we have seen before choosing a (pseudo-)orthonormal basis induces an lie algebra isomorphism

$$\mathfrak{so}(V, B) \longrightarrow \mathfrak{so}(p, q) \textit{ or } \mathfrak{so}(n, E)$$

when we consider symmetric bilinear forms.
In the symplectic case we get similar results:

$$\mathfrak{sp}(n, E) = \left\{X \in M_{2n}(E) \mid X^T J = -JX\right\}$$

and for non-degenerate skew-symmetric bilinear forms

$$\mathfrak{so}(V, B) \cong \mathfrak{sp}(n, E)$$

(after choosing a $B-$symplectic basis fo course).

unitary lie algebras

In the following complex cases the arguments are essentially the same. Note that instead of A^T we will now use $A^* = \bar{A}^T$. Then we can define

$$\mathfrak{u}(p, q) = \{X \in M_n(\mathbb{C}) \mid X^* I_{p,q} = -I_{p,q}X\}$$

this is a lie subalgebra of $\mathfrak{gl}(n, \mathbb{C})$ which can be seen by the same argument as for $\mathfrak{so}(V, B)$. Furthermore we can set

$$\mathfrak{su}(p, q) = \mathfrak{u}(p, q) \cap \mathfrak{sl}(n, \mathbb{C})$$

These matrices derive from hermitian forms on $End_{\mathbb{C}}(V)$. So by defining

$$\mathfrak{u}(V, B) = \{X \in End_{\mathbb{C}}(V) \mid B(Xv, w) = -B(v, Xw)\}$$

we achieve a coordinate-free representation via, i.e.

$$\mathfrak{u}(p, q) \cong \mathfrak{u}(V, B)$$

by choosing a pseudo-orthogonal basis of V,

quarternionic lie algebras

First of all we can also define the general linear lie algebra for quaternions by

$$\mathfrak{gl}(n, \mathbb{H}) = (M_n(\mathbb{H}), [\cdot, \cdot])$$

with the usual matrix commutator. Note that we need to consider this as a lie algebra over $\mathbb{R}$ since we have not defined what a lie algebra over a skew field should be. If we identify $\mathbb{H}^n$ with $\mathbb{C}^{2n}$ we simply define

$$\mathfrak{sl}(n,\mathbb{H}) = \{X \in \mathfrak{gl}(n,\mathbb{H}) \mid tr(X) = 0\}$$

We already stated this definition but since it is important that we use the identification above it is good to do this here again.

This real lie algebra is usually denoted by $\mathfrak{su}^*(2n)$.

To end the overview of lie algebras we define the unitary and orthogonal lie algebras in the same manner:

$$\mathfrak{sp}(p,q) = \{X \in \mathfrak{gl}(n,\mathbb{H}) \mid X^* I_{p,q} = -I_{p,q}X\}$$

This group consist of matrices satisfying

$$B(Xx,y) = -B(x,X^*y)$$

For the last example remember the map $\theta(A) = -JAJ$. With this we define

$$\mathfrak{so}^*(2n) = \left\{X \in \mathfrak{so}(2n,\mathbb{C}) \mid \theta(\bar{X}) = X\right\}$$

If we again identify $\mathbb{C}^{2n}$ and $\mathbb{H}^n$ we can see this as the matrices satisfying

$$C(Xx,y) = -C(x,X^*y)$$

where $C(x,y) = x^*jy$ in a quaternionian way.

connectness

If we consider these groups we can derive some information about their topology and how they look like.

connection of $SO(n)$ and $O(n)$

It is easy to see that $O(n)$ is not connected since the determinant is nonzero and changes signs.

I will give the proof that $SO(n)$ is connected only as a sketch. For details see [War]. An important proposition is the following:

Proposition 16. *Let H be a closed subgroup of a Lie group G. If H and G/H are connected then so is G.*

Since we know that $SO(1)$ is connected because it consists only of the 1×1 identity matrix our problem reduces to the question if $SO(n)/SO(n-1)$ is connected or not.

For this consider the action of $O(n)$ on S^{n-1} ($O(n)$ preserves norm). and note that $A \in O(n-1)$ can be embedded in $O(n)$ through

$$\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$$

Geometrically $O(n-1)$ leaves $\langle e_n\rangle$ fixed. Putting these things together one can find a diffeomorphism

$$SO(n)/SO(n-1) \longrightarrow S^{n-1}$$

and thus we achieve the required connectness. Proposition 16 then tells us that $SO(n)$ is connected for every n.

S^3 and unit quarternions

We want to see how $SO(3)$ looks like. Consider the totally imaginary quarternions

$$I\mathbb{H} = \{ai + bj + ck \mid a, b, c \in \mathbb{R}\}$$

because the absolute value is multiplicative in $\mathbb{H}$ and $\mathbb{H}$ is skew-field we see that $S^3 \subset \mathbb{H}$ is a group under multiplication and the map

$$(q, p) \mapsto qpq^{-1}$$

defines a group action $S^3 \times I\mathbb{H} \longrightarrow I\mathbb{H}$. (long computation). If we identify $I\mathbb{H} \cong \mathbb{R}^3$ every $q \in S^3$ defines a map

$$\sigma_q : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$

which satisfies

$$\sigma_1 = id \qquad \sigma_{q^{-1}} = \sigma_q^{-1} \qquad |\sigma_q(p)| = |p|$$

especially the last equation tells us that σ_q preserves norms. If we put everything together this results in an homomorphism

$$S^3 \longrightarrow SO(3)$$

but since q and $-q$ give the same map we end up with

$$SO(3) \cong \mathbb{RP}^3$$

moreover we get $\pi_1(SO(3)) = \mathbb{Z}_2$.

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