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Riemannian geometry seminar

Hopf-Rinow and Hadamard Theorems

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Introduction

Up till now we examined many properties of manifolds which can be considered „local "ones. They only depend on local properties of our manifold and sometimes prove to be false when we take a suffiecently large neighborhood.

In this talk we want to investigate the connection of certain local properties to global ones like, e.g being homeomorphic to the euclidian space $\mathbb{R}^n$ in case of non-positiv sectional curvature. To achieve this we only consider a special class of manifolds where $\exp_p$ is defined everywhere. These so called *complete manifolds* have intuitively neither holes nor boundary components which makes them in a sense „nice to handle ".

The most important theorem of this talk will be the Hopf-Rinow theorem that not only connects the topological property of being complete as a mertric space to the geometric notion of being geodesically complete but it tells us in addition that we can always connect two points with a length-minimizing geodesic.

That is on the one hand a powerfull tool to understand manifolds in regard to completeness. On the other hand it yields the strong theorem of *Hadamard*, i.e. manifolds with sectional curvature $K \leq 0$ are homeomorphic to $\mathbb{R}^n$.

Throughout this talk M will denote a connected riemannian manifold of dimension $dim(M) = n$ unless otherwise mentioned.

Hopf-Rinow

Since we want to have a look at global properties we need to make sure that M is „on its own", i.e M is no proper open subset of a bigger manifold. This can be characterized as follows

Definition 1. M is said to be extendible iff there exists a riemannian manifold M' such that M is isometric to a proper open subset of M'.

As mentioned before we want to consider manifolds that are in a sence nice enough to work with. With regard to the Hopf-Rinow theorem we want to take a look at manifolds where the exponential map is defined everywhere. This leads us to the next definition

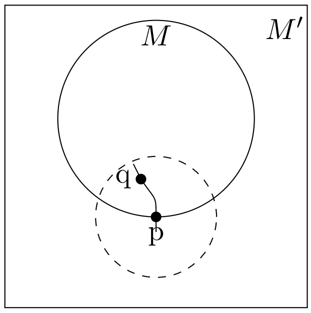
Definition 2. M is said to be geodesically complete iff for all $p \in M$ $\exp_p$ is defined for every $v \in T_pM$, i.e. all geodesics $\gamma(t)$ starting at p are defined for all $t \in \mathbb{R}$.

With this we can show that complete manifolds are also suitable to study global properties

Proposition 3. *If M is complete it is not extendible.*

Proof.

Assume the opposite. Then $\partial M \subset M'$ is non-empty. For $p \in \partial M$ choose $U' \subset M'$ as a normal neighborhood of p and choose $q \in U' \cap M$. Let $\psi(t)$ be the geodesic with $\psi(0) = p$ and $\psi(1) = q$. Then $\gamma(t) = \psi(1-t)$ is by isometry a geodesic in M für t small enough and $\gamma(0) = q$. But this geodesic is not defined for some $t \leq 1$ which contradicts the completeness.



□

To complete the preliminaries for the Hopf-Rinow Theorem we need to introduce a distance function on M. We will do this in the standard way und thus omit the proofs.

Definition 4. Let $p, q \in M$. Then

$$d(p,q) = \inf \{l(\gamma) \mid \gamma \text{ piecewise smooth and connection } p, q\}$$

Since M is connected this is well defined and has the desired properties i.e.

Proposition 5. *Let d be as above. Then*

i) $d : M \times M \longrightarrow \mathbb{R}$ *is a metric, thus* (M, d) *is a metric space*

ii) the topology induced from d coincides with the original topology

iii) $f(p) = d(p,q)$ *is continous for* $q \in M$

As a short remark we notice that $d(p,q) = l(\gamma)$ if there exists a minimizing geodesic from p to q.

Now we can state our main theorem from which it will be clear why completeness is important in our observations.

Theorem 1 (Hopf-Rinow)**.** *Let M be a riemannian manifold and $p \in M$ Then the following assertions are equivalent*

i) $\exp_p$ *is defined on all of* T_pM

ii) the closed and bounded sets of M are compact

iii) M is complete as a metric space

iv) M is geodesically complete

v) There exists a sequence of compact subsets $K_n \subset M$, $K_{n+1} \subset K_n$ *where* $M = \bigcup K_n$ *such that if* $q_n \notin K_n$ *then* $d(p, q_n) \longrightarrow \infty$

Furthermore any of the above implies:

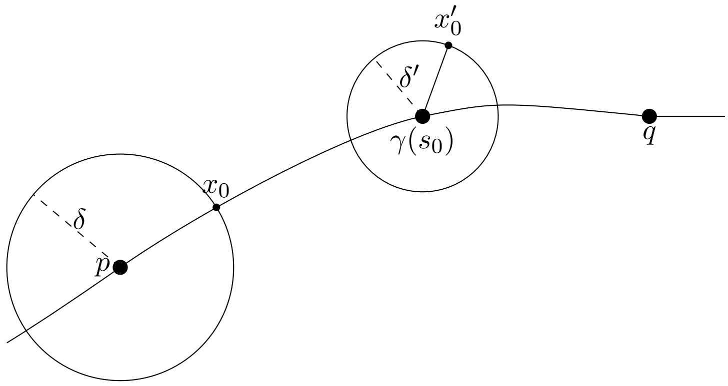
vi) for any $q \in M$ *there exists a geodesic* γ *joining* p *and* q *with* $l(\gamma) = d(p,q)$

Proof. $iv) \Rightarrow i)$ is definition and $ii) \Leftrightarrow v)$ is a general topological proof which is left for the reader. Since it comes in handy to have $vi)$ we will look consider that first

$i) \Rightarrow vi)$ Let $d(p,q) = r$ and let $B_\delta(p)$ be the normal ball at p where $S = S_\delta(p)$ is its boundary. Since $d(q,x)$ is continous it attains a minimum on S which we will denote by x_0. By definition of the exponential map we find a unit vector $v \in T_pM$ such that $x_0 = \exp_p(\delta v)$. Now we define $\gamma(t) = \exp_p(sv)$ and show that $\gamma(r) = q$. Since γ is a geodesic this will prove $i) \Rightarrow vi)$. We consider

$$d(\gamma(s), q) = r - s \tag{1}$$

and $A = \{s \in [0,r] \mid (1) \text{ is valid}\}$. Since $0 \in A$ and the continuity of the distance function A is not empty and closed. Thus we only need to check if A is open. Let $s_0 < r$ be in A and let δ' be sufficiently small. We want to show that $s_0 + \delta' \in A$. Similar to the above let $B_{\delta'}(\gamma(s_0))$ be the normal ball with boundary S'. Furthermore let x_0' be the minimum of $d(x,q)$ on S'.



By definition of the metric we have

$$r - s_0 = d(\gamma(s_0), q) = \delta' + \min_{x \in S'} d(x,q) = \delta' + d(x_0', q) \tag{2}$$

using the triangle inequalitiy we obtain

$$d(p, x_0') \geq d(p,q) - d(q, x_0') = r - (r - s_0 - \delta') = s_0 + \delta'$$

Now we see, that $\gamma(s_0 + \delta') = x_0'$ because the broken curve from p to x_0' via $\gamma(s_0)$ has length $s_0 + \delta'$ and hence needs to be a geodesic. So (2) turns out to be

$$r - s_0 = \delta' + d(\gamma(s_0) + \delta'), q) \quad \Leftrightarrow \quad d(\gamma(s_0) + \delta'), q) = r - (s_0 + \delta')$$

which is the desired result.

$i) \Rightarrow ii)$ For a closed and bounded set $A \subset M$ we always find a metric ball $B_\varepsilon(p)$ containing A. Since we now can use $vi)$ there is a ball $B_r(0) \subset T_pM$ such that $B_\varepsilon(p) \subset \exp_p(\overline{B_r(0)})$. Since exp is continous the image on the right is compact and hence A as a closed subset of a compact set is compact itself.

$ii) \Rightarrow iii)$ Let $P = (p_n)_n$ be a cauchy sequence in M. Then $\overline{\{p_n\}}$ is bounded and closed and thus compact. Since we can now find a convergent subsequence P must converge as well (being cauchy).

iii) $\Rightarrow$ *iv)* We suppose M is not geodesically complete. Then there exists a normalized geodesic γ that is defined for values $s < s_0$ but not for s_0. Let $(s_n)_n$ be a sequence such that $s_n \longrightarrow s_0$. For sufficiently large indices we have $|s_n - s_m| < \varepsilon$. This tells us that

$$d(\gamma(s_n), \gamma(s_m)) \leq |s_n - s_m| < \varepsilon$$

and hence that $\{\gamma(s_n)\}$ is cauchy.. Because M is complete there is $p_0 \in M : \gamma(s_n) \longrightarrow p_0$. Let (W, δ) be a totally normal neighborhood of p_0. Let N be big enough such that $|s_m - s_n| < \delta$, $\gamma(s_n), \gamma(s_m) \in W$ whenenver $m, n > N$. Then by definition there is a unique geodesic g connecting $\gamma(s_n)$ and $\gamma(s_m)$ with length less than δ. Wherever γ is defined it is obvious that $g = \gamma$. Since we know that $\exp_{\gamma(s_n)} : B_\delta(0) \longrightarrow M$ is a diffeo on its image which contains W we see that g extends γ for values beyond s_0.

□

From this theorem we get immediately 2 useful corollaries

Corollary 6. *If M is compact it is complete.*

Corollary 7. *A closed submanifold of a riemannian manifold is complete in the induces metric. In particular that holds for euclidian space.*

Hadamard

Aside of the corollaries above the Hopf-Rinow theorem has other interesting applications. We want to take a look at the properties of riemannian manifolds with sectional curvature $K \leq 0$ everywhere. The following lemma shows us that under this assumption the exponential map is a local diffeomorphism:

Lemma 8. *Let M be a complete riemannian manifold with $K(p, \sigma) \leq 0$ for all p, σ. Then the exponential map $exp_p : T_pM \longrightarrow M$ is a local diffeomorphism.*

Proof. Let J be a non-trivial Jacobi field along γ where $\gamma : [0, \infty) \longrightarrow M$ is a geodesic with $\gamma(0) = p$ and $J(0) = 0$. Since we have a non-positiv curvature the Jacobi equation yields

$$\begin{aligned} \langle J, J \rangle'' &= 2 \left\langle J', J' \right\rangle + 2 \left\langle J'', J \right\rangle \\ &= 2 \left\langle J', J' \right\rangle - 2 \left\langle R(\gamma', J)\gamma', J, \right\rangle \\ &= 2|J'|^2 - 2K(\gamma', J)|\gamma' \wedge J|^2 \geq 0 \end{aligned}$$

Thus we get that $\langle J, J \rangle'$ is an increasing function. Remember that $J'(0) \neq 0$ and $\langle J, J \rangle' (0) = 0$. Then around $t = 0$ we have

$$\langle J, J \rangle (t) > \langle J, J \rangle (0) = 0$$

Thus $\gamma(t)$ is not conjugate to $\gamma(0)$ along γ which implies that exp_p must be a local diffeomorphism by the inverse function theorem. □

Before we can state and proof the Hadamard theorem we need one more and in particular powerful lemma. Note that this lemma comes in handy in many situations aside from this talk.

Lemma 9. *Let M be a complete riemannian manifold, N a riemannian manifold and $f : M \longrightarrow N$ be a local diffeomorphism. If for all $p \in M,\ v \in T_pM$ we have $|df_p(v)| \geq |v|$ then f is a covering map.*

Proof. Since we already have a local diffeomorphism we only need to check that f has the path-lifting property, i.e. given a curve c in N and a point in $q \in M$ there exists a curve $\tilde{c}$ in M with $\tilde{c}(0) = q$ and $f \circ \tilde{c} = c$. Consider

$$A = \{t \in [0,1] \mid \exists \tilde{c} : [0,t] \longrightarrow M\}$$

Since f is a local diffeomorphism A is clearly open. Now let t_0 be the supremum of A and $(t_n)_n$ be an increasing series converging to t_0. Then $\{\tilde{c}(t_n)\}$ need to be contained in a compact set due to M being complete. Otherwise the distance from $\tilde{c}(t_n)$ and $\tilde{c}(0)$ would be arbitraily large but by assumption on $|df|$ we have

$$l(c|_{[0,t_n]}) = \int_0^{t_n} \left|\frac{dc}{dt}\right| dt = \int_0^{t_n} \left| df_{\tilde{c}(t)} \left(\frac{d\tilde{c}}{dt}\right)\right| dt \geq \int_0^{t_n} \left|\frac{d\tilde{c}}{dt}\right| dt \geq d(\tilde{c}(t_n), \tilde{c}(0))$$

this implies that $l(c|_{[0,t_0]})$ is arbitrarily large which is absurd.

Now $\{\tilde{c}(t_n)\}$ has an accumulation point $r \in M$. Let V be a neighborhood of r such that f is a diffeomorphism on V. Thus $c(t_0) \in V$ and therefor exists an interval $I \subset [0,1]$ such that $c(I) \subset f(V)$ and $t_0 \in I$. Choose n such that $\tilde{c}(t_n) \in V$ and consider the lift g on I that passes through r (remember that f is a diffeo on V). Because $g = \tilde{c}$ on $[0,t_n) \cap I$ g is an extension of $\tilde{c}$ and $t_0 \in A$. Hence A is closed and therefor $[0,1]$. $\square$

Now we can state and proof the theorem of Hadamard

Theorem 2. *Let M be a simply connected, complete manifold with sectional curvature $K(p,\sigma) \leq 0$ for all $p \in M$ and $\sigma \in T_pM$. Then M is diffeomorphic to $\mathbb{R}^n$ where $n = dim(M)$. More precisely* $\exp_p$ *is a diffeomorphism.*

Proof. This proof is simple now.

Since M is complete $\exp_p$ is a local diffeomorphism for all $p \in M$ and especially surjective. Introduce a riemannian metric on T_pM such that $\exp_p$ is a local isometry (the pullback). Because $\exp_p$ sends lines $t \mapsto tv$ to geodesics we find that the geodesics in T_pM are straight lines. Consider

$$\exp_0 : T_0T_pM \longrightarrow T_pM$$

We see that $\exp_0$ is the identity (while $T_0T_pM \cong T_pM$) and is hence defined everywhere. By the Hopf-Rinow theorem T_pM is a complete manifold.

Since $\exp_p$ must now be a covering and M simply connected the proof is finished. $\square$

References

[dC92] Manfredo Perdigao do Carmo. *Riemannian Geometry.* Birkhäuser, Boston, 1992.