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University of Heidelberg

Geometry of Lie Groups

The quotient manifold theorem

by
Stephan Schmitt

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A quick reminder of Distributions and Foliations

We start with reminding about the theory of distributions and foliations on manifolds, since we will need those notions in the main proof.

Definition 1.1. Let M be a smooth manifold. A Distribution on M of rank k is a rank-k subbundle of the tangent bundle TM. Often a rank-k distribution is described by specifying for each $p \in M$ a linear subspace $D_p \subset T_pM$ of dimension k, and letting $D = \bigcup_{p \in M} D_p$. It then follows from the local frame criterion for subbundles that D is a smooth distribution if and only if each point of M has a neighborhood U on which there are smooth vector fields $X_1, \ldots, X_k : U \to TM$ such that $X_1|_q, \ldots, X_k|_q$ form a basis for D_q at each $q \in U$. In this case, we say that D is the distribution spanned by the vector fields $X_1, \ldots, X_k$

Definition 1.2. Suppose $D \subset TM$ is a smooth distribution. A nonempty immersed submanifold $N \subset M$ is called an integral manifold of D if $T_pN = D_p$ at each point $p \in N$.

Definition 1.3. We say a smooth distribution D is involutive if given any pair of smooth sections of D, their Lie bracket is also a local section of D.

Definition 1.4. Let M be a smooth n-manifold and let $\mathcal{F}$ be any collection of k-dimensional submanifolds of M. A smooth chart (U, ϕ) for M is said to be flat for $\mathcal{F}$ if $\phi(U)$ is a cube in $\mathbb{R}^n$, and each submanifold in $\mathcal{F}$ intersects U in either the empty set or a countable union of k-dimensional slices of the form $x^{k+1} = c^{k+1}, \ldots, x^n = c^n$. We define a foliation of dimension k on M to be a collection $\mathcal{F}$ of disjoint, connected, nonempty, immersed k-dimensional submanifolds of M (called the leaves of the foliation), whose union is M, and such that in a neighborhood of each point $p \in M$ there exists a flat chart for $\mathcal{F}$.

We also need one specific proposition:

Proposition 1.5. *Let D be an involutive distribution on a smooth manifold M and let N be a connected integral manifold. If N is a closed subset of M, then it is a maximal connected integral manifold and is therefore a leaf of the foliation determined by D.*

Definitions and necessary results

In the following we denote with M a smooth manifold of dimension n. Further suppose we have a group G on M. We will write $\theta : G \times M \to M$ or $(g, p) \mapsto g \cdot p$. We will further assume that G acts on the left, allthough a similar argument can be made if G would act on the right.

Next we look at the orbit of a point $p \in M$, $G \cdot p = \{g \cdot p : g \in G\}$. We define M/G to be the set of orbits, which is obtained by defining an equivalence relation $p\ q$ if $\exists g \in G : g \cdot p = q$ and looking at the equivalence classes of that relation. This brings us to our first important lemma

Lemma 2.1. *For any contninuous action of a topological group G on a topological space M, the quotient map $\pi : M \to M/G$ is open.*

Proof. Define $g \cdot U \subset M$ for $g \in G$, $U \subset M$ by

$$g \cdot U = \{g \cdot x : x \in U\}$$

. Now if $U \subset M$ is open, $\pi^{-1}(\pi(U))$ is equal to the union of all sets of the form $g \cdot U$ as g ranges over G. Further they are open sets since $p \mapsto g \cdot p$ is a homeomorphism, i.e. $\pi^{-1}(\pi(U))$ is also open in M. Because π is the quotient map, the same is valid for $\pi(U)$ in M/G and therefore π is open. $\square$

The next step in preparing for the quotient manifold theorem involves proper group actions, i.e. actions in which the map $G \times M \to M \times M$ given by $(g,p) \mapsto (g \cdot p, p)$ is proper. We have the following result for proper actions.

Proposition 2.2. *If a Lie group acts continuously and properly on a manifold, then the orbit space is Hausdorff.*

Proof. Take a Lie group G acting continuously and properly on a manifold M. Let $\Theta : G \times M \to M \times M$ be the associated proper map, given by $\Theta(g,p) = (g \cdot p, p)$. We define the orbit relation $\mathcal{O} \subset M \times M$ by

$$\mathcal{O} = \Theta(M \times M) = \{(g \cdot p, p) \in M \times M \ : \ p \in M, g \in G\}$$

. Since proper continuous maps are closed $\mathcal{O}$ is a closed in $M \times M$. Our Lemma gives us the last piece, since the quotient map π is open a basic result in topology guarantees M/G is Hausdorff. □

Example 2.3. As the previous proposition implies, quotient spaces of smooth manifold are not necessarily Hausdorff. Take an irrational number α and let $\mathbb{R}$ act on $\mathbb{S}^1 \times \mathbb{S}^1$ by

$$t \cdot (w, z) = (\exp(2\pi i t)w, \exp(2\pi i \alpha t)z)$$

. This action is clearly smooth and it can be shown that it is free and produces dense orbits. This in turn implies the only smooth subsets of $\mathbb{S}^1 \times \mathbb{S}^1$ are the empty set and the set itself, in other words the orbit space has a trivial topology and is therefore not Hausdorff.

As can be seen from the definition, seeing that a particular action is proper is rather difficult, to solve this problem, we have the following proposition:

Proposition 2.4. *Let G be a Lie group acting continuously on M. Then the following are equivalent.*

- *The action is proper.*
- *If (p_i) is a sequence in M and (g_i) a sequence in G such that both (p_i) and $(g_i \cdot p_i)$ converge, then a subsequence of (g_i) converges.*
- *For every compact subset $K \in M$, the set $G_K = \{g \in G \ : \ (g \cdot K) \cap K \neq \varnothing\}$ is compact.*

Proof. Lee. □

Corollary 2.5. *Every continuous action by a compact Lie group on a manifold is proper.*

Proof. If (p_i) and (g_i) satisfy the previous hypotheses, then a subsequence of (g_i) converges, for the simple reason that every sequence in G has a convergent subsequence. □

This lets us classify the orbits of proper actions as follows.

Proposition 2.6. *Suppose θ is a proper smooth action of a Lie group G on a smooth manifold M. For $p \in M$, the orbit map $\theta^{(p)} : G \to M$ is a proper map and thus the orbit $G \cdot p = \theta^{(p)}(G)$ is closed in M. If in addition $G_p = \{e\}$, then $\theta^{(p)}$ is a smooth embedding and the orbit is a properly embedded submanifold.*

Proof. Assume $K \subset M$ is compact, then $(\theta^{(p)})^{-1}(K)$ is closed in G by continuity, further since it is contained in $G_{K \cup \{p\}}$, it is compact by the previous proposition. Therefore $\theta^{(p)}$ is proper and therefore closed. The statement follows from the properties of the quotient map and the characterizations of embeddings. □

This gives us another necessary condition for proper actions.

Corollary 2.7. *If a Lie group G acts properly on a manifold M, then each orbit is a closed subset of M, and each isotropy group is compact.*

Proof. The first statement follows immediately from the previous proposition. The second statement follows if we use the fact that the isotropy group of a point $p \in M$ is the set G_K for $K = \{p\}$ and an application of the characterization of proper actions we showed earlier. $\square$

Example 2.8. Let $\mathbb{R}^+$ be a group acting on $\mathbb{R}^n$ by

$$t \cdot (x^1, \dots, x^n) = (tx^i, \dots, tx^n).$$

We now have to ways to see this action is not proper: the isotropy group of the origin is all of $\mathbb{R}^+$, which is obviously not compact and the orbits of other points are open rays, which are not closed in $\mathbb{R}^n$.

The quotient manifold theorem

We will now prove the quotient manifold theorem, i.e. that smooth, free and proper group actions always lead to smooth manifolds as orbit spaces. The basic idea of the proof involves showing that G induces a foliation on M in terms of its orbits whose leaves are embedded submanifolds diffeomorphic to G. We then construct flat charts for the foliations to get coordinates on the orbit space.

Proposition 3.1. *Suppose G is a Lie group acting smoothly, freely and properly on a smooth manifold M. Then the orbit space M/G is a topological manifold of dimension equal to dim M - dim G and has a unique smooth structure with the property that the quotient map $\pi : M \to M/G$ is a smooth submersion.*

In this section we will use the following notation. Assume, without loss of generality, that G acts on the left. Denote by $\mathfrak{g}$ the Lie algebra of G, $k = \dim G$, $m = \dim M$ $n = m - k$. Let $\theta : G \times M \to M$ denote the action and $\Theta : G \times M \to M \times M$ the proper map $\Theta(g, p) = (g \cdot p, p)$.

Proof. The first step is comparatively easy, we start with the uniqueness of the smooth structure. Suppose M/G has two different smooth structures such that $\pi : M \to M/G$ is a smooth submersion, denote the different structures with $(M/G)_1$ and $(M/G)_2$ respectively. The identity map is smooth from $(M/G)_1$ to $(M/G)_2$ and of course, so is it's inverse, which proves that the structures are identical and thereby also uniqueness.

$$\begin{tikzcd} M \arrow[d, "\pi"] \arrow[dr, "\pi"] & \\ (M/G)_1 \arrow[r, "\mathrm{Id}"] & (M/G)_2 \end{tikzcd}$$

The next step is to show that M/G is a topological manifold and find a smooth structure for it. We say that a smooth chart (U, ϕ) for M is adapted to the G-action if it is a cubical chart with coordinate functions $(x, y) = (x^1, \ldots, x^k, y^1, \ldots, y^n)$, such that each G-orbit intersects U either in the empty set or in a single slice of the form $(y^1, \ldots, y^n) = (c^1, \ldots, c^n)$. The most important part of this proof is the claim that for each $p \in M$ there exists an adapted chart centered at p.

To prove the claim, note first that the G-orbits are properly embedded submanifolds of M diffeomorphic to G by our characterization of orbits. In fact, we will show that the orbits are integral manifolds of a smooth distribution on M.

Define a $D \subset TM$ by

$$D = \bigcup_{p \in M} D_p \qquad \text{where } D_p = T_p(G \cdot p).$$

Because every point is contained in exactly one orbit, and the orbits are submanifolds of dimension k, each D_p has dimension k. To see that D is a smooth distribution, for each $X \in \mathfrak{g}$ let $\hat{(}X)$ be the vector field on M defined by the infinitesimal generator of the flow $(t, p) \mapsto (\exp tX) \cdot p$. If $(X_1, \ldots, X_k)$ is a basis for $\mathfrak{g}$ then $(\hat{X}_1, \ldots, \hat{X}_k$ is a global frame for D, so D is smooth. Because the $G-$orbits are closed, we have that each connected component of an orbit is a leaf of the foliations determined by D, by the proposition about foliations.

Let $p \in M$ be arbitrary and let (U, ϕ) be a smooth chart for M centered at p that is flat for D with coordinate fucntions $(x, y) = (x^1, \ldots, x^k, y^1, \ldots, y^n)$, so each G-orbit intersects U either in the empty set or in a countable union of constant slices. It remains to show that we can find a cubical subset $U_0 \subset U$ centered at p and intersecting each G-orbit in at most a single slice, to prove the claim.

We start by assuming there is no such subset U_0. For each positive inter i, let U_i be the cubical subset of U consisting of points whose coordinates are all less than $1/i$ in absolute value. Let Y be the n-dimensional submanifold of M consisting of points in U whose coordinate representations are of the form $(0, y)$ and for each i let $Y_i = U_i \cap Y$. Since each k-slice U_i intersects Y_i in exactly one point, our assumption implies, we have distinct points $p_i, p_i' \in Y_i$ for each i, which fulfil $g_i \cdot p_i = p_i'$ for some $g_i \in G$, i.e. they are in the same orbit. By our choice of $\{Y_i\}$, both sequences (p_i) and $(p_i' = g_i \cdot p_i)$ converge to p. Because we required G to act properly, as proven in the earlier chapter, we may pass to a subsequence and assume that $g_i \to g \in G$.

$$g \cdot p = \lim_{i \to \infty} g_i \cdot p_i = \lim_{i \to \infty} p_i' = p$$

follows from continuity and since G acts freely, this implies $g = e$.

Now let $\theta^Y : G \times Y \to M$ be the restriction of the G-action to $G \times Y$. Note that $G \times Y$ and M both have dimension $k + n = m$. The restriction of θ^Y to $\{e\} \times Y$ is just the inclusion map $Y \to M$, and $T_pM = T_p(G \cdot p) \oplus T_pY$, it follows that $d(\theta^Y)_{(e,p)}$ is an isomorphism. Thus, there is a neighborhood W of (e, p) in $G \times Y$ such that $\theta^Y|_W$ is a diffeomorphism onto its image and hence injective. This however is contradicting the fact that $\theta^Y(g_i, p_i) = p_i' = \theta^Y(e, p_i')$ as soon as i is large enough that (g_i, p_i) and (e, p_i') are in W, because we are assuming $p_i \neq p_i'$.

We are left to show that M/G is indeed a smooth manifold, first we take care of the topological part. We have that M/G is Hausdorff by the previous chapter. Take $\{B_i\}$ as a countable basis for the topology of M, then $\{\pi(B_i)\}$ is a countable collection of open subsets of M/G, since π is open. By the properties of the quotient topology it follows that it is a basis for the quotient topology on M/G and therefore M/G is second-countable.

Let $q = \pi(p)$ be an arbitray point of M/G, and let (U, ϕ) be an adapted char for M centered at p, with $\phi(U)$ equal to an open cube in $\mathbb{R}^k \times \mathbb{R}^n$, which we write as $\phi(U) = U' \times U''$, where U' and U'' are open cubes in $\mathbb{R}^k$ and $\mathbb{R}^n$ respectively. Define $V = \pi(U)$, since π is open, V is also open. Take further coordinate functions of ϕ and denote them by $(x^1, \ldots, x^k, y^1, \ldots, y^n)$ as before, let $Y \subset U$ be the submanifold $\{x^1 = \cdots = x^k = 0\}$. Looking at $\pi : Y \to V$ we have an bijective map, by the definition of an adapted chart. We can say even more, take $W \subset Y$ open, then

$$\pi(W) = \pi(\{(x, y) : (0, y) \in W\}),$$

which is open in M/G and thus $\pi|_Y : Y \to V$ is a homeomorphism. Now let $\sigma : V \to Y \subset U$ be a local section of π, i.e. $\sigma = (\pi|_Y)^{-1}$.

Finally define $\eta : V \to U''$ by sending the equivalence class of a point (x, y) to y; this is well defined, again by definition of an adapted chart. More formally, we have $\eta = \pi'' \circ \phi \circ \sigma$, where $\pi'' : U' \times U'' \to U'' \subset \mathbb{R}^n$ is the projection onto the second factor. Because σ is a homeomorphism from V to Y and $\pi'' \circ \phi$ is a homeomorphism from Y to U'', it follows that η is a homeomorphism. This shows that M/G is locally Euclidean, and thus completes the proof that we have indeed a topological manifold of dimension n.

The last claim in the theorem involves us showing that M/G has a smooth structure such that π is a smooth submersion. We use the atlas consisting of all charts (V, η) constructed in the last step. We have the coordinate representation $\pi(x, y) = y$ for π with respect to any such chart for M/G and the corresponding adapted chart for M. The projection of the second argument is certainly a smooth submersion, so it remains to be seen that any two such charts are indeed smoothly compatible.

Take (U, ϕ) and $(\tilde{U}, \tilde{\phi})$ two adapted charts for M and denote their corresponding charts for M/G as (V, η) and $(\tilde{V}, \tilde{\eta})$. We first consider the case where both adapted charts are centered at the same point in M. We write the adopted coordinates as (x, y) and $(\tilde{x}, \tilde{y})$. Two points with the same y-coordinate are in the same orbit, since our coordinates are adapted to the G-action, but this also means that they have the same $\tilde{y}$-coordinate. This means we can write the transition map between these coordinates as $(\tilde{x}, \tilde{y}) = (A(x, y), B(y))$, where A and B are smooth maps defined on some neighborhood of the origin. The transition map $\tilde{\eta} \circ \eta^{-1}$ is just $\tilde{y} = B(y)$, which is clearly smooth. Now assume they are not centered at the same point, instead assume that for $p \in U, \tilde{p} \in \tilde{U}$ we have adapted charts (U, ϕ)

and $(\tilde{U}, \tilde{\phi})$ such that $\pi(p) = \pi(\tilde{p})$. We can add constant vectors to modify both charts in a such a way, that they are centered at p and $\tilde{p}$. Since p and $\tilde{p}$ are in the same orbit, there is an element $g \in G$ such that $g \cdot p = \tilde{p}$. Because $\theta_p : M \to M$ is a diffeomorphism taking orbits to orbits, we have another chart centered at p, namely $\tilde{\phi}' = \tilde{\phi} \circ \theta_g$. Moreover, $\tilde{\sigma}' = \theta_g^{-1} \circ \tilde{\sigma}$ is the local scetion corresponding to $\tilde{\phi}'$ and therefore $\tilde{\eta}' = \pi'' \circ \tilde{\phi}' \circ \tilde{\sigma}' = \pi'' \circ \tilde{\phi} \circ \theta_g \circ \theta_g^{-1} \circ \tilde{\sigma} = \pi'' \circ \tilde{\phi} \circ \tilde{\sigma} = \tilde{\eta}$. The we are in the same situation as before and the two charts are smoothly compatible.

□

The quotient manifold theorem is very powerful, however we can say even more. First a quick reminder:

Definition 3.2. A fiber bundle with two topological spaces M and F, over M with model fiber F, is a topological space E together with a surjective continuous map $\pi : E \to M$ with the property that for each $x \in M$, there is a neighborhood U of x in M coupled with a homeomorphism $\Phi : \pi^{-1}(U) \to U \times F$, called a local trivialization of E over U, such that the following diagram commutes:

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\Phi"] \arrow[dr, "\pi"'] & & U \times F \arrow[dl, "\pi_1"] \\ & U & \end{tikzcd}$$

We call E the total space of the bundle, M is it's base and π is its projection.

Corollary 3.3. *With the perquisites of the previous theorem, the manifold M is the total space of a smooth fibre bundle with base M/G, model fiber G and projection $\pi : M \to M/G$.*

Proof. Rough sketch:

We have to check the local trivialization property, to do this show that for any smooth local section $\sigma : U \to M$ of π, the map $(g, x) \to g \cdot \sigma(x)$ is a diffeomorphism from $G \times X$ to $\pi^{-1}(U)$. The claim then follows. See also [2]. □

Covering Manifolds

After proving our main theorem, we now move on to give a small example of it's application. We will investigate what we can say about our quotient manifold, if the corresponding Lie group is discrete. First recall the following two facts:

Proposition 4.1. *The covering space of a smooth manifold is itself a smooth manifold.*

Proposition 4.2. *For any smooth covering $\pi : E \to M$ the automorphism group is a discrete Lie group acting smoothly and freely on the covering space E.*

Proof. For proofs, see [Lee]. □

The first of the preceding proposition answers the question what the covering space of a smooth manifold looks like, but it is also often interesting to know what kind of space is covered by a smooth manifold, a corner stone in investigating this, is knowing when an action is proper:

Lemma 4.3. *Suppose a discrete Lie group Γ acts continuously and freely on a manifold E. The action is proper if and only if the following conditions both hold:*

- *Every point $p \in E$ has a neighborhood U such that for each $g \in \Gamma$, $(g \cdot U) \cap U = \varnothing$ unless g=e.*
- *If $p, p' \in E$ are not in the same Γ-orbit, there exist neighborhoods V of p and V' of p' such that $(g \cdot V) \cap V' = \varnothing$ $\forall g \in \Gamma$*

Proof. Suppose that the action is free and proper, let $\pi : E \to E/\Gamma$ denote the quotient map. We know that E/Γ is Hausdorff, based on the previous chapter. If $p, p' \in E$ are in different orbits, we choose disjoint neighborhoods W of $\pi(p)$ and W' of p', and then $V = \pi^{-1}(W)$ and $V' = \pi^{-1}(W')$ satisfy the conclusion of the second condition. To prove the first condition, take $p \in E$ and define V to be a precompact neighborhood of p. By our first characterization of proper actions in the previous section the set $\Gamma_{\overline{V}}$ is a compact subset of Γ and hence finite because Γ is discrete. Writing $\Gamma_{\overline{v}} = \{e, g_1, \ldots, g_m\}$. Shrinking V is necessary, we may assume that $g_i^{-1} \cdot p \notin \overline{V}$ (implying $p \notin g_i \cdot \overline{V}$) for $i = 1, \ldots, m$. Then the open subset

$$U = V \setminus (g_1 \cdot \overline{V} \cup \cdots \cup g_m \cdot \overline{V})$$

satisfies the conclusion of the first part.

Conversely assume that both of the points hold. Take a sequence (g_i) in Γ and a sequence (p_i) in E, such that $p_i \to p$ and $g_i \cdot p_i \to p'$. If p and p' are in different orbits, there are again disjoint neighborhoods V, V' as before; however for large enough i, we have $p_i \in V$ and $g_i \cdot p_i \in V'$, which contradicts the fact that $(g_i \cdot V) \cap V' = \varnothing$. Thus, p and p' are in the same orbit, so there exists $g \in \Gamma$ such that $g \cdot p = p'$. If that's the case we have $g^{-1} g_i \cdot p_i \to p$. Choose a neighborhood U of p as before and let i be large enough that p_i and $g^{-1} g_i \cdot p_i$ are both in U. Because $(g^{-1} g_i \cdot U) \cap U \neq \varnothing$, it follows that $g^{-1} g_i = e$. So $g_i = g$ when i is large enough, which certainly converges. By our first characterization, properness follows. □

A continuous discrete group action satisfying the first condition is also called properly discontinuous.

Proposition 4.4. *Let M be a smooth manifold, $\pi : E \to M$ be a smooth covering map. With the discrete topology, the automorphism group $Aut_\pi(E)$ acts smoothly, freely and properly on E.*

Proof. We already know that the action is smooth and free, we will now improve the theorem and show that it is also proper using our preceeding lemma. Taking $e \in E$ arbitrarily, choose $W \subset M$ to be an evenly covered neighborhood of $\pi(p)$. If U is the component of $\pi^{-1}(W)$ containing p, then it is easy to check that U satisfies the first condition. Now take p, p' with different orbits, then just as in the preceeding proof, we can choose disjoint neighborhoods W, W' of $\pi(p), \pi(p')$ respectively. It follows then that $V = \pi^{-1}(W)$ and $V' = \pi^{-1}(W')$ satisfy the second condition. □

With that we are able to answer the question we posed at the begining of the section and use the quotient manifold theorem to prove a sort of inverse of the preceeding proposition.

Proposition 4.5. *Suppose E is a connected smooth manifold and Γ is a discrete Lie group acting continuously, freely and properly on E. Then the orbit space E/Γ has a unique smooth structure such that $\pi : E \to E/\Gamma$ is a smooth normal covering map.*

Proof. It follows from the quotient manifold theorem that E/Γ has a unique smooth manifold structure such that π is a smooth submersion.Because a smooth covering map is in particular a smooth submersion, any other smooth manifold structure on E making π into a smooth covering map must be equal to this one. We also get that π is a local diffeomorphism since $\dim E/\Gamma = \dim E - \dim \Gamma = \dim E$. It remains to show that π is a normal covering map.

To see this take $p \in E$, by our preceding lemma, p has a neighborhood U in E, that satisfies

$$(g \cdot U) \cap U = \varnothing,$$

as long as $g \in \Gamma$ and $g \neq e$.

We can assume U is connected, since otherwise we just shrink it. Defining $V = \pi(U)$, which is open due to our lemma. Further we can reduce the problem to showing that π is a homeomorphism from V to $\pi^{-1}(V)$, since $\phi^{-1}(V)$ is a disjoint union of open subsets $g\dot{U}$, $g \in \Gamma$. We have the following commuting diagram

$$\begin{tikzcd} U \arrow[rr, "g"] \arrow[dr, "\pi"'] & & g \cdot U \arrow[dl, "\pi"] \\ & V & \end{tikzcd}$$

Since $g : U \to g\dot{U}$ is a homeomorphism, it suffices to show that $\pi : U \to V$ is a homeomorphism. We already know that it is surjective, continuous and open. To see injectivity, assume $\pi(q) = \pi(q')$ for $q, q' \in U$, which means that $q = g \cdot q$ for some $g \in \Gamma$. But as we said before this can only happen if $g = e$, i.e. $q = q'$. Therefore π is a smooth covering map, it is further normal since Γ acts transitively on fibers by definition and its elements act as automorphism of π. ☐

References

[1] John M. Lee, *Introduction to Smooth Manifolds*, Graduate Texts in Mathematics(Springer), 2013.

[2] J.J. Duistermaat, J.A.C. Kolk, *Lie Groups*, Springer, 1999.