Wintersemester 2015/2016

University of Heidelberg

Geometric Structures on Manifolds

Geometric Manifolds

by

Stephan Schmitt

Contents

Introduction, first Definitions and Results

Manifolds - The Group way

The keystone of working mathematically in Differential Geometry, is the basic notion of a Manifold, when we usually talk about Manifolds we mean a Topological Space that, at least locally, looks just like Euclidean Space. The usual formalization of that Concept is well known, we take charts to 'map out' the Manifold, in this paper, for sake of Convenience we will take a slightly different approach to formalize the Concept of 'locally euclidean', to formulate it, we need some tools, let us introduce them now:

Definition 1.1. Pseudogroups
A pseudogroup on a topological space X is a set $\mathcal{G}$ of homeomorphisms between open sets of X satisfying the following conditions:

- The Domains of the elements $g \in \mathcal{G}$ cover X
- The restriction of an element $g \in \mathcal{G}$ to any open set contained in its Domain is also in $\mathcal{G}$.
- The Composition $g_1 \circ g_2$ of two elements of $\mathcal{G}$, when defined, is in $\mathcal{G}$
- The inverse of an Element of $\mathcal{G}$ is in $\mathcal{G}$.
- The property of being in $\mathcal{G}$ is local, that is, if $g : U \to V$ is a homeomorphism between open sets of X and U is covered by open sets U_α such that each restriction $g|_{U_\alpha}$ is in $\mathcal{G}$, then $g \in \mathcal{G}$

Definition 1.2. $\mathcal{G}$-Manifolds
Let $\mathcal{G}$ be a pseudogroup on $\mathbb{R}^n$. An n-dimensional $\mathcal{G}$-manifold is a Hausdorff space with countable basis M with a $\mathcal{G}$-atlas on it. A $\mathcal{G}$ -atlas is a collection of $\mathcal{G}$-compatible coordinate charts whose domain cover M. A coordinate chart, or local coordinate System is a pair (U_i, ϕ_i), where U_i is open in M and $\phi_i : U_i \to \mathbb{R}^n$ is a homeomorphism onto its image. Compatibility meaning, whenever two charts (U_i, ϕ_i) and (U_j, ϕ_j) intersect, the transition map, or coordinate change

$$\gamma_{i,j} = \phi_i \circ \phi_j^{-1} : (U_i \cap U_j) \to \phi_i(U_i \cap U_j)$$

is in $\mathcal{G}$.

Definition 1.3. Stiffening
If $\mathcal{H} \subset \mathcal{G}$ are pseudogroups, an $\mathcal{H}$-Atlas is automatically also an $\mathcal{G}$-atlas, the $\mathcal{H}$-structure is called a $\mathcal{H}$-stiffening of the $\mathcal{G}$-structure

Lemma 1.4. *Given a set $\mathcal{G}_0$ of homeomorphisms between open subsets of X, there is a unique minimal pseudogroup $\mathcal{G}$ on X, that contains $\mathcal{G}_0$, we say that $\mathcal{G}$ is generated by $\mathcal{G}_0$.*

As you might have noticed, this formulation of a Manifold is not far removed from the usual Definition. The Definition via pseudogroup has some useful properties however. Before we move on, we give 2 well known examples of Manifolds with the above definition.

Example 1.5. Differentiable Manifolds
If $\mathcal{C}^r$, for $r \geq 1$, is the pseudogroup of $\mathcal{C}^r$ diffeomorphisms between open sets of $\mathbb{R}^n$ a $\mathcal{C}^r$-manifold is called a differentiable manifold(of class $\mathcal{C}^r$). A $\mathcal{C}^r$-isomorphism is called diffeomorphism. $\mathcal{C}^\infty$ manifolds are also called smooth manifolds.

Example 1.6. Analytic Manifolds
Let $\mathcal{C}^\omega$ be the pseudogroup of real analytic diffeomorphisms between open subsets of $\mathbb{R}^n$. A $\mathcal{C}^\omega$-manifold is called a real analytic manifold. Real analytic diffeomorphisms are uniquely determinded by their restriction to any open set; this will be essential in the study of the developing map. It is a deep theorem that every smooth manifold admits a unique real analytic stiffening.

Geometric Structures

We can now finally move on to define what we are looking for, Geometric structures on Manifolds:

Definition 1.7. It is convenient to slightly broaden the Definition of a $\mathcal{G}$-Manifold, by allowing $\mathcal{G}$ to be a pseudogroup on any connected Manifold X, not just $\mathbb{R}^n$. Note that, as long as $\mathcal{G}$ acts transitively, this does not give any new type of Manifold.

Many important pseudogroups come from group actions on Manifolds. Given a Group G acting on a Manifold X, let $\mathcal{G}$ be the pseudogroup generated by restrictions of Elements of G. Thus every $g \in \mathcal{G}$ agrees locally with elements of G: the Domain of G can be covered with open sets U_α such that $g|_{U_\alpha} = g_\alpha|_{U_\alpha}$ for $g_\alpha \in G$. A $\mathcal{G}$-manifold is also called a (G,X)-manifold.

We will now give some examples before moving on to the Develpoing Map.

Example 1.8. Euclidean Manifolds

If G is the group of isometries of Euclidean Space $\mathbb{E}^n$, a $(G, \mathbb{E}^n)$-manifold is called Euclidean, or flat, manifold. The only compact two-dimensional manifolds that can be given Euclidean structures are the torus and the Klein bottle, but they have many such structures. We will discuss the euclidean torus further in the next section as an introduction for the Developing map.

Example 1.9. Hyperbolic Manifolds

If G is the group of isometries of hyperbolic space $\mathbb{H}^n$, a $(G, \mathbb{H}^n)$-manifold is a hyperbolic manifold. A 3 dimensional example would be the Seifert-Weber dodecahedral space, which will we discuss more below.

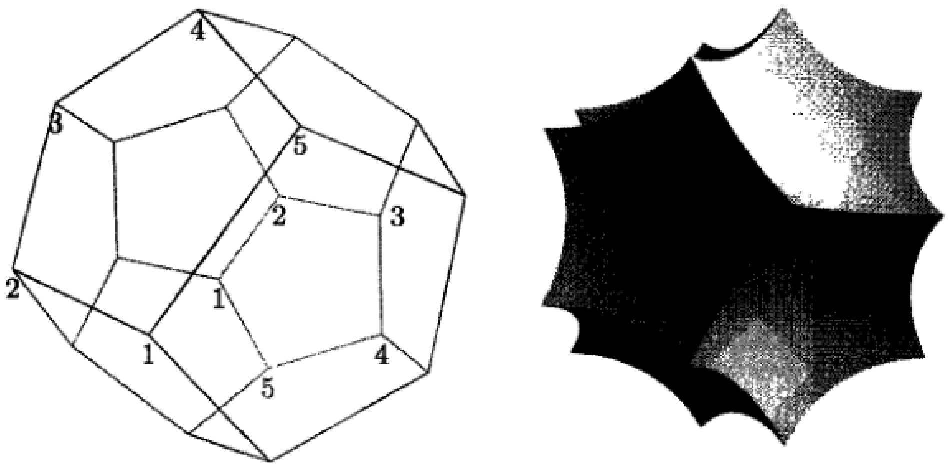


Figure 1.22. The Seifert–Weber dodecahedral space. If opposite faces of a dodecahedron are glued by three-tenths of a clockwise revolution, the edges are glued in quintuples, and the resulting space is the Seifert–Weber dodecahedral space. The gluing can be realized geometrically if we use a hyperbolic dodecahedron with dihedral angles of 72°—the solid shown on the right, in the Poincaré ball model.

Example 1.10. Seifert-Weber dodecahedral space

If the opposite faces of a dodecahedral are glued together using clockwise twists by three-tenths of a revolution(see Picture), a bit of chasing around the diagram shows that edges are identified in six groups of five. All twenty vertices are glued together, and small spherical triangles around the vertices(obtained by intersection of the dodecahedron with tiny spheres) are arranged in the pattern of a regular icosahedron after gluing. The resulting space is a manifold known as the Seifert-Weber dodecahedral space.

Note that the angles of a Euclidean dodecahedron are much larger than the 72° angles needed to do the gluing geometrically. In this case, we can use the three-dimensional hyperbolic space $\mathbb{H}^3$, which can be mapped into the interior of a three-dimensional ball, just as in two dimensions. This description in the ball is the well known Poincare ball model of Hyperbolic space.

The Developing Map and Completeness

An introductory discussion of the torus

We now come to the really interesting part of this discussion, the Developing Map. We start with a discussion of the torus, probably the simplest surface next to the sphere. It is well known that we can describe a torus by gluing a square, in a way where we glue the opposite sites, see also the figure below.

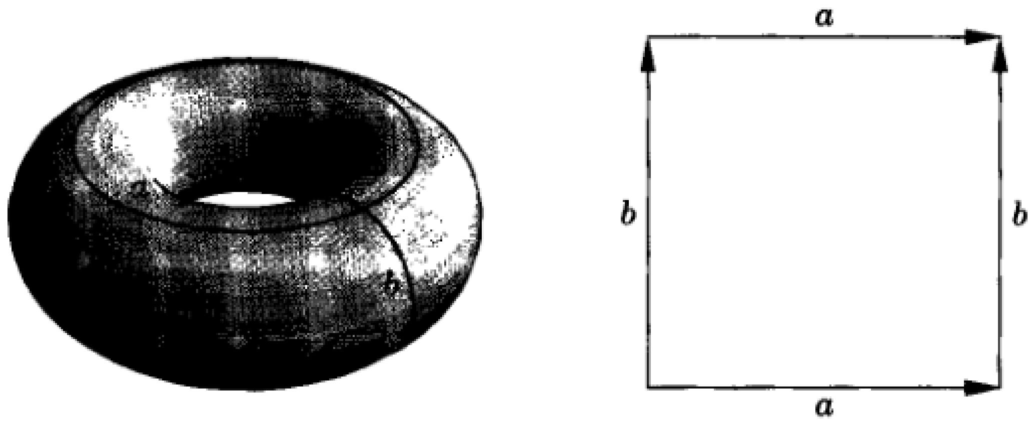


Figure 1.1. The square torus. A torus can be obtained, topologically, by gluing together parallel sides of a square. Conversely, if you cut the torus on the left along the two curves indicated, you can unroll the resulting figure into the square on the right.

Perhaps surprising, we can also construct a torus in a different way of gluing: Take a regular hexagon, again identify the parallel sides.

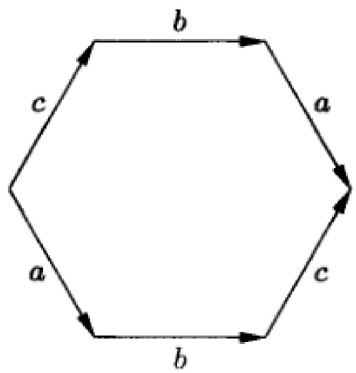


Figure 1.2. The hexagonal torus. Here is another gluing pattern of a polygon which yields a torus. This pattern reveals a different kind of symmetry from the first.

Now, this is curious, if we look at the completed, glued structure, both seem to look the same, however, the 'hexagon-torus' has a sixfold symmetry which obviously is not compatible to that of the 'square-torus'. Now, the different description of the torus are closely related to common tilings of the Euclidean Plane $\mathbb{E}^2$. How do we do this? Take a collection of infinite squares or hexagon, labeled as before. Start with a single on of those, and keep adding layers and layers of the corresponding polygon, every time identifying the edges of the new polygon, with the correspondingly labeled edge of the old ones. Making sure that the local picture near each vertex looks like the local picture in the original pattern, when the edges of a single polygon were identified, we find that each new tile fits in exactly one way. We end up with a tiling of the Euclidean plane by congruent squares and hexagons.

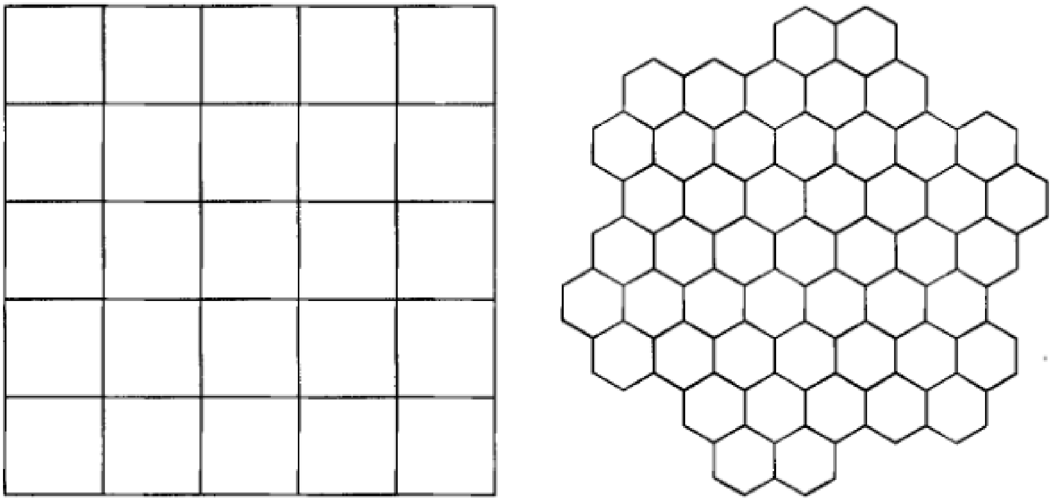


Figure 1.3. Tiling the plane with tori. These tilings of the plane arise from the two descriptions of the torus by gluing polygons. They show the universal covering space of the torus, obtained by "unrolling" the torus.

We find another thing, this tiling of Euclidean space shows that the Euclidean Space is a covering space for the 'hexagon-' and 'square-torus'. The covering map for the square tiling is the map that identifies corresponding points in each square, taking them all to the same point on the glued-up torus. Even more is true: since $\mathbb{E}^2$ is simply connected, its the Universal Cover of the torus.

Definition of the Developing map

As before with the gluing, we need to formalize the process of 'unrolling' a manifold to get a tiling of a spaces, we discussed in the previous subsection, to make it mathematically really useful. We will now give a proper Definition and generalize the concept. The formalization of tilling is called **developing** a space.

Let X be a connected, real analytic manifold and G a group of real analytic diffeomorphisms acting transitively on X. An element of G is then completely determined by its restriction to any open subset of X. We will now look at a (G,X)-Manifold. Because of the importance of the Developing map, we will treat the definition of it more carefully and rigorous:

Let $\phi : U \to X$ be a chart for an (G,X)-manifold M and let $\alpha : [a, b] \to M$ be a curve whose initial point $\alpha(a)$ is in U. Then there is a partition

$$a = x_0 < x_1 < \cdots < x_m = b$$

and a set $\{\phi_i : U_i \to X\}_{i=1}^m$ of charts for M such that $\phi_1 = \phi$ and U_i contains $\alpha([x_{i-1}, x_i])$ for each $i = 1, \dots, m$. Let g_i be the element of G that agrees with $\phi_i\phi_{i+1}^{-1}$ on the connected component of $\phi_{i+1}(U_i \cap U_{i+1})$ containing $\phi_{i+1}\alpha(x_i)$. Let α_i be the restriction of α to the interval $[x_{i-1}, x_i]$. Then $\phi_i\alpha_i$ and $g_i\phi_{i+1}\alpha_{i+1}$ are curves in X and

$$g_i\phi_{i+1}\alpha(x_i) = \phi_i\phi_{i+1}^{-1}\phi_{i+1}\alpha(x_i) = \phi_i\alpha(x_i).$$

Thus $g_i\phi_{i+1}\alpha_{i+1}$ begins where $\phi\alpha_i$ ends, and so we can define a curve $\hat{\alpha} : [a, b] \to X$ by the formula

$$\hat{\alpha} = (\phi_1\alpha_1)(g_1\phi_2\alpha_2)(g_1g_2\phi_3\alpha_3)\dots(g_1 \dots g_{m-1}\phi_m\alpha_m).$$

Note 2.1. *For convenience, we will refer to $\hat{\alpha}$ simply as α if the context is clear.*

We claim that $\hat{\alpha}$ does not depend on the choice of the charts $\{\phi_i\}$ once a partition of $[a, b]$ has been fixed. Take another set of charts,$\{\theta_i : V_i \to X\}$ for M such that $\theta_1 = \phi$ and $\alpha([x_{i-1}, x_i]) \in V_i \forall i = 1, \dots, m$. Let h_i be the element of G that agrees with $\theta_i\theta_{i+1}^{-1}$ on the component of $\theta_{i+1}(V_i \cap V_{i+1})$ containing $\theta_{i+1}\alpha(x_i)$. As $U_i \cap V_i$ contains $\alpha([x_{i-1}, x_i])$, it is enough to show that

$$g_1 \dots g_{i-1}\phi_i = h_1 \dots h_{i-1}\theta_i$$

on the component of $U_i \cap V_i$ which for each i contains $\alpha([x_{i-1}, x_i])$. This is true by the hypothesis for $i = 1$. We proceed inductively: Suppose that it is true for $i - 1$. Define f_i as the element of G that agrees with $\theta_i\phi_i^{-1}$ on the component of $\phi_i(U_i \cap V_i)$ containing $\phi_i\alpha([x_{i-1}, x_i])$. On the one hand, we have that f_i agrees with

$$\theta_i(\theta_{i-1}^{-1}h_{i-2}^{-1} \dots h_i^{-1})(g_1 \dots g_{i-2}\phi_{i-1})\phi_i^{-1}$$

on the component of $\phi_i(U_{i-1} \cap V_{i-1} \cap U_i \cap V_i)$, which contains $\phi_i\alpha(x_{i-1})$. On the other hand, $(h_{i-1}^{-1} \dots h_1^{-1})(g_1 \dots g_{i-1})$ agrees on the component $\phi_i(U_{i-1} \cap V_{i-1} \cap U_i \cap V_i)$ containing $\phi_i\alpha(x_{i-1})$, with

$$(\theta_i\theta_{i-1}^{-1})(h_{i-2}^{-1} \dots h_1^{-1})(g_1 \dots g_{i-2})(\phi_{i-1}\phi_i^{-1}).$$

Hence

$$f_i = (h_{i-1}^{-1} \dots h_1^{-1})(g_1 \dots g_{i-1})$$

since they agree on a non-empty subset of a rigid metric space(for a more detailed treatment of this step refer to [2]). Therefore

$$\begin{aligned}
(g_1 \dots g_{i-1})\phi_i &= (h_1 \dots h_{i-1})(h_{i-1}^{-1} \dots h_1^{-1})(g_1 \dots g_{i-1})\phi_i \\
&= (h_1 \dots h_{i-1})f_i\phi_i \\
&= (h_1 \dots h_{i-1})\theta_i
\end{aligned}$$

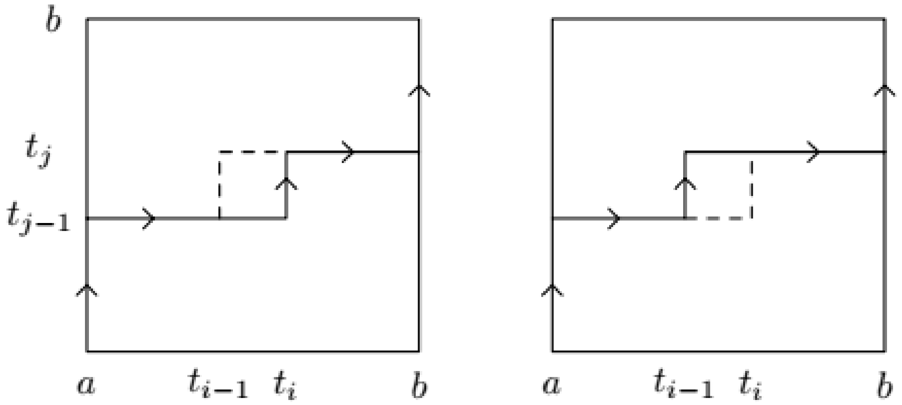
on the component of $U_i \cap V_i$ containing $\alpha([x_{i-1}, x_i])$. Therefore, by induction, the analytic continuation is independent of the choice of charts.

The next step is to show that $\hat{\alpha}$ does not depend on the partition of $[a, b]$. Take $\{s_i\}$ with charts $\{\theta_i : V_i \to X\}$. Then $\{r_i\} = \{s_i\} \cup \{x_i\}$ is a partition of $[a, b]$ containing both partitions. Since the charts ϕ_i and θ_i can both be used in turn for the partition $\{r_i\}$, we can deduce that all three partitions determine the same curve $\hat{\alpha}$. The curve $\hat{\alpha} : [a, b] \to X$ is called the continuation of $\phi\alpha_1$ along α.

We also have the following:

Theorem 2.2. *Let $\phi : U \to X$ be a chart for an (X,G)-manifold M, let $\alpha, \beta : [a, b] \to M$ be curves with the same initial point in U and the same terminal point in M, and let $\hat{\alpha}, \hat{\beta}$ be the continuations of $\phi\alpha_1, \phi\beta_1$ along respectively α, β. If α and β are homotopic by a homotpy that keeps their endpoints fixed, then $\hat{\alpha}$ and $\hat{\beta}$ have the same endpoints, and they are homotopic by a homotopy that keeps their endpoints fixed.*

Proof. This is clear if α and β differ only along a subinterval (c, d) such that $\alpha([c, d])$ and $\beta([c, d])$ are contained in a simply connected coordinate neighborhood in U. In the general case, let $H : [a, b]^2 \to M$ be a homotopy from α to β that keeps the endpoints fixed. As $[a, b]$ is compact, there is a partition $a = x_0 < x_1 \cdots < x_m = b$ such that $H([x_{i-1}, x_i] \times [x_{j-1}, x_j])$ is contained in a simply connected coordinate neighborhood U_{ij} for each i, j. Let α_{ij} be the curve in M defined by applying H to the curve in $[a, b]^2$ and β_{ij} the curve in M defined by applying H to the curve in $[a, b]^2$ illustrated below.



(a) (b)

Then by the first remark, $\hat{\alpha}_{ij}$ and $\hat{\beta}_{ij}$ have the same endpoints arnd are homotopic by a homotopy keeping their endpoints fixed. By composing all these homotopies starting at the lower right-hand corner of $[a, b]^2$, proceeding right to left along each row of rectangles $[x_{i-1}, x_i] \times [x_{j-1}, x_j]$, and ending at the top left-hand corner of $[a, b]^2$, we find that $\hat{\alpha}$ and $\hat{\beta}$ are homotopic by a homotopy keeping their endpoints fixed. ☐

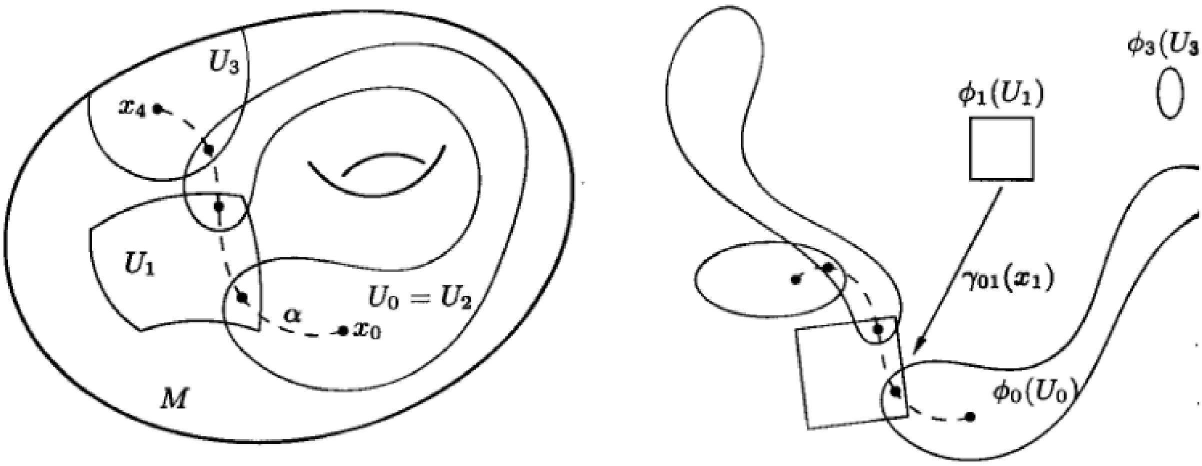


Figure 3.15. Analytic continuation. Here M is an affine torus (left). The path α (dashed) is subdivided at points $x_0, x_1, \ldots, x_4$ (marked by dots) so that each segment $[x_i, x_{i+1}]$ lies entirely in a coordinate patch U_i. The analytic continuation of ϕ_0 along α is ϕ_0 itself on a neighborhood of the first segment, $\gamma_{01}(x_1)\phi_1$ on a neighborhood of the second, and so on. The analytic continuation can be thought of as a multivalued map from M to $X = \mathbf{E}^2$, or as a map from the universal cover $\tilde{M}$ to X.

We can now define the Developing Map:

Definition 2.3. For a fixed base point and initial chart ϕ_0 the developing map of a (G,X)-manifold M is the map $D : \tilde{M} \to X$ that agrees with the analytic continuation of ϕ_0 along each path, in a neighborhood of the path's endpoint. In Symbols,

$$D = \phi_0^\sigma \circ \pi$$

in a neighborhood of $\sigma \in \tilde{M}$. If we change the initial data(basepoint and the initial chart), the developing map changes by composition in the range with an element of G.

Although G acts transitively on X in the cases of primary interest, this condition is not necessary for the definition of D. For example, if G is the trivial group and X is closed, then closed (G,X)-manifolds are precisely the finite-sheeted covers of X, and D is the covering projection.

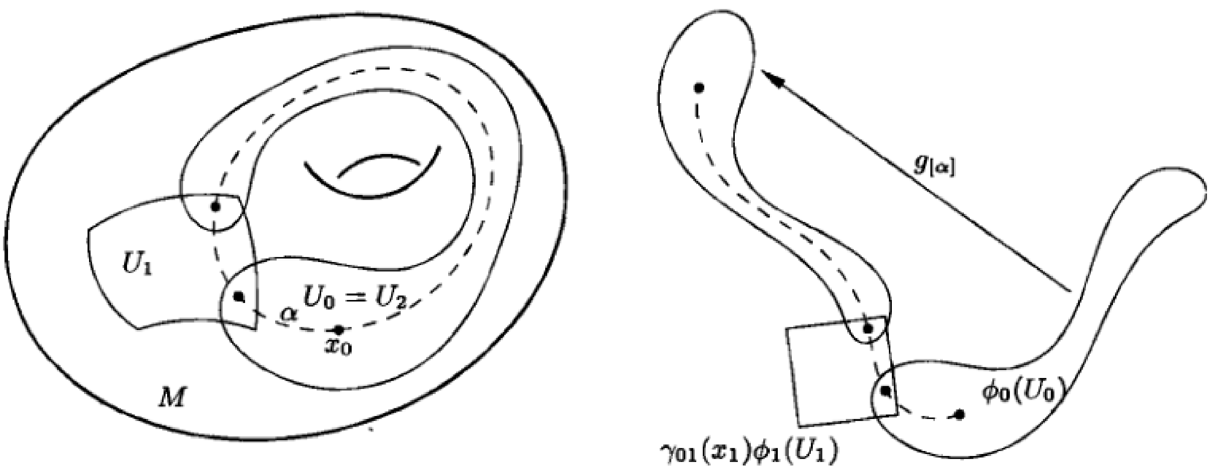


Figure 3.16. The holonomy around a path. For the torus of Figure 3.15, analytic continuation around the loop α requires two coordinate changes: from ϕ_0 to $\gamma_{01}(x_1)\phi_1$ to $\gamma_{01}(x_1)\gamma_{12}(x_2)\phi_2$. Therefore the holonomy around α is $g_{[\alpha]} = \gamma_{01}(x_1)\gamma_{12}(x_2) = \gamma_{01}(x_1)\gamma_{01}^{-1}(x_2)$.

Now let σ be an element of the fundamental group of M. Analytic continuation along a loop representing σ gives a germ ϕ_0^σ that is comparable to ϕ_0, since they are both defined at the basepoint(Figure above).

Take $g_\sigma \in G$, such that $\phi_0^\sigma = g_\sigma \phi_0$, g_σ is called the **holonomy** of σ. It follows immediately from the Definition, that

$$D \circ T_\sigma = g_\sigma \circ D$$

, where $T_\sigma : \tau \mapsto \sigma\tau$ is the covering transformation associated with σ. Applying this equation to a product, we see that the map $H : \sigma \mapsto g_\sigma$ from $\pi_1(M)$ into M is a group homomorphism, which we call the holonomy of M. Its image is the holonomy group of M. Note that H depends on the choice involved in the construction of D: when D changes, H changes by conjugation in G.

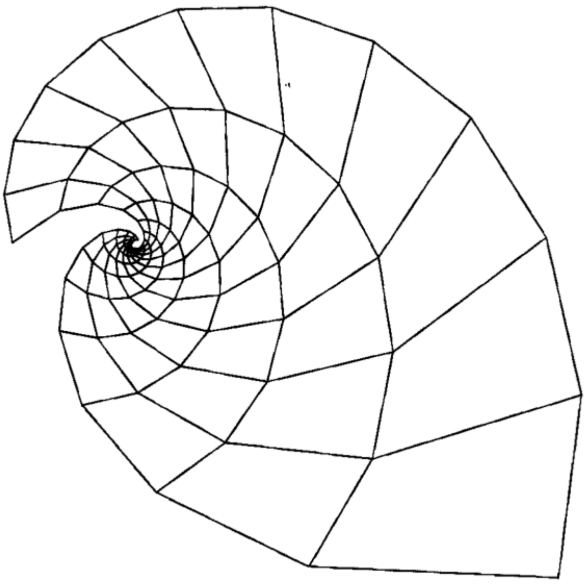


Figure 3.17. Developing the affine torus. The developing map of an affine torus constructed from a quadrilateral (Example 3.3.4) generally omits a single point of the plane.

Developing map and Manifolds, Completeness

Developing Manifolds

Now that we have the developing map, the question is: What does it tell us about our Manifold? And by extension, what does the holonomy tell us about our manifold?

In general, the holonomy of M need not determine the (G,X)-structure on M, but there is an important special case in which it does:

We say that M is a complete (G,X)-manifold if $D : \tilde{M} \to X$ is a covering map. Since a covering of a simple connected space is a homeomorphism we see that if M is complete and X simply connected, we can identify $\tilde{M}$ with X, via the developing map(and often will do so). This identification is canonical up to composition in the range with an element of G. We also get the following result:

Theorem 3.1. *If G is a group of analytic diffeomorphisms of a simply connected space X, any complete (G,X)-manifold may be reconstructed from its holonomy group* Γ*, as the quotient space* X/Γ*.*

Because of the preceeding theorem, it is often worthwhile to replace a not simply connected space X, with its universal cover$\tilde{X}$. The group of Homeomorphisms of X is a covering group acting on $\tilde{X}$. Its elements act as lifts of elements of G. We get

$$1 \to \pi_1(X) \to \tilde{G} \to G \to 1$$

Note that there is a one-to-one correspondence between $(\tilde{G}, \tilde{X})$-structures and (G,X)-structures, but the holonomy for a $(\tilde{G}, \tilde{X})$ contains more information.

some completeness results

We continue by exploring the relationship between Developing maps and completeness:

Lemma 3.2. *Let G act transitively on an analytic manifold X. Then X admits a G-invariant Riemannian metric if and only if, for some* $x \in X$*, the image of* G_x *in* $GL(T_xX)$ *has compact closure.*

Proof. One direction is clear: If G preserves a metric, G_x maps to a subgroup of $O(T_xX)$ which is compact. To prove the converse, fix a $x \in X$ and assume the image of G_x has compact closure H_x. Let Q be any positive definite form on T_xX. Using the Haar measure on H_x, average the set of transforms of g^*Q, for $g \in H_x$, to obtain a quadratic form on T_xX, invariant under H_x. Propagate this to every other point $y \in X$ by pulling back any element $g \in G$ that takes y to x; the pullback is independent of the choice of g. The resulting Riemannian metric is invariant under G. □

Theorem 3.3. *Let G be a Lie group, acting analytically and transitively on a manifold X, such that the stabilizer* G_x *of x is compact, for some (hence all)* $x \in X$*. Then every closed (G,X)-manifold M is complete.*

Proof. Transitivity implies that the given condition at one point x is equivalent to the same condition everywhere. So we fix $x \in X$ and look at the tangent space T_xX. There is an analytic homomorphism of G_x to the linear group of T_xX, whose image is compact.

Using the preceding lemma, we can pull back the invariant metric from X to M on any (G,X)-manifold M. The resulting Riemannian metric on M is invariant under any (G,X)-map. Now in a Riemannian Manifold, we can find for any point y, a ball $B_\varepsilon(y)$ or radius $\varepsilon > 0$ that is a homeomorphic image of the round ball under the exponential map(ball-like) and convex. If M is closed, we can choose ε uniformly by compactness. We may also assume that all ε-balls in X are contractible and convex, since G is a transitive group of isometries.

Then for any $y \in \tilde{M}$, the ball $B_\varepsilon(y)$ is mapped homeomorphically by D, for if $D(y) = D(y')$ for $y \neq y'$ in the ball, the geodesic connecting y to y' maps to a self-intersecting geodesic, contradicting the convexity of ε-balls in X. Furthermore, D is an isometry between $B_\varepsilon(y)$ and $B_\varepsilon(D(y))$ by definition.

Now take $x \in X$ and $y \in B^{-1}_{\varepsilon/2}(x)$. The ball $B_\varepsilon(y)$ maps isometrically, and thus must properly contain a homeomorphic copy of $B_{\varepsilon/2}(x)$. The entire inverse image $D^{-1}(B_{\varepsilon/2}(x))$ is then a disjoint union of such homeomorphic copies. Therefore D evenly covers X, so it's a covering projection, and M is complete.

□

We give another theorem classify completeness on a (G,X)-Manifold.

Theorem 3.4. *Let G be a transitive group of real analytic diffeomorphisms of X with compact stabilizers G_x. Fix a G-invariant metric on X and let M be a (G,X)-manifold with metric inherited from X. The following are equivalent:*

1. *M is a complete (G,X)-manifold.*
2. *For some $\varepsilon > 0$, every closed ε-ball is in M is compact.*
3. *For all $a > 0$, every closed a-ball in M is compact.*
4. *There is some family of compact subsets S_t of M, for $t \in \mathbb{R}_{>0}$, such that $\bigcup_{t\in\mathbb{R}_{>0}} S_t = M$ and S_{t+a}, contains the neighborhood of radius a about S_t.*
5. *M is complete as a metric space.*

Proof. Hopf-Rinow. For a full proof see [1]. □

Some selected results

We will now present some selected, nice results, we start with a little bit of group theory.

Discrete Groups

According to Proposition 3.1, when G is a group of analytic diffeomorphisms of a simply connected manifold X, complete (G,X)-manifolds (up to isomorphism) are in one-to-one correspondence with certain subgroups of G (up to conjugacy by elements of G). There are certain traditional fallacies concerning the characterization of the groups that are holonomy groups for complete (G, X)-manifolds, so it is worth going through the definitions carefully.

Definition 4.1. Group actions: Let Γ be a group acting on a topological space X by homeomorphisms. We will normally consider the action to be effective; this means that the only element of r that acts trivially is the identity element of r, so in effect we can see Γ as a group of homeomorphisms of X. Here are other properties that the action might have

- The action is free if no point of X is fixed by an element of Γ other than the identity.
- The action is discrete if Γ is a discrete subset of the group of homeomorphisms of X, with the compact-open topology.
- The action has discrete orbits if every $x \in X$ has a neighborhood U such that, the set of $\gamma \in \Gamma$ mapping x inside U is finite.
- The action is wandering if every $x \in X$ has a neighborhood U, such that the set of $\gamma \in \Gamma$ for which $\gamma(U) \cap U \neq \emptyset$ is finite
- Assume X is locally compact. The action of r is properly discontinuous if for every compact subset K of X the set of $\gamma \in \Gamma$ such that $\gamma(K) \cap K \neq \emptyset$ is finite.

We will now state some necesarry group theoretic theorems:

Theorem 4.2. *Let* Γ *be a group acting freely on a connected (Hausdorff) manifold X, and assume the action is wandering. Then the quotient space* X/Γ *is a (possibly non-Hausdorff) manifold and the quotient map is a covering map.*

Proof. Given $x \in X$ take a neighborhood U of x that intersects only finitely many of its translates γU. Using the Hausdorffness of X and the freeness of the action, choose a smaller neighborhood of x whose translates are all disjoint. Then each translate maps homeomorphically to its image in the quotient, so the image is evenly covered. Since x was arbitrary, this implies the claim. $\square$

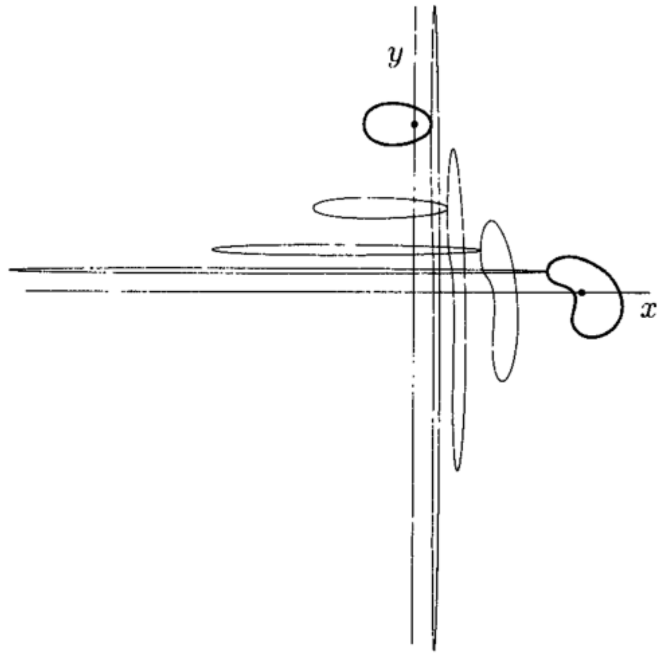


Figure 3.22. Action with non-Hausdorff quotient. The quotient of $\mathbf{R}^2 \setminus 0$ by the action generated by the linear map $(x, y) \longmapsto (2x, \frac{1}{2}y)$ is a manifold, because the action is wandering and free, but it is not Hausdorff: every neighborhood of the image of $(1, 0)$ intersects every neighborhood of the image of $(0, 1)$. This is because the iterates of any neighborhood of a point on the x-axis accumulate along the y-axis, and vice versa.

Theorem 4.3. *Let* Γ *be a group acting on a manifold X. The quotient space* X/Γ *is a manifold with* $X \to X/\Gamma$ *a covering projection if and only if* Γ *acts freely and properly discontinuously.*

Proof. If the action is free and properly discontinuous, we just have to check that the quotient is Hausdorff, by the previous proposition and the well known fact that the last three properties defined in the beginning of the chapter are strictly stronger than the previous one. Suppose x and y are points in X on distinct orbits. Let K be a union of two disjoint compact neighborhoods of x and y which contain no translates of x or y. Then $K \setminus \bigcup_{\gamma \neq 1} \gamma K$ is still a union of a neighborhood of x with a neighborhood of y, and these neigerborhoods project to disjoint neighborhoods in X/Γ.

For the converse, suppose that X/Γ is Hausdorff and that $p : X \to X/\Gamma$ is a covering projection. For any pair of points $(x_1, x_2) \in X \times X$, we will find neighborhoods U_1 of x_1 and U_2 of x_2 such that

$\gamma(U_1) \cap U_2 \neq \emptyset$ for at most one $\gamma \in \Gamma$. If x_2 ist not on the orbit of x_1 this follows from the Hausdorff property of X/Γ because $p(x_1)$ and $p(x_2)$ have disjoint neighbordhoods. If x_2 has the form γx_1, this follows from the fact that p is a covering projection - we take for U_1 a neighborhood of x_1 that projects homeomorphically to the quotient space and let $U_2 = \gamma U_1$.

Now let K be any compact subset of X. Since $K \times K$ is compact, there is a finite covering of $K \times K$ by product neighborhoods of the form $U_1 \times U_2$ where U_1 hast at most one image under Γ intersecting U_2. Therefore the set of elements $\gamma \in \Gamma$ such that $\gamma K \cap K \neq \emptyset$ is finite and Γ acts freely and properly discontinuously.

□

Theorem 4.4. *Suppose that Γ acts on the spaces X and Y, and that $f : X \to Y$ is a proper, surjective, equivariant map. Then the action on X is properly discontinuous if and only if the action on Y is.*

Proof. Images and inverse images of compact sets under f are compact. Every compact subset of X is contained in a set of the form $f^{-1}(K)$, where $K \subset Y$ is compact, so it suffices to consider such sets to check proper discontinuity in X. The proposition now follows because $K \cap \gamma K \neq \emptyset$ if and only if $f^{-1}(K) \cap \gamma f^{-1}(K) \neq \emptyset$ □

Corollary 4.5. *Suppose G is a Lie group and X is a manifold on which G acts transitively with compact stabilizers G_x. Then any discrete subgroup of G acts properly discontinuously on X.*

Proof. The map $G \to X = G/G_x$ is proper. Apply the previous proposition and the fact that discrete subgroups of Lie groups are properly discontinuous. □

Now, this means that in the cases most of interest to us, the different definitions are equivalent. Taking this, as well as the fact that the Holonomy characterizes certain manifolds(Theorem 3.1), the relation betwen stabilizers and completeness(Theorem 3.3) and finally Theorem 4.3 we get the following, important corollary.

Corollary 4.6. *Suppose G is a Lie group acting transitively, analytically and with compact stabilizers on a simply connected manifold X. If M is a closed differentiable manifold, (G,X)-structures on M (that is, (G,X)-stiffenings of M up to diffeomorphism) are in one-to-one correspondence with conjugacy classes of discrete subgroups of G that are isomorphic to $\pi_1(M)$ and act freely on X with quotient M. If M is not closed, we get the same correspondence if we look only at complete (G,X)-structures on M.*

Note that the condition 'with quotient M' is indeed necessary, because the fundamental group of M, $\pi_1(M)$, does not determine the diffeomorphism class of M.

Consider for example the punctured torus and the three dimensional punctured sphere. Both have the same fundamental group, but do not agree diffeomorphically.

References

[1] William P. Thurston, *Three-Dimensional Geometry and Topology*, Princeton University Press, 1997.

[2] John G. Ratcliffe, *Foundations of Hyperbolic Manifolds*, Springer, 2005.