Wintersemester 2014/2015

University of Heidelberg

Differential Forms seminar

Differential Forms in $\mathbb{R}^n$

by

Stephan Schmitt

Sven Grützmacher

Contents

Differential Forms in $\mathbb{R}^3$ **1**

Differential Forms in $\mathbb{R}^n$ **2**

Exterior Product 3
Pullback 4
exterior derivative 6
divergence and gradient 8
laplacian 8

Differential Forms in $\mathbb{R}^3$

We will start with an introduction to Differential Forms in $\mathbb{R}^3$, after that we will naturally generalize the concept to $\mathbb{R}^n$ and proof important properties about them.

Definition 1.1. Let $p \in \mathbb{R}^3$, denote the set $\{q-p|q \in \mathbb{R}^3\}$ by $\mathbb{R}^3_p$. This space is called the tangent space(of $\mathbb{R}^3$ in p). The vectors $e_i; i = 1,2,3$ form a canonical basis for $\mathbb{R}^3_0$. Their translates for p, denoted by $(e_i)_p$, form a Basis for $\mathbb{R}^3_p$. Consider now the dual $(\mathbb{R}^3_p)^*$ of $\mathbb{R}^3_p$. This space is called the Co-Tangent Space. We obtain a basis for this space by taking $dx^i, i = 1,2,3$, where $x^i : \mathbb{R}^3 \to \mathbb{R}$ is the map assigning a point p its i-th coordinate.

Proposition 1.2. *The set* $\{(dx^i)_p; i = 1,2,3\}$ *is the dual basis of* $\{(e_i)_p\}$

Proof.

$$(dx^i)_p(e_j) = \frac{\partial x_i}{\partial x_j} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

□

Let us now introduce two essential maps working on those spaces:

Definition 1.3. A map $v : \mathbb{R}^3 \to \mathbb{R}^3_p$, given by

$$v(p) = \sum_{i=1}^{3} a_i(p) e_i$$

where $a_i : \mathbb{R}^3 \to \mathbb{R}$ is called Vector Field. It is called differentiable, if the a_i are.

Definition 1.4. A map $\omega : \mathbb{R}^3 \to (\mathbb{R}^3_p)^*$,

$$\omega(p) = \sum_{i=1}^{3} a_i(p)(dx^i)_p$$

is called exterior form of degree 1 and Differential 1-Form, if the a_i are differentiable.

As the name suggests, there are forms of higher degree:

Definition 1.5. Denote by $\Lambda^2(\mathbb{R}^3_p)^*$ the set of all alternating, bilinear maps $\mathbb{R}^3_p \times \mathbb{R}^3_p \to \mathbb{R}$. If $\omega^1, \omega^2 \in (\mathbb{R}^3_p)^*$, define $\omega^1 \wedge \omega^2 \in \Lambda^2(\mathbb{R}^3_p)^*$ as

$$\omega^1 \wedge \omega^2(v_1, v_2) = det(\omega^i(v_j))$$

.

We need a basis for this space, conveniently the set $\{(dx^i \wedge dx^j)_p; i = 1,2,3, i < j\}$ is one. This result will be proven later in a more general manner. Also notice the following as an imediate result of this definition:

Proposition 1.6.

$$(dx^i \wedge dx^j)_p = -(dx^j \wedge dx^i)_p; i \neq j$$

and

$$dx^i \wedge dx^i = 0$$

Definition 1.7. A map $\omega : \mathbb{R}^3 \to \Lambda^2(\mathbb{R}^3_p)^*$,

$$\omega(p) = \sum_{i<j} a_{ij} dx^i \wedge dx^j; i,j = 1,2,3$$

where a_{ij} are real functions, is called exterior form of degree 2 or differential 2-form, if the a_{ij} are differentiable.

Differential Forms in $\mathbb{R}^n$

We will now generalize these concepts to $\mathbb{R}^n$. The definitions of tangent and co-tangent space are the exact same with n in place of 3. We will start with the n-dimensional variant of $\Lambda^2(\mathbb{R}^n_p)^*$:

Definition 2.1. Let $\Lambda^k(\mathbb{R}^n_p)^*$ denote the space of alternating, k-linear maps from $\mathbb{R}^n_p \times \cdots \times \mathbb{R}^n_p \to \mathbb{R}$. Analogous to 3 dimensions if $\omega^i \in (\mathbb{R}^n_p)^*$; $i = 1, \ldots, k$ we get an element $\omega^1 \wedge \cdots \wedge \omega^k \in \Lambda^k(\mathbb{R}^n_p)^*$ by setting

$$\omega^1 \wedge \cdots \wedge \omega^k(v_1 \ldots v_k) = det(\omega^i(v_j))); i, j = 1, \ldots, k$$

Again we need a Basis for this space:

Proposition 2.2. *The Set*

$$\{dx^{i_1} \wedge \cdots \wedge dx^{i_k}; i_1 < i_2 < \cdots < i_k; i_j \in \{1, \ldots, n\}\}$$

forms a Basis for $\Lambda^k(\mathbb{R}^n_p)^*$.

Proof. The elements of this Set are linearly independet. To see this, take:

$$\sum_{i_1 < \cdots < i_k} a_{i_1 \ldots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k} = 0$$

and apply it to

$$(e_{j_1} \ldots e_{j_k}), j_1 < \cdots < j_k, j_l \in \{1, \ldots, n\}$$

this gives

$$\sum_{i_1 < \cdots < i_k} a_{i_1 \ldots i_k} dx^{i_1} \wedge dx^{i_k}(e_{j_1}, \ldots, e_{j_k}) = a_{j_1 \ldots j_k} = 0$$

Because:

Start by looking at the determinant $\rho = |dx^{i_l}(e_{j_n})|$. Assuming $i_1 > j_1$ it follows that $i_k > \cdots > i_1 > j_1$. This in turn implies $dx^{i_l}(e_{j_1}) = 0$; $l = 1, \ldots, k$. In turn assuming $i_1 < j_1$ gives $j_k > \cdots > j_1 > i_1$, implying: $dx^{i_1}(e_{j_n}) = 0$; $n = 1, \ldots, k$.

Both giving us $\rho = 0$ if $i_1 \neq j_1$. Now assume $i_1 = j_1$ and $j_2 \neq i_2$. Using the exact same Argument as above it follows that $\rho = 0$ if $i_2 \neq j_2$. Iterating gives us the desired

$$(dx^{i_1} \wedge \cdots \wedge dx^{i_k})(e_{j_1}, \ldots, e_{j_k}) = 0$$

if $\exists j_l \neq i_l$. It remains to see that the above expression gives 1 if $i_1 = j_1, \ldots, i_k = j_k$. But this is an immediate result of the definition of forms.

We will now show $f \in \Lambda^k(\mathbb{R}^n_p)^* \Rightarrow f = \sum_{i_1 < \cdots < i_k} a_{i_1 \ldots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}$

Define

$$g = \sum_{i_1 < \cdots < i_k} f(e_{i_1}, \ldots, e_{i_k}) dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

Clearly:

$$g(e_{i_1} \ldots e_{i_k}) = f(e_{i_1} \ldots e_{i_k}); \forall i_1, \ldots i_k$$

It follows that $f = g$. Setting $f(e_{i_1}, \ldots, e_{i_k}) = a_{i_1 \ldots i_k}$ the result follows. □

Definition 2.3. A map $\omega : \mathbb{R}^n \to \Lambda^k(\mathbb{R}^n_p)^*$, given by

$$f = \sum_{i_1 < \cdots < i_k} a_{i_1 \ldots i_k}(p)(dx^{i_1} \wedge \cdots \wedge dx^{i_k})_p; i_j \in \{1, \ldots, n\}$$

where the $a_{i_1 \dots i_k}$ are real functions in $\mathbb{R}^n$, if they are differentiable, this map is called differential k-form.

In the following we will denote $(i_1, \dots, i_k), i_1 < \dots < i_k; i_k \in \{1, \dots, n\}$ with I. Therefore ω can be written

$$\omega = \sum_I a_I dx^I$$

Further we will set the convention to refer to differentiable functions as differential 0-forms.

Let us now look at an example for forms in $\mathbb{R}^4$

Example 2.4. Let $a_i, a_{ij}, \dots$ denote real functions.
0-forms: functions in $\mathbb{R}^4$
1-forms: $a_1 dx^1 + a_2 dx^2 + a_3 dx^3 + a_4 dx^4$
2-forms: $a_{12} dx^1 \wedge dx^2 + a_{13} dx^1 \wedge dx^3 + a_{14} dx^1 \wedge dx^4 + a_{23} dx^2 \wedge dx^3 + a_{24} dx^2 \wedge dx^4 + a_{34} dx^3 \wedge dx^4$
3-forms: $a_{123} dx^1 \wedge dx^2 \wedge dx^3 + a_{124} dx^1 \wedge dx^2 \wedge dx^4 + a_{134} dx^1 \wedge dx^3 \wedge dx^4 + a_{234} dx^2 \wedge dx^3 \wedge dx^4$
4-forms: $a_{1234} dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4$

From now on we will restrict ourself to differential k-forms, or simply k-forms. In the following we will define operations of k-forms to see their structure.

Exterior Product

Definition 2.5. Let $\alpha = \sum_I a_I dx^I, \beta = \sum_I b_I dx^I$ denote two k-forms, their sum is definded to be $\alpha + \beta = \sum_I (a_I + b_I) dx^I$.

Now let $\alpha = \sum_I a_I dx^I$ be a k-form and $\beta = \sum_J b_J dx^J$ an s-form. Their exterior product is defined to be: $\alpha \wedge \beta = \sum_{IJ} a_I b_J dx^I \wedge dx^J$, which is an (s+k)-form. Notice that this agrees, by definition of the determinant, with our previous definition of $\wedge$.

Example 2.6. Let us look at the explicit exterior product of two forms, therefore let $\alpha = x_1 dx^1 + x_2 dx^2 + x_3 dx^3$ denote a 1-form in $\mathbb{R}^3$ and $\beta = x_1 dx^1 \wedge dx^2 + dx^1 \wedge dx^3$ a 2-form. Using $dx^i \wedge dx^i = 0$ and $dx^i \wedge dx^j = -(dx^j \wedge dx^i), i \neq j$, we obtain:

$$\alpha \wedge \beta = x_2 dx^2 \wedge dx^1 \wedge dx^3 + x_3 x_1 dx^3 \wedge dx^1 \wedge dx^2 = (x_1 x_3 - x_2) dx^1 \wedge dx^2 \wedge dx^3$$

Proposition 2.7. *Let α be a k-form, β a s-form and γ an r-form, then:*

a) $\alpha \wedge (\beta \wedge \gamma) = (\alpha \wedge \beta) \wedge \gamma$

b) $\alpha \wedge \beta = (-1)^{ks} (\beta \wedge \alpha)$

c) $\alpha \wedge (\beta + \gamma) = \alpha \wedge \beta + \alpha \wedge \gamma$, *if* $r = s$

Proof. a) and c) trivially follow from definition, let us look at b):
Denote

$$\alpha = \sum_I a_I dx^I, \beta = \sum_J b_J dx^J$$

.

Then:

$$\alpha \wedge \beta = \sum_{IJ} a_I b_J dx^{i_1} \wedge \dots \wedge dx^{i_k} \wedge dx^{j_1} \wedge \dots \wedge dx^{j_s}$$

Switching one indice, we obtain

$$\alpha \wedge \beta = \sum_{IJ} b_J a_I (-1) dx^{i_1} \wedge \dots \wedge dx^{i_{k-1}} \wedge dx^{j_1} \wedge dx^{i_k} \wedge \dots \wedge dx^{j_s}$$

repeating this k-times, we get:

$$\sum_{IJ} b_J a_I (-1)^k dx^{j_1} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k} \wedge dx^{j_2} \wedge \cdots \wedge dx^{j_s}$$

Since J has s elements, we repeat this process for all $j_l \in J$ and obtain

$$\alpha \wedge \beta = (-1)^{ks}(\beta \wedge \alpha)$$

□

Pullback

So far we introduced the tangent space and co-tangent space with maps giving you elements of those spaces. Another important question is, given $\mathbb{R}^n$ and $\mathbb{R}^m$ can we somehow get differential forms in $\mathbb{R}^n$ from those in $\mathbb{R}^m$. The next definition will give us a possibility to do so in a very natural way. All we need is a differentiable function.

Definition 2.8. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a differentiable function. This induces a map, denoted by f^*, called the pullback of f, that takes k-forms in $\mathbb{R}^m$ to k-forms in $\mathbb{R}^n$. For a k-form ω in $\mathbb{R}^m$ $(f^*w)_p(v_1, \dots, v_k) = \omega(f(p))(df_p(v_1), \dots, df_p(v_k))$. Where $df_p : \mathbb{R}^n_p \to \mathbb{R}^m_{f(p)}$ is the differential of f at point $p \in \mathbb{R}^n$ and $v_1, \dots, v_k \in \mathbb{R}^n_p$. In the special case of g being a 0-form(a differentiable function), we set: $f^*g = g \circ f$

As always, we need some properties for this new map, after that we will try to make more sense of this map.

Proposition 2.9. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a differentiable function, α, β k-forms on $\mathbb{R}^m$ and $g : \mathbb{R}^m \to \mathbb{R}$ a 0-form in $\mathbb{R}^m$, then:*

a) $f^*(\alpha + \beta) = f^*\alpha + f^*\beta$

b) $f^*(g\alpha) = f^*(g)f^*(\alpha)$

c) If $\alpha^1, \dots, \alpha^k$ are 1-forms in $\mathbb{R}^m$, $f^*(\alpha^1 \wedge \cdots \wedge \alpha^k) = f^*(\alpha^1) \wedge \cdots \wedge f^*(\alpha^k)$

Proof. These proofs will be rather short:

□

a)

$$\begin{aligned} f^*(\alpha + \beta)(p)(v_1, \dots, v_k) &= (\alpha + \beta)(f(p))(df_p(v_1), \dots, df_p(v_k)) \\ &= (f^*\alpha)(p)(v_1, \dots, v_k) + (f^*\beta)(p)(v_1, \dots, v_k) \\ &= (f^*\alpha + f^*\beta)(p)(v_1, \dots, v_k) \end{aligned}$$

b)

$$\begin{aligned} f^*(g\alpha)(p)(v_1 \dots v_k) &= (g\alpha)(f(p))(df_p(v_1), \dots, df_p(v_k)) \\ &= (g \circ f)(p) f^*\alpha(p)(v_1, \dots, v_k) \\ &= f^*g(p) f^*\alpha(p)(v_1, \dots, v_k) \end{aligned}$$

c)

$$\begin{aligned} f^*(\alpha^1 \wedge \cdots \wedge \alpha^k) &= (\alpha^1 \wedge \cdots \wedge \alpha^k)(df(v_1), \dots, df(v_k)) \\ &= det(\alpha^i(df(v_j)) = det(f^*\alpha^i(v_j)) \\ &= (f^*\alpha^1 \wedge \cdots \wedge f^*\alpha^k)(v_1, \dots, v_k) \end{aligned}$$

Note that the last part of this proposition is also true for k-forms, and will be proven so later. For now we try to make sense of the pullback by interpreting it as a substitution of variables, to justify this interpretation:

Example 2.10. Let $(x_1, \ldots, x_n)$ be coordinates in $\mathbb{R}^n$ and $(y_1, \ldots, y_m)$ in $\mathbb{R}^m$, define $f : \mathbb{R}^n \to \mathbb{R}^m$ by $y_1 = f_1(x_1, \ldots, x_n), \ldots, y_m = f_m(x_1, \ldots, x_n)$. Now let $\omega = \sum_I a_I dy^I$ be a k-form in $\mathbb{R}^m$.

Using the last Proposition we obtain

$$f^*\omega = \sum_I f^*(a_I)(f^*dy^{i_1}) \wedge \cdots \wedge (f^*dy^{i_k})$$

since

$$f^*(dy^i)(v) = dy^i(df(v)) = d(y_i \circ f(v)) = df^i(v)$$

we finally get

$$f^*\omega = \sum_I a_I(f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n))df^{i_1} \wedge \cdots \wedge df^{i_k}$$

where f_i and df^i are functions of x_j. Thus, applying f^* to ω is equivalent to 'substitue' the variables y_i and their differentials in ω by x_k and dx^k obtained abvore.

Another thing we have yet to mention: Subsets of $\mathbb{R}^n$. Up until now we only referred to $\mathbb{R}^n$ directly, sometimes however it is more convenient to use the concepts we introduced only on Subsets of $\mathbb{R}^n$. As long as these are open, everything done up to now extends trivially.

To showcase these last two facts see:

Example 2.11. Let ω be the 1-form in $\mathbb{R}^2 - \{0, 0\}$ given by

$$\omega = -\frac{y}{x^2 + y^2}dx + \frac{x}{x^2 + y^2}dy$$

Next define

$$U = \{r > 0; 0 < \theta < 2\pi\}$$

and let $f : U \to \mathbb{R}^2$ be the map

$$f(r, \theta) = \begin{cases} x &= rcos\theta \\ y &= rsin\theta \end{cases}$$

We can now compute $f^*\omega$. For that, notice

$$dx = cos\theta dr - rsin\theta d\theta$$

$$dy = sin\theta dr + rcos\theta d\theta$$

which gives us

$$f^*\omega = -\frac{rsin\theta}{r^2}(cos\theta dr - rsin\theta d\theta) + \frac{rcos\theta}{r^2}(sin\theta dr + rcos\theta d\theta) = d\theta$$

We can now proof another important fact, the Pullback actually commutes with the exterior product:

Proposition 2.12. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a differentiable map. Then*

a) $f^*(\alpha \wedge \beta) = (f^*\alpha) \wedge (f^*\beta)$, where α is a k-form and β a s-form in $\mathbb{R}^m$

b) $(f \circ g)^*\alpha = g^*(f^*\alpha)$, $g : \mathbb{R}^p \to \mathbb{R}^n$ differentiable

Proof. Setting $(y_1, \ldots, y_m) = (f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n)) \in \mathbb{R}^m$,$(x_1, \ldots, x_n) \in \mathbb{R}^n, \alpha = \sum_I a_I dy^I$, $\beta = \sum_J b_J dy^J$

We get:

a)

$$f(\alpha \wedge \beta)^* = f^*(\sum_{IJ} a_I b_J dy^I \wedge dy^J) = \sum_{IJ} a_I(f_1, \dots, f_m) b_J(f_1, \dots, f_m) df^I \wedge df^J$$

We can now seperate this and get:

$$\sum_I a_I(f_1, \dots, f_m) df^I \wedge \sum_J b_J(f_1, \dots, f_m) df^J = f^*\alpha \wedge f^*\beta$$

b)

$$\begin{aligned}(f \circ g)^*\alpha &= \sum_I a_I((f \circ g)_1, \dots, (f \circ g)_m) d(f \circ g)^I \\ &= \sum_I a_I(f_1(g_1, \dots, g_n), \dots, f_m(g_1, \dots, g_n)) df^I(dg^1, \dots, dg^n) \\ &= g^*(f^*(\alpha))\end{aligned}$$

□

exterior derivative

Unitl now we have seen some properties of k-forms in R^n. Yet we haven't covered why those forms are also called : „differential “forms. This name implies some form of differentiability we haven't seen so far. To motivate the following definition we take a look at 0-forms which we understand quite well.

Example 2.13. Let $g : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a 0-form (and thus a smooth function). Then its differential is given by

$$dg = \sum_{i=1}^n \frac{\partial g}{\partial x_i} dx^i$$

which is a 1-form

We generalize this idea to k-forms now

Definition 2.14. Let $w = \sum_I a_I dx^I$ be a k-form in $\mathbb{R}^n$. The exterior differential is

$$dw = \sum_I da_I \wedge dx^I$$

We will see later that this generalizes some concepts of differential calculus. To show this we will take a look at a little example und some properties of exterior differentiation:

Example 2.15. Let $w = xydx + e^z dy + xdz$. Then

$$\begin{aligned}dw &= d(xy) \wedge dx + d(e^z) \wedge dy + d(x) \wedge dz \\ &= (ydx + xdy) \wedge dx + e^z dz \wedge dy + dx \wedge dz \\ &= -xdx \wedge dy + dx \wedge dz - e^z dy \wedge dz\end{aligned}$$

Proposition 2.16. *Let w be a k-form, φ a s-form in $\mathbb{R}^n$ and $f : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ smooth. Then*

a) d is $\mathbb{R}$-linear

b) $d(w \wedge \varphi) = dw \wedge \varphi + (-1)^k w \wedge d\varphi$

c) $d(dw) = d^2w = 0$

d) $d(f^*w) = f^*(dw)$

Proof. a) follows directly from the definition

b) Sei $w = \sum a_I dx^I$ und $\varphi = \sum b_J dx^J$. Dann gilt

$$\begin{aligned} d(w \wedge \varphi) &= d\left(\sum_{IJ} (a_I b_J) dx^I \wedge dx^J\right) \\ &= \sum_{IJ} d(a_I b_J) \wedge dx^I \wedge dx^J \\ &= \sum_{IJ} b_J da_I \wedge dx^I \wedge dx^J + \sum_{IJ} a_I db_J \wedge dx^I \wedge dx^J \\ &= dw \wedge \varphi + (-1)^k \sum_{IJ} a_I dx^I \wedge db_J \wedge dx^J \\ &= dw \wedge \varphi + (-1)^k w \wedge d\varphi \end{aligned}$$

c) we will give this proof in two steps.
First let w be a 0-form and thus a smooth function $\mathbb{R}^n \longrightarrow \mathbb{R};\ x \mapsto f(x_1, \ldots, x_n)$. Then

$$d(df) = d\left(\sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx^i\right) = \sum_{i=1}^{n} d\left(\frac{\partial f}{\partial x_i}\right) \wedge dx^i = \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_j \partial x_i} dx^j \wedge dx^i = \sum_{i,j=1}^{n} \partial_{ji} f dx^j \wedge dx^i$$

Since f is smooth the order of differtiation does not matter, e.g. $\partial_{ij} f = \partial_{ji} f$.
Furthermore $dx^j \wedge dx^i = -dx^i \wedge dx^j$. Thus we get

$$d(df) = \sum_{i<j} (\partial_{ij} f - \partial_{ji} f) dx_i \wedge dx^j = 0$$

Now let $w = \sum a_I dx^I$. Through linearity it suffices to consider $w = a_I dx^I$ where $a_I \neq 0$. Using b) we get

$$dw = da_I \wedge dx^I + a_I d(dx^I)$$

while $d(dx^I) = d(1) \wedge dx^I = 0$ we can compute

$$d(dw) = d(da_I \wedge dx^I) = d(da_I) \wedge dx^I + da_I \wedge d(dx^I) = 0$$

since we have already shown $d^2 = 0$ in the cases above.

d) we use the same method as in c). So let $g : \mathbb{R}^n \longrightarrow \mathbb{R};\ y \mapsto g(y_1, \ldots, y_n)$ be a smooth function. Then

$$f^*(dg) = f^*\left(\sum_i \frac{\partial g}{\partial y_i} dy^i\right) = \sum_{ij} \frac{\partial g}{\partial y_i} \frac{\partial f_i}{\partial x_j} dx^j = \sum_j \frac{\partial (g \circ f)}{\partial x_j} dx^j = d(g \circ f) = d(f^* g)$$

Now again let $w = \sum a_I dx^I$ be a k-form. Since f commutes with the exterior product we obtain

$$\begin{aligned} d(f^*w) = d\left(\sum_I f^*(a_I) f^*(dx^I)\right) &= \sum_I d(f^*(a_I)) \wedge f^*(dx^I) \\ = \sum_I f^*(da_I) \wedge f^*(dx^I) = f^*\left(\sum_I da_I \wedge dx^I\right) &= f^*(dw) \end{aligned}$$

□

The first properties certainly justify the name „differential“form since we got a product rule and linearity. Property c) is probably the most important one which can be easily seen if you have some basic knowledge in Ko-/Homology-theory.

Since in $\mathbb{R}^n$ we can identify smooth vector fields with smooth 1-forms via the scalarproduct induced isomorphism we can take a look at some concepts of differential calculus to see in which way they're connected to 1-forms.

divergence and gradient

The divergence and the gradient are commonly known objects in the first calculus lectures. There they're often defined by concrete formulas. Yet they can naturally be defined using the exteroir derivative

Definition 2.17. Let v be a vector field in $\mathbb{R}^n$. We can identify v with a map $\mathbb{R}^n \longrightarrow \mathbb{R}^n$. Then the divergence of v is a function $\mathbb{R}^n \longrightarrow \mathbb{R}$ defined as follows:

$$div(v)(p) = trace(dv)_p$$

Since we know what dv_p looks like concerning the standard basis we deduce the known formula for $v = \sum a_i e_i$

$$div(v) = \sum \frac{\partial a_i}{\partial x_i}$$

Definition 2.18. Given a smooth function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ we define a vector field $grad(f) : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ through

$$< grad(f)(p), u >= df_p(u), u \in \mathbb{R}^n_p$$

If we choose the standard basis for $\mathbb{R}^n_p$ we get

$$graf(f) = \sum \frac{\partial f}{\partial x_i} e_i$$

which can be easily derived from the initial definition by

$$< grad(f)(p), u >= df_p(u) = \sum \frac{\partial f}{\partial x_i} dx^i(u) = \sum \frac{\partial f}{\partial x_i} u_i$$

laplacian

A combination of the two concepts above is the laplacian which involves second derivatives.

Definition 2.19. Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a smooth function. Then the lapacian is

$$\triangle f = div(grad(f));\ \mathbb{R}^n \longrightarrow \mathbb{R}$$

As before we can describe it in local coordinates:

$$\triangle f = div \left(\sum_i \frac{\partial f}{\partial x_i} e_i \right) = \sum_{ij} \frac{\partial^2 f}{\partial x_i \partial x_j}$$