Curvature of Riemannian Manifolds

Seminar Riemannian Geometry
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1 Motivation

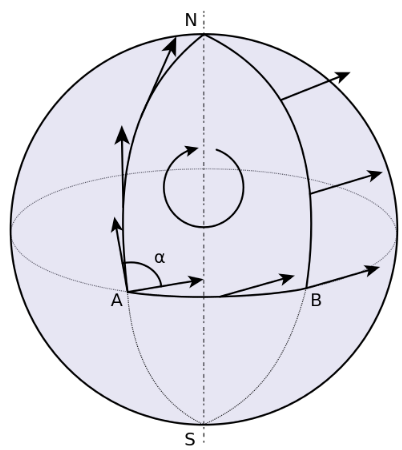

Figure 1: A vector parallel transported along a closed curve on a curved manifold.[1]

The aim of this talk is to define the *curvature* of Riemannian Manifolds and meeting some important simplifications as the *sectional, Ricci* and *scalar curvature*. We have already noticed, that a vector transported parallel along a closed curve on a Riemannian Manifold M may change its orientation. Thus, we can determine whether a Riemannian Manifold is curved or not by transporting a vector around a loop and measuring the difference of the orientation at start and the endo of the transport. As an example take Figure 1, which depicts a parallel transport of a vector on a two-sphere. Note that in a non-curved space the orientation of the vector would be preserved along the transport.

2 Curvature

In the following we will use the Einstein sum convention and make use of the notation:

$\mathfrak{X}(M)$	space of smooth vector fields on M
$\mathscr{D}(M)$	space of smooth functions on M

2.1 Defining Curvature and finding important properties

This rather geometrical approach motivates the following definition:

Definition 2.1 (Curvature)**.** The *curvature* of a Riemannian Manifold is a correspondence that to each pair of vector fields $X, Y \in \mathfrak{X}\,(M)$ associates the map $R(X,Y) : \mathfrak{X}(M) \to \mathfrak{X}(M)$ defined by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z + \nabla_{[X,Y]} Z \tag{1}$$

∇ is the Riemannian connection of M.

The last term in (1) is needed in order for R to be linear, as we will see soon. [X,Y] is the Lie bracket, which resembles the derivative of Y along the flow generated by X and thus being equal to the Lie derrivative $\mathfrak{L}_X$ introduced in an earlier talk. We have already seen a geometrical interpretation of (1), now we will get to know two more possible interpretations. As $R(X,Y)Z = 0$ if $M = \mathbb{R}^n$ Euclidean, we can say that $R(X,Y)Z$ effectively measures, how much M differs from being *flat*. To see this just plug $Z = (z_1, ..., z_n) \in \mathbb{R}^n$ into Definition 2.1.

Another interpretation is possible, consider a coordinate basis $\{x_i\}$ around a point $p \in M$. Using $[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}] = 0$, the last term in (1) drops out and the *curvature* R can be interpreted as measuring the non-commutativity of the covariant derivative.

In the now following proposition we would like to study some important properties of the *curvature* R.

Proposition 2.2 (Properties of the Riemannian Curvature)**.** .

a) *Bilinearity in* $\mathfrak{X}(M) \times\ \mathfrak{X}(M)$*: Let* $f, g \in \mathscr{D}(M)$. $X_1, X_2, Y_1, Y_2 \in \mathfrak{X}(M)$

1. $R(fX_1 + gX_2, Y_1) = fR(X_1, Y_1) + gR(X_2, Y_1)$
2. $R(X_1, fY_1 + gY_2) = fR(X_1, Y_1) + gR(X_1, Y_2)$

b) *Linearity* $R(X,Y) : \mathfrak{X}(M) \to \mathfrak{X}(M)$*: Let* $X, Y, Z, W \in \mathfrak{X}(M)$ $f, g \in \mathscr{D}$

1. $R(X,Y)(Z + W) = R(X,Y)Z + R(X,Y)W$
2. $R(X,Y)(fZ) = fR(X,Y)Z$

Proof. As the proof of (b.) can be found in [dCa], we will focus only on bilinearity. It suffices to show that R satisfies

$$R(X_1 + X_2, Y)Z = R(X_1, Y)Z + R(X_2, Y)Z$$
$$R(fX, Y)Z = fR(X,Y)Z$$

The first equation is trivially true(inserting in the Definition 2.1), for the second

$$\begin{aligned}
R(fX,Y)Z &= \nabla_{fX}\nabla_Y Z - \nabla_Y \nabla_{fX} Z + \nabla_{[fX,Y]} Z \\
&= f\nabla_X\nabla_Y Z - \nabla_Y f \nabla_X Z + \nabla_{f[X,Y]-(Yf)X} Z \\
&= f\nabla_X\nabla_Y Z - f\nabla_Y\nabla_X Z - (Yf)\nabla_X Z + f\nabla_{[X,Y]}Z + (Yf)\nabla_X Z \\
&= fR(X,Y)Z
\end{aligned}$$

□

The next Proposition will be the famous *1. Bianchi identity*, named after his founder Luigi Bianchi (1902). It is valid for symmetric connections.(here *Levi-Civita connection*) .

Proposition 2.3 (1. Bianchi Identity)**.**

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$$

Proof. Just explicitly calculate the expression above,

$$\begin{aligned}
&R(X,Y)Z + R(Y,Z)X + R(Z,X)Y \\
&\qquad = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z + \nabla_{[X,Y]} Z \\
&\qquad + \nabla_Y\nabla_Z X - \nabla_Y\nabla_Z X + \nabla_{[Y,Z]} X \\
&\qquad + \nabla_X\nabla_Z Y - \nabla_Z\nabla_X Y + \nabla_{[Z,X]} Y \\
&\qquad = \nabla_Y[X,Z] + \nabla_Z[Y,X] + \nabla_X[Z,Y] \\
&\qquad - \nabla_{[X,Z]} Y - \nabla_{[Y,X]} Z - \nabla_{[Z,Y]} X \\
&\qquad = [Y,[X,Z]] + [Z,[Y,X]] + [X,[Z,Y]] \\
&\qquad = 0
\end{aligned}$$

The last equation is true due to *Jacobi identity* for vector fields. □

We will now get to know some useful symmetry properties of R. Be aware that from now on we will denote $\langle R(X,Y)Z,T\rangle = (X,Y,Z,T)$.

Proposition 2.4 (Useful identities of R)**.** *Let* $X,Y,Z,T \in \Xi(M)$

i) $(X,Y,Z,T) + (Y,Z,X,T) + (Z,X,Y,T) = 0$

ii) $(X,Y,Z,T) = -(Y,X,Z,T)$

iii) $(X,Y,Z,T) = -(Y,X,T,Z)$

iv) $(X,Y,Z,T) = (Z,T,X,Y)$

Proof. Before we begin proving all properties, we observe that *i)* is just the *first Bianchi identity* again and that *ii)* is just a consequence of our definition of the curvature endomorphism. We will now show *iii)* and *iv)* will follow from *i) - iii)*.

Observe that the third statment is equivalent to

$$\begin{aligned}(X,Y,Z,Z) &= 0 \\ &= \langle \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z, Z\rangle\end{aligned}$$

Note that the following holds

$$\begin{aligned}\langle \nabla_Y \nabla_X Z, Z\rangle &= Y\langle \nabla_X Z, Z\rangle - \langle \nabla_X Z, Z\rangle \text{ and,} \\ \langle \nabla_{[X,Y]} Z, Z\rangle &= \frac{1}{2}[X,Y]\langle Z,Z\rangle\end{aligned}$$

Using this and the product rule to calculate,

$$\begin{aligned}(X,Y,Z,Z) &= Y\langle \nabla_X Z, Z\rangle - X\langle \nabla_Y Z, Z\rangle + \frac{1}{2}[X,Y]\langle Z,Z\rangle \\ &= \frac{1}{2}Y(X,\langle Z,Z\rangle) - \frac{1}{2}X(Y,\langle Z,Z\rangle) + \frac{1}{2}[X,Y]\langle Z,Z\rangle \\ &= -\frac{1}{2}[X,Y]\langle Z,Z\rangle + \frac{1}{2}[X,Y]\langle Z,Z\rangle \\ &= 0\end{aligned}$$

To see that *iv)* holds we will use *i)* and perform cyclic permutations, leaving us with 4 equations. Summing up and using *iii)*, which we just showed, we come to the claimed equality:

$$\begin{aligned}(X,Y,Z,T) + (Y,Z,X,T) + (Z,X,Y,T) &= 0 \\ (T,X,Y,Z) + (T,X,Y,Z) + (T,X,Y,Z) &= 0 \\ (Z,T,X,Z) + (Z,T,X,Z) + (Z,T,X,Z) &= 0 \\ (Y,Z,X,T) + (Y,Z,X,T) + (Y,Z,X,T) &= 0 \\ \\ \Rightarrow 2(Z,X,Y,T) + 2(T,Y,Z,X) &= 0 \\ (Z,X,Y,T) = (Y,T,Z,X)\end{aligned}$$

□

We now try to find a expression for the *curvature* R locally in a coordinate system (U, $\boldsymbol{x}$) at point $p \in M$. The procedure is as usual denote $\frac{\partial}{\partial x_i} = X_i$. We put

$$R(X_i, X_j)X_k = R^l_{ijk} X_l$$

By R^l_{ijk} we mean the components of the curvature R expressed in the chart $(U, \boldsymbol{x})$. Our aim is now to express these components in terms of the *Christoffel Symbols* Γ^i_{jk}, the coefficients of the Riemannian connection. In order to do so, write:

$$X = u^i X_i,\ Y = v^j X_j,\ Z = w^k X_k$$

As we have already seen in Proposition 2.2, R is linear. Thus the components u^i, v^j, w^k can be pulled out of the expression above. We write:

$$R(X,Y)Z = R^l_{ijk} u^i v^j w^k X_l$$

We can now simply calculate:

$$\begin{aligned}
R(X_i, X_j)X_k &= \nabla_{X_j}\nabla_{X_i}X_k - \nabla_{X_i}\nabla_{X_j}X_k \\
&= \nabla_{X_j}(\Gamma^l_{ik}X_l) - \nabla_{X_i}(\Gamma^l_{jk}X_l) \\
&\overset{\text{pr}}{=} (\nabla_{X_j}\Gamma^l_{ik} - \nabla_{X_i}\Gamma^l_{jk})X_l \\
&+ (\Gamma^l_{ik}\Gamma^s_{jl} - \Gamma^l_{jk}\Gamma^s_{il})X_s \\
&= (\frac{\partial}{\partial x_j}\Gamma^s_{ik} - \frac{\partial}{\partial x_i}\Gamma^s_{jk} + \Gamma^l_{jk}\Gamma^s_{jl} - \Gamma^l_{jk}\Gamma^s_{il})X_s \\
\text{Thus we can identify: } \Rightarrow R^s_{ijk} &= \Gamma^l_{jk}\Gamma^s_{jl} - \Gamma^l_{jk}\Gamma^s_{il} + \frac{\partial}{\partial x_j}\Gamma^s_{ik} - \frac{\partial}{\partial x_i}\Gamma^s_{jk}
\end{aligned}$$

Please notice the double summation in the fourth equality. At this point it is important to emphazise, that Γ^i_{jk} is *not* a tensor on M, as the covariant derivative is not *linear* in all its arguments. However, we notice that the value of $R(X,Y)Z$ solely depends on the values of the function R^l_{ijk} at the point p, and the values of X,Y,Z at p. The curvature, being linear in all arguments, is a tensor. Thus, we conclude that the Christoffels in the last statement above must have been combined in such a way that the additional term in the transformation rule for Γ^i_{jk} drops out, which can be shown by direct calculation of the expression for R^i_{ijk}.

Put,

$$\langle R(X_i, X_j)X_k, X_s\rangle = R^l_{ijk}g_{ls} = R_{ijks}$$

we can now express the identies of Proposition 2.4 via:

$$\begin{aligned}
R_{ijks} + R_{jkis} + R_{kijs} &= 0 \\
R_{ijks} &= -R_{jiks} \\
R_{ijks} &= -R_{ijsk} \\
R_{ijks} &= R_{ksij}
\end{aligned}$$

2.2 Sectional curvature

We will now talk about the *sectional curvature*, an important simplification of the *Riemannian Curvature Tensor* as it completely determines it. As we want the *sectional curvature* K to be independent of our choice of coordinates, a normalisation factor is required:

Definition 2.5. Let V be vector space, $x, y \in V$

$$|x \wedge y| \equiv \sqrt{|x|^2|y|^2 - \langle x, y\rangle}$$

$|x \wedge y|$ is the area of a parallelogram in V spanned by x and y.

We now will show that the *sectional curvature* K, which we will define, is with Definition 2.5 as claimed, independent on the choice of the vectors x, y. We formulate:

Proposition 2.6. *Let $\sigma \subset T_pM$ be a two-dimensional subspace of the tangent space, let the vectors $x, y \in \sigma$ be linearly independent. Then,*

$$K(x,y) = \frac{(x,y,x,y)}{|x \wedge y|^2}$$

is independent of the choice of $x, y \in \sigma$.

Proof. We could explicitly calculate that for another basis $\{x', y'\} \in \sigma$, $K(x,y) = K(x',y')$. Being lazy, we observe that we can pass from the basis $\{x, y\}$ of σ to any other basis, denoted by $\{x', y'\}$, by simply iterating following transformations, however, it is clear that $K(x,y)$ stays invariant under these transformations.

1.) $\{x,y\} \to \{y,x\}$

2.) $\{x,y\} \to \{\lambda x, y\}$

3.) $\{x,y\} \to \{x + \lambda y, y\}$

In other words, when we change basis of σ both the numerator and denominator change by the square determinant of the transformation matrix. □

Finally we are able to define the *sectional curvature.*

Definition 2.7 (Sectional curvature)**.** At a fixed point $p \in M$ and a two-dimensional subspace lying in the tangent space, $\sigma \in T_pM$, $K(x,y) = K(\sigma) \in \mathbb{R}$, $\{x,y\}$ being any basis of σ, is called *sectional curvature* of σ at p.

As stated earlier, we will now show that indeed, if $K(\sigma)$ is known for all σ, R is completely determined. We will state this fact, in the following important Lemma:

Lemma 2.8. *V vector space with inner product $\langle\, , \rangle$, dim $V \geq 2$.*
We define the tri-linear mappings, $R : V \times V \times V \to V$, and $R' : V \times V \times V \to V$ in such a way that Proposition 2.4 holds by

$$(x,y,z,t) = \langle R(x,y)z, t\rangle, \ \ (x,y,z,t)' = \langle R'(x,y)z, t\rangle$$

In the case that x, y are linearly independent,

$$K(\sigma) = \frac{(x,y,x,y)}{|x \wedge y|^2}, \ \ K'(\sigma) = \frac{(x,y,x,y)'}{|x \wedge y|^2},$$

again σ is a two-dimensional subspace spanned by x, y.
If $\forall\, \sigma \subset V, K(\sigma) = K'(\sigma)$, then $R = R'$.

Proof. Observe that we only have to show that $\forall x, y, z, t \in V$ $(x,y,z,t) = (x,y,z,t)'$ holds. As $K(\sigma) = K'(\sigma)$ by hypothesis, the first line holds $\forall x, y \in V$

$$(x,y,x,y) = (x,y,x,y)'$$
$$(x+z, y, x+z, y) = (x+z, y, x+z, y)'$$

Using linearity of R and R':

$$(x,y,x,y)+2(x,y,z,y)+(z,y,z,y)=(x,y,x,y)'+2(x,y,z,y)'+(z,y,z,y)'$$
$$\Rightarrow (x,y,z,y)=(x,y,z,y)'$$

Using same trick as before and what we just obtained, but now in the second and fourth argument:

$$(x,y+t,z,y+t)=(x,y+t,z,y+t)'$$
$$\Rightarrow (x,y,z,t)+(x,t,z,y)=(x,y,z,t)'+(x,t,z,y)'$$
$$\overset{2.4.\text{iv})}{\Rightarrow} (x,y,z,t)-(x,y,z,t)'=(y,z,x,t)+(y,z,x,t)'$$

We realize that $(x,y,z,t)-(x,y,z,t)'$ is invariant under cyclic permutations of the first three arguments. Using now the *Bianchi identity* 2.4.i):

$$3[(x,y,z,t)-(x,y,z,t)']=0$$
$$\Rightarrow (x,y,z,t)=(x,y,z,t)' \qquad \forall\, x,z,y,t\in V$$

□

Now we want to characterize an important class of Riemannian Manifolds, those with constant curvature. We will do this by means of the components R_{ijkl} of the curvature in an orthonormal basis. Manifolds with constant curvature do play an important role in the development of Riemannian Geometry but also in physics. We will study them in the following Lemma and Corollary.

Lemma 2.9. *M is a Riemannian Manifold, p point of M.*
Be $R': T_pM\times T_pM\times T_pM\to T_pM$ a tri-linear mapping $\forall\, X,Y,W,Z\in T_pM$:

$$\langle R'(X,Y,W),Z\rangle=\langle X,W\rangle\langle Y,Z\rangle-\langle Y,W\rangle\langle X,W\rangle$$

M with curvature R *has constant* sectional curvature K_0 *if and only if $R=K_0R'$.*

Proof. Set $K(p,\sigma)=K_0\ \forall\,\sigma\in T_pM$, denote:

$$\langle R'(X,Y,W),Z\rangle=(X,Y,W,Z)'$$

The notation above makes sense since R' fullfills Proposition 2.4. Now for all pairs of vectors $X,Y\in T_pM$, we have:

$$(X,Y,X,Y)'=\langle X,X\rangle\langle Y,Y\rangle-\langle X,Y\rangle^2=|X\wedge Y|^2$$

Thus,

$$R(X,Y,X,Y)=K_0|X\wedge Y|^2=K_0R'(X,Y,X,Y)$$

Using Lemma 2.8, which implies that $\forall\, X,Y,W,Z$,

$$R(X,Y,W,Z)=K_0R'(X,Y,W,Z)$$
$$\Rightarrow R=K_0R'$$

The Converse is trivial. □

It is convenient to express a condition for $K(p, \sigma = K_0)$ in terms of the coefficients R_{ijkl}:

Corollary 2.10. *Let M be a n - dimensional Riemannian Manifold, p point of M. $\{e_1, ..., e_n\}$ is an orthonormal basis of the tangent space T_pM. The coefficients of the curvature R expressed in the basis: $R_{ijkl} = \langle R(e_i, e_j)e_k, e_l\rangle$, $i, j, k, l = 1, ..., n$. Then $K(p, \sigma) = const = K_0$ $\forall \sigma \in T_pM$, if and only if:*

$$R_{ijkl} = K_0(\delta_{ij}\delta_{kl} - \delta_{il}\delta_{jk})$$

So the searched expression is $K(p, \sigma) = K_0$, if and only if $R_{ijij} - R_{ijji} = K_0$ $\forall i \neq j$, $R_{ijkl} = 0$ else.

2.3 Ricci and scalar curvature

Another geometric object, with great importance especially for general relativity, is the *Ricci Curvature* and the *scalar curvature*, which in some sense condense the information encoded in the *curvature R*. We define:

Definition 2.11 (Ricci Curvature and scalar curvature)**.** Let M be an n-dim Riemannian Manifold, p a point on M. Consider a unit vector $x = z_n$ with $\{z_1, ..., z_{n-1}\}$ being the orthonormal basis of the hyperplane in T_pM orthogonal to x, then the *Ricci Curvature* is given by:

$$Ric_p(x) = \frac{1}{n-1}\sum_{i=1}^{n-1}\langle R(x, z_i)x, z_i\rangle = Ric_p(x, x)$$

and the *scalar curvature*K_s is given by:

$$K_s(p) = \frac{1}{n}\sum_{j=1}^{n} Ric_p(z_j) = \frac{1}{n(n-1)}\sum_{j=1}^{n}\sum_{i=1}^{n-1}\langle R(z_i, z_j)z_i, z_j\rangle$$

So the *Ricci Curvature* $Ric_p(x)$ is the sum of the sectional curvature of planes spanned by x and other elements of an orthonormal basis. Both do not $Ric_p(x)$ and $K_s(p)$ depend on the choice of the corresponding orthonormal basis, we will prove this by noting that $Ric_p(x, y) = \frac{1}{n-1}\sum_{i=1}^{n-1}\langle R(x, z_i)y, z_i\rangle$ as the trace of the linear map from $T_pM \to T_pM$.

Proof. Define the bilinear form Q on T_pM: $x, y \in T_pM$:

$$Q(x, y) = tr(z \to R(x, z)y)$$

Choose x to be a unit vector, then extend x to an orthonormal basis of T_pM $\{z_1, ..., z_{n-1}, z_n = x\}$. Then the trace of the mapping $z \to R(x, z)y$ is given by:

$$\begin{aligned} Q(x, y) &= \sum_{i=1}^{n-1}\langle R(x, z_i)y, z_i\rangle \\ &\overset{2.4.iv)}{=} \sum_{i=1}^{n-1}\langle R(y, z_i)x, z_i\rangle \\ &= Q(y, x) \end{aligned}$$

We note Q is symmetric, observe $Q(x,x) = (n-1)Ric_p(x)$; This completes the proof that $Ric_p(x)$ is intrinsically defined.
Notice Q, being a bilinear form, on T_pM corresponds to a linear and self- adjoint mapping K_s:

$$\langle K_s(x), y\rangle = Q(x,y)$$

And again, take an orthonormal basis $\{z_1, ..., z_n\}$, the trace of $K(x)$ is given now:

$$\begin{aligned}
tr(K_s) &= \sum_{j=1}^{n} \langle K(z_j), z_j\rangle \\
&= \sum_{j=1}^{n} Q(z_j, z_j) \\
&= (n-1)\sum_{j=1}^{n} Ric_p(z_j) \\
&= n(n-1)K_s(p)
\end{aligned}$$

This completes the proof. $\square$

(This procedure in general is called *contraction*. By this, we mean choosing two indices of a tensor of rank (a,b), one contra- and one covariant, set them equal and finally sum over. Thus, obtaining a tensor of rank $(a-1, b-1)$. The Ricci tensor emerges by contraction out of the curvature tensor. The Ricci Scalar by contraction of the Ricci Tensor)

As always, it is convenient to express the *Ricci Tensor*, the bilinear form $\frac{1}{n-1}Q$, locally in a coordinate system. As before $X_i = \frac{\partial}{\partial x_i}$, we will denote by $g_{ij} = \langle X_i, X_j\rangle$ and g^{ij} denotes the inverse of the matrix, $g_{ik}g^{kl} = \delta^i_j$. We now express the coefficients of the *Ricci Tensor* in the basis X_i:

$$\frac{1}{n-1}R_{ik} = \frac{1}{n-1}\sum_{j=1}^{n} R^j_{ijk} = \frac{1}{n-1}\sum_{s,j=1}^{n} R_{ijks}g^{sj}$$

$$K_s = \frac{1}{n(n-1)}\sum_{i,k=1}^{n} R_{ik}g^{ik}$$

For the last statement observe, that for $A : T_pM \to T_pM$, a linear, self-adjoint mapping and $B : T_pM \times T_pM \to \mathbb{R}$ the associated bilinear form ($B(X,Y) = \langle A(X), Y\rangle$), then $tr(A) = \sum_{ik} B(X_i, X_k)g^{ik}$.

This talk will conclude with an important Lemma, which will be useful in later talks.

Lemma 2.12. *Be f: $A \subset \mathbb{R}^2$ a parametrized surface, (s,t) the usual coordinates of $\mathbb{R}^2$, a vector field $V(s,t)$ along f. For each (s,t):*

$$\frac{D}{\partial t}\frac{D}{\partial s}V - \frac{D}{\partial s}\frac{D}{\partial t}V = R\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right)V$$

Proof. Calculate the left side of the equation, can be found in [dCa]. $\square$

References

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