An example of arithmetic groups

Seminar Differentialgeometrie: Geometry of Lie Groups
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1 Lattices in locally compact Hausdorff groups

The definition of a lattice in a topological group relies on the notion of *Haar measure*.

Definition 1 (Left invariant Haar measure)**.** Let G be a topological group. A *left invariant Haar measure* on G is a non-trivial, regular Borel measure μ on G such that

$$\mu(gE) = \mu(E)$$

for all Borel sets E in G and $g \in G$.

A *right invariant Haar measure* is defined in complete analogy, so that we have

$$\mu(Eg) = \mu(E)$$

for Borel sets E in G and $g \in G$. We have the following theorem for locally compact Hausdorff groups:

Theorem 2. *Let G be a locally compact Hausdorff group. Then there exists a left invariant Haar measure on G which is unique up to scalar multiples.*

Proof. [AM2012, Theorem 2.1.1] □

The same result also holds for right invariant Haar measures. In general the construction of Haar measure is rather technical and not very intuitive. In a few cases however we can give explicit descriptions of it:

Example 3. The Haar measure on $\mathbf{R}^n$ is the Lebesgue measure on $\mathbf{R}^n$ since the Lebesgue measure is a regular Borel measure invariant under translations.

Example 4. Let Γ be a discrete group. Then the measure which assigns to any set its cardinality is a left and right invariant Haar measure on Γ. It is called the *counting measure*.

Remark 5. If G is a Lie group a left invariant Haar measure can be constructed by choosing a non-zero left invariant n-form which induces a left invariant volume form on G. For more details see [AM2012, Section 2.1].

1.1 The modular function

In general a left invariant Haar measure is not necessarily a right invariant Haar measure.

Example 6. Let

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \mid a, b \in \mathbf{R} \text{ and } a \neq 0 \right\}.$$

By identifying G with the corresponding open set in the (a, b) plane in $\mathbf{R}^2$ a left invariant Haar measure on G is given by

$$\mu(E) = \int_E \frac{1}{a^2} da db$$

and a right invariant Haar measure by

$$\mu(E) = \int_E \frac{1}{|a|} da db$$

whereby E denotes a measurable set in G. The two measures are not constant multiples of each other, so in particular not every left invariant Haar measure on G is a right invariant Haar measure.

Now let G be a locally compact Hausdorff group, μ be a left invariant Haar measure on G and $g \in G$. Then the mapping

$$E \mapsto \mu(Eg)$$

defined on the algebra of Borel sets on G also defines a left invariant Haar measure on G. By the uniqueness part of Theorem 2 there exists a constant $\Delta_G(g)$ such that

$$\mu(Eg) = \Delta_G(g)\mu(E)$$

for all Borel sets E in G. This defines a function

$$\begin{aligned} \Delta_G : G &\to \mathbf{R}_{>0} \\ g &\mapsto \Delta_G(g) \end{aligned}$$

which is also a group homomorphism.

Definition 7. The function Δ_G is called the *modular function* of G. G is called *unimodular* if $\Delta_G \equiv 1$.

Proposition 8. *Compact and abelian topological groups are unimodular.*

Proof. The case of an abelian topological group is trivial. For the compact case see [AM2012, Corollary 2.2.2]. $\square$

1.2 Haar measure on quotients

Let G be a locally compact Hausdorff group and H a closed subgroup of G. Then G/H equipped with the quotient topology is again a locally compact Hausdorff space.

Definition 9. A *left G-invariant Haar measure* on G/H is a regular Borel measure μ on G/H such that

$$\mu(gE) = \mu(E)$$

for all $g \in G$ and all Borel sets E in G/H.

We may now define the notion of a lattice:

Definition 10. Let G be a locally compact Hausdorff group. A discrete subgroup Γ of G such that G/Γ carries a finite left G-invariant Haar measure is called a *lattice* in G.

In general, we have the following theorem for left G-invariant Haar measures:

Theorem 11. *A left G-invariant Haar measure on G/H is unique up the scalar multiples and exists if and only if*

$$\Delta_G(h) = \Delta_H(h)$$

for alle $h \in H$.

Proof. [AM2012, Theorem 2.3.5]. □

As a consequence of the previous theorem one can show that if a locally compact Hausdorff group admits a lattice it must be unimodular (cf. [AM2012, Proposition 2.4.2]).

2 $\mathrm{SL}_n(\mathbf{Z})$ is a lattice in $\mathrm{SL}_n(\mathbf{R})$

Proposition 12. $\mathrm{SL}_n(\mathbf{Z})$ *is a discrete subgroup of* $\mathrm{SL}_n(\mathbf{R})$*.*

Proof. As the topology on $\mathrm{SL}_n(\mathbf{R})$ originates from the Euclidean topology, $\mathrm{SL}_n(\mathbf{Z})$ must be discrete. Furthermore, the inverse of an integer matrix g in $\mathrm{GL}_n(\mathbf{R})$ is again an integer matrix if and only if $|\det(g)| = 1$, so $\mathrm{SL}_n(\mathbf{Z})$ is also a group. □

To show that $\mathrm{SL}_n(\mathbf{R})/\mathrm{SL}_n(\mathbf{Z})$ has finite volume we will first introduce the Iwasawa decomposition of $\mathrm{SL}_n(\mathbf{R})$.

2.1 The Iwasawa decomposition of $\mathrm{SL}_n(\mathbf{R})$

Let $g \in \mathrm{SL}_n(\mathbf{R})$ and denote by $v_1, \dots, v_n$ the column vectors of g. Following the Gram-Schmidt process, we inductively define for each $i = 1, \dots, n$ a vector v_i' such $v_i' \perp v_j'$ for $j = 1, \dots, i-1$ and

$$v_i' = v_i + \sum_{j=1}^{i-1} u_{ij} v_j$$

with $u_{ij} \in \mathbf{R}$. The u_{ij} are uniquely determined and give rise to the the unipotent matrix

$$\tilde{u} = \begin{pmatrix} 1 & u_{12} & \dots & \dots & u_{1n} \\ 0 & 1 & u_{23} & \dots & u_{2n} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 & u_{n-1n} \\ 0 & \dots & \dots & 0 & 1 \end{pmatrix}.$$

If we set

$$\tilde{a} = \begin{pmatrix} \frac{1}{|v_1'|} & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \frac{1}{|v_n'|} \end{pmatrix},$$

then the columns of the matrix $g\tilde{u}\tilde{a}$ form an orthonormal basis of $\mathbf{R}^n$, i. e. we have $g\tilde{u}\tilde{a} \in \mathrm{O}(n)$. For the determinant of $\tilde{a}$ we obtain

$$\begin{aligned} |\det(\tilde{a})| &= |\det(g)||\det(\tilde{u})||\det(\tilde{a})| \\ &= |\det(g)\det(\tilde{u})\det(\tilde{a})| \\ &= |\det(g\tilde{u}\tilde{a})| \\ &= 1 \end{aligned}$$

and as all entries of $\tilde{a}$ are strictly positive we must have $\det(\tilde{a}) = 1$, so in particular we have $g\tilde{u}\tilde{a} \in \mathrm{SO}(n)$. If we set $k := g\tilde{u}\tilde{a}, a := \tilde{a}^{-1}$ and $u := \tilde{u}^{-1}$ we get the decomposition

$$g = kau,$$

which is called the *Iwasawa decomposition of* g.

Proposition 13 (Iwasawa decomposition)**.** *For every element $g \in \mathrm{SL}_n(\mathbf{R})$ there are unique elements $k \in \mathrm{SO}(n)$, a a diaongal matrix in $\mathrm{SL}_n(\mathbf{R})$ with positive entries and n an upper triangular matrix with 1's on the diagonal such that*

$$g = kau.$$

2.2 $\mathrm{SL}_n(\mathbf{R}) = \mathcal{S}_{\frac{2}{\sqrt{3}},\frac{1}{2}}\,\mathrm{SL}_n(\mathbf{Z})$

In this section we will prove that $\mathrm{SL}_n(\mathbf{R}) = \mathcal{S}_{\frac{2}{\sqrt{3}},\frac{1}{2}}\,\mathrm{SL}_n(\mathbf{Z})$, whereby $\mathcal{S}_{t,C}$ denotes the Siegel set for the parameters t and C.

Definition 14 (Siegel sets)**.** Given $t > 0$ and $C > 0$ the *Siegel set* $\mathcal{S}_{t,C}$ is the set of elements g in $SL_n(\mathbf{R})$ such that

$$\frac{a_i}{a_{i+1}} \leq t \text{ for all } i = 1, \dots, n-1 \text{ and } |u_{ij}| \leq C \text{ for all } 1 \leq i < j \leq n$$

if $g = kau$ is the Iwasawa decomposition of g.

To do so, we will require some results for lattices in $\mathbf{R}^n$.

2.2.1 Lattices in $\mathbf{R}^n$

Proposition 15. *Let $L \subset \mathbf{R}^n$ be a lattice. Then there exists a basis $v_1, \dots, v_n$ of $\mathbf{R}^n$ such that L is the integral linear span of $v_1, \dots, v_n$, i. e. $L = \mathbf{Z}v_1 + \dots + \mathbf{Z}v_n$.*

Proof. Let $L \subset \mathbf{R}^n$ be a lattice and denote by W its $\mathbf{R}$-linear span in $\mathbf{R}^n$. As L is lattice, $\mathbf{R}^n/L$ must have finite volume and therefore this must also be true for $\mathbf{R}^n/W$. As $\mathbf{R}^n/W$ is a vector space this can only be true in case $\dim \mathbf{R}^n/W = 0$, i. e. we must have $\mathbf{R}^n/W = \{0\}$ which means $W = \mathbf{R}^n$.

Now let $w_1, \ldots, w_n$ be a basis of $\mathbf{R}^n$ contained in L and denote by L' its integral span, i. e. we have

$$L' = \mathbf{Z}w_1 + \ldots + \mathbf{Z}w_n.$$

As L is a subgroup of $\mathbf{R}^n$, L' is contained in L and as an integral span it is also a lattice in $\mathbf{R}^n$.

As a consequence L/L' must be finite, so L must be finitely generated and torsionfree. Therefore, L must be a free abelian group on k generators. As L' has finite index in L and is free on n generators we have $k = n$.

As the linear span of L is $\mathbf{R}^n$ it follows that L must the integral span of a basis of $\mathbf{R}^n$. □

Lemma 16. *Let $L \subset \mathbf{R}^n$ be a lattice, then there exists an integral basis $v_1, \ldots, v_n$ of L such that*

$$\|v_i'\|^2 \geq \frac{3}{4}\|v_{i-1}\|^2,$$

with v_i' denoting the orthogonal projection of v_i onto the orthogonal complement of the span of $v_1, \ldots, v_{i-1}$.

Proof. First let $n = 2$. Let $v_1 \in L$ be a non-zero element of smallest norm and $v_2 \in L \backslash \mathbf{Z}v_1$. Then v_1 and v_2 form an integral basis of L. We have $v_2 = v_2' + \lambda v_1$ for some $\lambda \in \mathbf{R}$. Choose $a \in \mathbf{Z}$ such, that $\mu = \lambda - a \in [-\frac{1}{2}, \frac{1}{2}]$. Then the projection of $w = v_2 - av_1$ to the orthogonal complement $v_1^\perp$ is v_2' and w lies in L.

Since v_1 has smallest norm by hypothesis, the norm of $v_2 - av_1$ is bounded from below by $\|v_1\|$. This yields

$$\|v_1\|^2 \leq \|w\|^2 = \|v_2'\|^2 + |\mu|\|v_1\|^2 \leq \|v_2'\|^2 + \frac{1}{4}\|v_1\|^2,$$

which implies

$$\|v_2'\|^2 \geq \frac{3}{4}\|v_1\|^2,$$

which is exactly the claimed estimate.

Now assume that the statement has been proven for $n-1$ basis vectors. Let v_1 be an element of L of the smallest norm. As L is isomorphic to $\mathbf{Z}^n$, v_1 must be part of an integral basis of L. The projection of L to $v_1^\perp$ is again a lattice L' (cf. [Ven2010, Lemma 5]). By the induction hypothesis we find an integral basis $v_2', \ldots, v_n'$ of L' satisfying the asserted inequalities. Let $v_2, \ldots, v_n \in L$ be such that v_i is projected to v_i'. Then $v_1, \ldots, v_n$ is an integral basis of L and satisfies the inequalities of the lemma. □

With this background we may now prove the following theorem:

Theorem 17. *Given an element $g \in \mathrm{SL}_n(\mathbf{R})$ there exists an element $\gamma \in \mathrm{SL}_n(\mathbf{Z})$ such that*

$$g\gamma \in \mathcal{S}_{\frac{2}{\sqrt{3}}, \frac{1}{2}}.$$

Proof. By identifying an element $g \in \mathrm{GL}_n(\mathbf{R})$ with its columns $v_1, \ldots, v_n$ we can view the elements of $\mathrm{GL}_n(\mathbf{R})$ as bases for $\mathbf{R}^n$. Taking the integral span of $v_1, \ldots, v_n$ we obtain a map

$$\mathrm{GL}_n(\mathbf{R}) \to \mathcal{L},$$

where $\mathcal{L}$ denotes the space of lattices in $\mathbf{R}^n$. Restricting this map to $\mathrm{SL}_n(\mathbf{R})$ we obtain a map

$$\mathrm{SL}_n(\mathbf{R}) \to \mathcal{L}^0$$

where $\mathcal{L}^0$ denotes the space of unimodular lattices in $\mathbf{R}^n$, i. e. the lattices in $\mathbf{R}^n$ with covolume 1. By Proposition 15 this map must further be surjective. The base changes of a lattice in $\mathbf{R}^n$ are exactly the the linear transformations of the basis vectors arising from elements of $\mathrm{SL}_n(\mathbf{Z})$, so in particular we obtain an isomorphism

$$\mathrm{SL}_n(\mathbf{R})/\mathrm{SL}_n(\mathbf{Z}) \to \mathcal{L}^0.$$

Now let $g \in \mathrm{SL}_n(\mathbf{R})$ and L be the lattice in $\mathbf{R}^n$ given by the integral linear span of the columns of g. By Lemma 16 there exists a basis $v_1, \dots, v_n$ of L such that

$$|v'_{i+1}| \geq \frac{3}{4}|v_i|^2$$

for all $1 \leq i \leq n-1$. By the isomorphism above there must be a $\delta \in \mathrm{SL}_n(\mathbf{Z})$ such that $g\delta$ has the Iwasawa decomposition

$$g\delta = kav$$

with

$$\frac{a_i}{a_{i+1}} = \frac{\|v'_i\|}{\|v'_{i+1}\|} \leq \frac{\|v_i\|}{\|v'_{i+1}\|} \leq \frac{2}{\sqrt{3}}$$

for all $i = 1, \dots, n-1$. Now given an arbitrary element $v \in N$ we can always find an element $\theta \in N$ with all integer entries sucht that

$$v\theta = 1 + \sum_{i<j} u_{ij}E_{ij} \text{ and } |u_{ij}| < \frac{1}{2} \text{ for all } i<j.$$

This can be verified by an induction over the size of the matrices. Setting $\gamma = \delta\theta$ we obtain an element of $\mathrm{SL}_n(\mathbf{Z})$ such that

$$g\gamma \in \mathcal{S}_{\frac{2}{\sqrt{3}},\frac{1}{2}}.$$

□

In order to show that $\mathrm{SL}_n(\mathbf{Z})/\mathrm{SL}_n(\mathbf{R})$ has finite volume it will now suffice to show that Siegel sets have finite volume.

2.3 Siegel sets

Before examining the volume of Siegel sets we will calculate the Haar measure on $\mathrm{SL}_n(\mathbf{R})$.

2.3.1 Haar measure on $\mathrm{SL}_n(\mathbf{R})$

We will first show that $\mathrm{SL}_n(\mathbf{R})$ is unimodular. We have the togologial group isomorphism

$$\mathrm{GL}_n(\mathbf{R})^+ \xrightarrow{\sim} \mathrm{SL}_n(\mathbf{R}) \times \mathbf{R}^+$$
$$g \mapsto \left(\sqrt[n]{\det(g)}g, \det(g)\right)$$

with $\mathrm{GL}_n(\mathbf{R})^+$ denoting the subgroup of $\mathrm{GL}_n(\mathbf{R})$ of matrices with positive determinant. Both $\mathrm{GL}_n(\mathbf{R})^+$ and $\mathbf{R}^+$ are unimodular with Haar measures

$$d\mu(g) = \frac{dg}{(\det g)^n}.$$

on $\mathrm{GL}_n(\mathbf{R})^+$ and

$$d\mu(g) = \frac{dg}{|g|}.$$

on $\mathbf{R}^+$. We have the following general result:

Lemma 18. *Let A and B be topological groups with left Haar measures da and db, respectively. Then the left Haar measure on $A \times B$ is $da \times db$. Furthermore if Δ_A and Δ_B are the modular functions on A and B, then the modular function on $A \times B$ is given by $\Delta_{A\times B}(a,b) = \Delta_A(a)\Delta_B(b)$ for all $(a,b) \in A \times B$.*

Proof. The product measure $da \times db$ defines a left invariant regular Borel measure on $A \times B$ and must therefore be the left invariant Haar measure. It is then easy to verify that the modular function of $A \times B$ is just the product of the modular functions of A and B. □

As a consequence we get

Corollary 19. *The group $\mathrm{SL}_n(\mathbf{R})$ is unimodular.*

Now let $K = SO(n)$, A be the group of diagonal matrices in $SL_n(\mathbf{R})$ with positive entries and N the group of upper triangular matrices with 1's on the diagonal. By the Iwasawa decomposition we have a homeomorphism

$$SL_n(\mathbf{R}) \to KNA,$$

which induces a homeomorphism

$$\mathrm{SL}_n(\mathbf{R})/K \to NA.$$

By Corollary 19 and Proposition 8 the groups $\mathrm{SL}_n(\mathbf{R})$ and K are unimodular, so there exists an $\mathrm{SL}_n(\mathbf{R})$-invariant measure on $\mathrm{SL}_n(\mathbf{R})/K$ by Theorem 11. Using [AM2012, Proposition 2.3.6] with the homeomorphism above this measure can be pushed forward to a left invariant Haar measure on NA.

By the characterisation of the Haar measure on quotients in [AM2012, Theorem 2.3.5] we then obain that

$$dg = dk d(na)$$

with dk denoting the Haar measure on K and $d(na)$ denoting the Haar measure on NA. As N and A are unimodular groups (A is abelian and for N see [AM2012, Exercise 2.1.6]) and A normalizes N, i. e. $ana^{-1} \in N$ for all $a \in A$ and $n \in N$, we get by [Lan1985, Page 40] that

$$d(na) = dnda,$$

with dn denoting the Haar measure on N and da the Haar measure on A. So the Haar measure dg on $\mathrm{SL}_n(\mathbf{R})$ is given by

$$dg = dkdnda.$$

2.3.2 $\mathcal{S}_{t,C}$ has finite volume

Proposition 20. *Let $t, C > 0$. The volume of the Siegel set $\mathcal{S}_{t,C}$ in $\mathrm{SL}_n(\mathbf{R})$ is finite.*

Proof. For $g \in \mathcal{S}_{t,C}$ let $g = kau$ be the Iwasawa decomposition. Setting $v = aua^{-1}$, v is an upper triangular matrix of the form

$$v = 1 + \sum_{i<j} \frac{a_i}{a_j} E_{ij}.$$

In k, v, a coordinates the Haar measure on dg on $\mathrm{SL}_n(\mathbf{R})$ is given by

$$dg = \left(\prod_{i<j} \frac{a_i}{a_j} \right) dkdvda.$$

The volume of $\mathcal{S}_{t,C}$ is then given by

$$\left(\int\limits_{\mathrm{SO}(n)} dk\right)\left(\int\limits_{\frac{a_i}{a_{i+1}}\leq t}\left(\prod_{i<j}\frac{a_i}{a_j}\right)\frac{da_1}{a_1}\cdots\frac{da_{n-1}}{a_{n-1}}\right)\left(\int\limits_{|u_{ij}|\leq C} du_{ij}\right).$$

As $\mathrm{SO}(n)$ is compact the integral $\int_{\mathrm{SO}(n)} dk$ is finite. Furthermore as $|u_{ij}| \leq C$ for all $i < j$ the integral $\int_{|u_{ij}|\leq C} du_{ij}$ must also be finite. For the remaing integral consider the change of variables

$$\alpha_1 = \frac{a_1}{a_2}, \ldots, \alpha_{n-1} = \frac{a_{n-1}}{a_n}.$$

This yields

$$\begin{aligned}\int\limits_{\frac{a_i}{a_{i+1}}\leq t}\left(\prod_{i<j}\frac{a_i}{a_j}\right)\frac{da_1}{a_1}\cdots\frac{da_{n-1}}{a_{n-1}} &= \int\limits_{\alpha_i\leq t}\alpha_1^{m_1}\cdots\alpha_{n-1}^{m_{n-1}}d\alpha_1\cdots d\alpha_{n-1}\\ &= \prod_{i=1}^{n-1}\int\limits_0^t \alpha_i^{m_i}d\alpha_i\end{aligned}$$

for non-negative integers m_i, so in particular this integral must also be finite which proves the assertion. □

2.4 $\mathrm{SL}_n(\mathbf{Z})$ is a non-cocompact lattice in $\mathrm{SL}_n(\mathbf{R})$

As we have

$$\mathrm{SL}_n(\mathbf{Z}) = \mathcal{S}_{\frac{2}{\sqrt{3}},\frac{1}{2}}\mathrm{SL}_n(\mathbf{Z})$$

and

$$\mathrm{vol}(\mathcal{S}_{\frac{2}{\sqrt{3}},\frac{1}{2}}) < \infty$$

the quotient $\mathrm{SL}_n(\mathbf{R})/\mathrm{SL}_n(\mathbf{Z})$ must have finite volume, so with Proposition 12 we get that $\mathrm{SL}_n(\mathbf{Z})$ is a lattice in $\mathrm{SL}_n(\mathbf{R})$. Furthermore $\mathrm{SL}_n(\mathbf{R})/\mathrm{SL}_n(\mathbf{Z})$ is not compact. This is a consequence of the following theorem:

Theorem 21 (Mahler Criterion)**.** *A sequence $(g_m) \in \mathrm{SL}_n(\mathbf{R})/\mathrm{SL}_n(\mathbf{Z})$ does not have a convergent subsequence if and only if there exists a sequence $(v_m) \in \mathbf{Z}^n$ with $(v_m) \neq 0$ and $g_m(v_m) \to 0$ for $m \to \infty$.*

Proof. [Ven2010, Theorem 14]. □

So for example if we take for the sequence (g_m) the diagonal matrices

$$g_m = (2^{-m}, 2^m, 1, \ldots, 1)$$

and $v_m = (1, 0, \ldots, 0)$ for all $m \in \mathbf{N}$ then $g_m(v_m)$ tends to zero. By Theorem 21 then, the sequence (g_m) can not have a convergent subsequence, so in particular $\mathrm{SL}_n(\mathbf{R})/\mathrm{SL}_n(\mathbf{Z})$ can not be compact.

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