Jacobi fields

Introduction to Riemannian Geometry
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July 15, 2015

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In the following let M always denote a Riemannian manifold of dimension n.

1 The Jacobi equation

Jacobi fields provide a means of describing how fast the geodesics starting from a given point $p \in M$ tangent to $\sigma \subset T_pM$ spread apart, in particular we will see that the spreading is determined by the curvature $K(p, \sigma)$.

To understand the abstract definition of a Jacobi field as a vector field along a geodesic satisfying a special differential equation we will first have another look at the exponential mapping introduced in [Car92][Chap. 3, Sec. 2].

Observation 1. Let $p \in M$ and $v \in T_pM$ so, that $\exp_p$ is defined at $v \in T_pM$. Furthermore let $f : [0,1] \times [-\varepsilon, \varepsilon] \to M$ be the parametrized surface given by

$$f(t,s) = \exp_p tv(s)$$

with $v(s)$ being a curve in T_pM satisfying $v(0) = v$. In the proof of the Gauss Lemma ([Car92][Chap. 3, Lemma 3.5]) we have seen that

$$\frac{\partial f}{\partial s}(t,0) = (d\exp_p)_{tv(0)}(tv'(0)). \tag{1}$$

We now want to examine the thusly induced vector field $J(t) = \frac{\partial f}{\partial s}(t,0)$ along the geodesic $\gamma(t) = \exp_p(tv), 0 \leq t \leq 1$. Since γ is a geodesic we must have $\frac{D}{\partial t}\frac{\partial f}{\partial t} = 0$ for all (t,s). Using this and [Car92][Chap. 4, Lemma 4.1] we get

$$\begin{aligned} 0 = \frac{D}{\partial s}\left(\frac{D}{\partial t}\frac{\partial f}{\partial t}\right) &= \frac{D}{\partial t}\frac{D}{\partial s}\frac{\partial f}{\partial t} - R\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right)\frac{\partial f}{\partial t} \\ &= \frac{D}{\partial t}\frac{D}{\partial t}\frac{\partial f}{\partial s} + R\left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}\right)\frac{\partial f}{\partial t}. \end{aligned}$$

which shows that J statisfies the differential equation $\frac{D^2 J}{dt^2} + R(\gamma'(t), J(t))\gamma'(t) = 0$.

We now use the just obtained differential equation to define Jacobi fields along a geodesic:

Definition 2. Let $\gamma : [0, a] \to M$ be a geodesic in M. A vector field J along γ is called a *Jacobi field* if it satisfies the *Jacobi equation*

$$\tfrac{D^2}{dt^2} J(t) + R(\gamma'(t), J(t))\gamma'(t) = 0$$

for all $t \in [0, a]$.

Let us first observe an important property of Jacboi fields:

Proposition 3. *Let $\gamma : [0, a] \to M$ be a geodesic in M and J a Jacobi field along γ. Then J is determined by its initial conditions $J(0)$ and $\frac{DJ}{dt}(0)$.*

Proof. We first choose an orthonormal basis $\{e_i\}_{i=1,\dots,n}$ of T_pM which we can extend to parallel, orthonormal fields $e_1(t), \dots, e_n(t)$ along γ. Then for the Jacobi field J we get

$$J(t) = \sum_{i=1}^{n} f_i(t) e_i(t)$$

for smooth functions $f_i \in C^\infty, i = 1, \dots, n$. This yields

$$\frac{D^2}{dt^2} J(t) = \sum_{i=1}^{n} f_i^{''}(t) e_i(t)$$

and

$$R(\gamma', J)\gamma' = \sum_{i=1}^{n} \langle R(\gamma', J)\gamma', e_i\rangle e_i = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} f_j \langle R(\gamma', e_j)\gamma', e_i\rangle \right) e_i$$

so the Jacobi equation is equal to the system

$$f_i'' + \sum_{j=1}^{n} f_j \langle R(\gamma', e_j)\gamma', e_i\rangle = 0 \text{ for all } i = 1, \dots, n.$$

Since this is a linear system of the second order, given the initial conditions $J(0)$ and $\frac{DJ}{dt}(0)$ there exists a C^∞ solution of the system defined on $[0, a]$. $\square$

Before considering a first example we make two remarks:

Remark 4. The proof of Proposition 3 showed that the Jacobi equation is equal to a linear system of the second order. For a given geodesic γ there exist therefore exactly $2n$ linearly independent Jacobi fields along γ.

Remark 5. Let γ be a geodesic. Then we have

$$\frac{D^2}{dt^2} \gamma'(t) + R(\gamma'(t), \gamma'(t))\gamma'(t) = 0 + 0 = 0,$$

so γ' defines a Jacobi field along γ. Note that, as γ is a geodescis, the field γ' never vanishes. Analogous one can see that the field $t \mapsto t\gamma'(t)$ is also a Jacobi field along γ which vanishes if and only if $t = 0$.

We will now give an example of Jacobi fields on manifolds of constant curvature:

Example 6 (Jacobi fields on manifolds of constant curvature)**.** Let M be a Riemannian manifold of constant sectional curvature K and let $\gamma : [0, l] \to M$ be a normalized geodesic on M. Further let J be a Jacobi field along γ, normal to γ'.

Applying [Car92][Chap. 4, Lemma 3.4] and using the fact that $|\gamma'| = 1$ as well as that J is normal to γ' we have for every vector field T along γ

$$\langle R(\gamma', J)\gamma', T\rangle = K\left\{\langle \gamma', \gamma'\rangle\langle J, T\rangle - \langle \gamma', T\rangle\langle J, \gamma'\rangle\right\} = K\langle J, T\rangle = \langle KJ, T\rangle,$$

which yields

$$R(\gamma', J)\gamma' = KJ, \tag{2}$$

i. e. the Jacobi equation is of the form

$$\frac{D^2}{dt^2}J + KJ = 0.$$

Now let $w(t)$ be a parallel field along γ with $\langle \gamma'(t), w(t)\rangle = 0$ and $|w(t)| = 1$ for all $t \in [0, l]$. Then

$$J(t) = \begin{cases} \frac{\sin(t\sqrt{K})}{\sqrt{K}} w(t), & \text{if } K > 0, \\ tw(t) & \text{if } K = 0, \\ \frac{\sinh(t\sqrt{-K})}{\sqrt{-K}} w(t) & \text{if } K < 0 \end{cases}$$

is a solution for Equation (2) with initial conditions $J(0) = 0$ and $J'(0) = w(0)$. This can be easily verified, for example in the case $K > 0$ we have

$$\begin{aligned} \frac{D^2}{dt^2}J(t) + KJ(t) &= \frac{D}{dt}\left(\cos(t\sqrt{K})w(t) + \frac{\sin(t\sqrt{K})}{\sqrt{K}}\underbrace{\frac{D}{dt}(w(t))}_{=0}\right) + K\frac{\sin(t\sqrt{K})}{\sqrt{K}}w(t) \\ &= -\sqrt{K}\sin(t\sqrt{K})w(t) + \sqrt{K}\sin(t\sqrt{K})w(t) \\ &= 0 \end{aligned}$$

for all $t \in [0, l]$ just as desired.

1.1 The form of Jacobi fields

So far we have seen one systematic way to construct a Jacobi field along a geodesic using the exponential mapping. The next proposition and corollary will show that all possible Jacobi fields J with $J(0) = 0$ along a geodesics are essentially of this form.

Proposition 7. *Let $\gamma : [0, a] \to M$ be a geodesic and let J be a Jacobi field along γ with $J(0) = 0$. Put $\frac{DJ}{dt}(0) = w$ and $\gamma'(0) = v$. Consider w as an element of $T_{av}\left(T_{\gamma(0)}M\right)$ and construct a curve $v(s)$ in $T_{\gamma(0)}M$ with $v(0) = av$ and $v'(0) = aw$. Put $f(t, s) = \exp_p(\frac{t}{a}v(s)), p = \gamma(0)$ and define a Jacobi field $\overline{J}$ by $\overline{J}(t) = \frac{\partial f}{\partial s}(t, 0)$. Then $\overline{J} = J$ on $[0, a]$.*

Proof. According to Proposition 3 it suffices to show that $J(0) = \overline{J}(0)$ and $\frac{DJ}{dt}(0) = \frac{D\overline{J}}{dt}$. Using Equation (1) we have

$$\overline{J}(t) = \frac{\partial f}{\partial s}(t, 0) = (d\exp_p)_{tv}(tw), \tag{3}$$

which yields $\overline{J}(0) = (d\exp_p)_0(0) = 0 = J(0)$ as we have $(d\exp_p)_0(w) = w$ for all w as seen in the proof of [Car92][Chapter 3, Proposition 2.9].

Furthermore we have

$$\begin{aligned}\frac{D}{dt}\frac{\partial f}{\partial s}(t,0) &= \frac{D}{dt}((d\exp_p)_{tv}(tw)) = \frac{D}{dt}(t(d\exp_p)_{tv}(w))\\ &= (d\exp_p)_{tv}(w) + t\frac{D}{dt}((d\exp_p)_{tv}(w)),\end{aligned}$$

so we get

$$\frac{D\overline{J}}{dt}(0) = \frac{D}{dt}\frac{\partial f}{\partial s}(0,0) = (d\exp_p)_0(w) = w = \frac{DJ}{dt}(0)$$

just as desired. □

Corollary 8. *Let $\gamma : [0,a] \to M$ be a geodesic. If J is a Jacobi field along γ with $J(0) = 0$ then we have*

$$J(t) = (d\exp_p)_{t\gamma'(0)}\left(tJ'(0)\right).$$

Proof. The statement follows immediately from Proposition 7 and Equation (3). □

An analogous construction than that in Proposition 7 can also be obtained for Jacobi fields J that do not satisfy $J(0) = 0$. For details see [Car92][Chap. 5, Exercise 2].

1.2 Relationship between the spreading of geodesics and curvature

We will now use the previously introduced Jacobi fields to obtain a relationship between the spreading of geodesics originating from the same point and the curvature at this point.

Proposition 9. *Let $p \in M$ and $\gamma : [0,a] \to M$ be a geodesic with $\gamma(0) = p, \gamma'(0) = v$. Let $w \in T_v(T_pM)$ with $|w| = 1$ and let J be a Jacobi field along γ given by*

$$J(t) = (d\exp_p)_{tv}(tw).$$

Then the Taylor expansion of $|J(t)|^2$ about $t = 0$ is given by

$$|J(t)|^2 = t^2 - \frac{1}{3}\langle R(v,w)v, w\rangle t^4 + R(t)$$

where $\lim_{t\to 0}\frac{R(t)}{t^4} = 0$.

Proof. We have $J(0) = (d\exp_p)_0(0) = 0$ and $J'(0) = w$ hence the first three coefficients of the Taylor expansion are

$$\begin{aligned}\langle J,J\rangle(0) &= \langle J(0),J(0)\rangle = \langle 0,0\rangle = 0\\ \langle J,J\rangle'(0) &= \left(\langle J',J\rangle + \langle J,J'\rangle\right)(0) = 2\langle J,J'\rangle(0) = 2\langle J(0),J'(0)\rangle = 2\langle 0,w\rangle = 0,\\ \langle J,J\rangle''(0) &= 2\langle J',J'\rangle(0) + 2\langle J'',J\rangle(0) = 2\langle w,w\rangle = 2\end{aligned}$$

As J is a Jacobi field we have $J''(0) = -R(\gamma',J)\gamma'(0) = 0$ which yields

$$\langle J,J\rangle'''(0) = 6\langle J',J''\rangle(0) + 2\langle J''',J\rangle(0) = 0$$

To calculate the fourth coefficient observe that

$$\frac{D}{dt}\left[R(\gamma',J)\gamma'\right](0) = R(\gamma',J')\gamma'(0),$$

since for any W we have

$$\frac{d}{dt}\langle R(\gamma',W)\gamma',J\rangle = \frac{d}{dt}\langle R(\gamma',J)\gamma',W\rangle = \langle\frac{D}{dt}R(\gamma',J)\gamma',W\rangle + \langle R(\gamma',J)\gamma',W'\rangle,$$

so we get

$$\begin{aligned}
\langle \frac{D}{dt}(R(\gamma', J)\gamma'), W\rangle &= \frac{d}{dt}\langle R(\gamma', W)\gamma', J\rangle - \langle R(\gamma', J)\gamma', W'\rangle \\
&= \langle \frac{D}{dt} R(\gamma', W)\gamma', J\rangle + \langle R(\gamma', W)\gamma', J'\rangle - \langle R(\gamma', J)\gamma', W'\rangle \\
&= \langle R(\gamma', J')\gamma', W\rangle + \langle \frac{D}{dt} R(\gamma', W)\gamma', J\rangle - \langle R(\gamma', J)\gamma', W'\rangle
\end{aligned}$$

which for $t = 0$ yields the desired identity. Together with the Jacobi equation we obtain that $J'''(0) = -R(\gamma', J')\gamma'(0)$ so we get

$$\begin{aligned}
\langle J, J\rangle''''(0) &= 8\langle J', J'''\rangle(0) + 6\langle J'', J''\rangle(0) + 2\langle J'''', J\rangle(0) \\
&= -8\langle J', R(\gamma', J')\gamma'\rangle(0) \\
&= -8\langle R(v, w)v, w\rangle
\end{aligned}$$

just as desired. □

From Proposition 9 we can now draw an important corollary:

Corollary 10. *If $\gamma : [0, l] \to M$ is parametrized by arc length, and $\langle w, v\rangle = 0$, the expression $\langle R(v, w)v, w\rangle$ is the sectional curvature at p with respect to the plane σ generated by v and w. Therefore in this situation*

$$|J(t)|^2 = t^2 - \frac{1}{3}K(p, \sigma)t^4 + R(t)$$

and

$$|J(t)| = t - \frac{1}{6}K(p, \sigma)t^3 + \tilde{R}(t) \text{ with } \lim_{t\to 0} \frac{\tilde{R}(t)}{t^3} = 0. \tag{4}$$

Proof. The first statement is an immediate application of Proposition 9 and for the second statement we just have to compare the coefficients of the Taylor expansion with the coefficients of the Taylor expansion raised to the power of two. □

With this knowledge we can now make a statement about the relation between geodesics and curvature:

Remark 11 (Relation between geodesics and curvature)**.** Let

$$f(t, s) = \exp_p tv(s), t \in [0, \delta], s \in (-\varepsilon, \varepsilon)$$

be a parametrized surface where δ is chosen so small that $\exp_p tv(s)$ is defined and $v(s)$ is a curve in T_pM with $|v(s)| = 1, v(0) = v$ and $v'(0) = w, |w| = 1$.

Our first observation is that the rays $t \mapsto tv(s), t \in [0, \delta]$ starting from the origin $0 \in T_pM$ deviate from the ray $t \mapsto tv(0)$ with the velocity

$$\left|(\frac{\partial}{\partial s} tv(s))(0)\right| = \left|t(\frac{\partial}{\partial s} v(s))(0)\right| = |tv'(0)| = |tw| = t.$$

On the other hand Equation (4) tells us that the geodesics $t \mapsto \exp_p(tv(s))$ deviate from the geodesic $\gamma(t) = \exp_p tv(0)$ with a velocity that differs from t by a term of the third order of t given by $-\frac{1}{6}K(p, \sigma)t^3$.

In particular we get that locally the geodesics spread apart less than the rays in T_pM if $K_p(\sigma) > 0$ and more apart if $K_p(\sigma) < 0$ and that for small t the value $K(p, \sigma)t^3$ furnishes and approximation for the extent of this spread with an error of order t^3.

2 Conjugate points

We will now explore the relationship between the singularities of the exponential mapping and Jacobi fields and then derive some further properties of Jacobi fields. We start with a central definition:

Definition 12. Let $\gamma : [0, a] \to M$ be a geodesic. The point $\gamma(t_0)$ is said to be *conjugate* to $\gamma(0)$ along γ for $t_0 \in (0, a]$, if there exists a Jacobi field J along γ, not identically zero, with $J(0) = 0 = J(t_0)$. The maximum number of such linearly independent fields is called the *multiplicity* of the conjugate point $\gamma(t_0)$.

If we expand the definition naturally to $\gamma(0)$ we immediately get that $\gamma(t_0)$ is conjugate to $\gamma(0)$ if and only if $\gamma(0)$ is conjugate to $\gamma(t_0)$.

Lemma 13. *Let $\gamma : [0, a] \to M$ be a geodesic and $J_1, \ldots, J_k$ be Jacobi fields along γ with $J_i(0) = 0$ for $i = 1, \ldots, k$. Then $J_1, \ldots, J_k$ are linearly independent if and only if $J_1'(0), \ldots, J_k'(0)$ are linearly independent.*

Proof. We first assume that $J_1, \ldots, J_k$ are linearly independent Jacobi fields with $J_i(0) = 0$ for $i = 1, \ldots, k$. If $J_1'(0), \ldots, J_k'(0)$ were not linearly independent we would have

$$\lambda_1 J_1'(0) + \ldots + \lambda_k J_k'(0) = 0, \exists i \in \{1, \ldots, k\} \text{ s. t. } \lambda_i \neq 0.$$

Without loss of generality assume that $\lambda_1 \neq 0$ then we have

$$J_1'(0) = (-\frac{\lambda_2}{\lambda_1})J_2'(0) + \ldots + (-\frac{\lambda_k}{\lambda_1})J_k'(0) = \left((-\frac{\lambda_2}{\lambda_1})J_2 + \ldots + (-\frac{\lambda_k}{\lambda_1})J_k\right)'(0).$$

Since $J_1(0) = \left[(-\frac{\lambda_2}{\lambda_1})J_2 + \ldots + (-\frac{\lambda_k}{\lambda_1})J_k\right](0)$ Proposition 3 yields $J_1 = (-\frac{\lambda_2}{\lambda_1})J_2 + \ldots + (-\frac{\lambda_k}{\lambda_1})J_k$ which would be a contradiction.

For the converse assume that $J_1, \ldots, J_k$ are linearly dependent, i. e.

$$\lambda_1 J_1 + \ldots + \lambda_k J_k = 0, \exists i \in \{1, \ldots, k\} \text{ s. t. } \lambda_i \neq 0.$$

Once again assuming that $\lambda_1 \neq 0$ we infer $J_1 = (-\frac{\lambda_2}{\lambda_1})J_2 + \ldots + (-\frac{\lambda_k}{\lambda_1})J_k$ which would imply that $J_1'(0) = (-\frac{\lambda_2}{\lambda_1})J_2'(0) + \ldots + (-\frac{\lambda_k}{\lambda_1})J_k'(0)$ which cannot be the case. □

Remark 14. As M is a manifold of dimension n we have that $\dim T_pM = n$ for all $p \in M$. Hence along any geodesic $\gamma : [0, a] \to M$ we get from Lemma 13 that there exist exactly n linearly indepedent Jacobi fields along γ which vanish at $\gamma(0)$.

Furthermore we have seen in Remark 4 that for a geodesic $\gamma : [0, a] \to M$ the field $J(t) = t\gamma'(t)$ is a Jacobi field along γ that never vanishes for $t \neq 0$. Hence this Jacobi does not satisfy $J(0) = 0 = J(t_0)$ for any $t_0 \in (0, a]$ so the multiplicity of any conjugate point can never exceed $n - 1$.

Let us now consider conjugate points on the sphere $\mathbf{S}^n = \{x \in \mathbf{R}^{n+1} \mid |x| = 1\}$:

Example 15. From [Car92][Chaper 6] we know that the sphere has constant sectional curvature 1. As we have seen in the proof of Example 6 the Jacobi equation is then of the form $\frac{D^2}{dt^2}J + J = 0$ and for every geodesic γ of $\mathbf{S}^n$ we know that $J(t) = (\sin t)w(t)$ with $w(t)$ being a parallel field along γ with $\langle \gamma'(t), w(t) \rangle = 0$ and $|w(t)| = 1$ is a Jacobi field along γ. We have

$$J(0) = (\sin 0)w(0) = 0 = (\sin \pi)w(\pi) = J(\pi),$$

i. e. the point $\gamma(\pi)$ is conjugate to $\gamma(0)$.

As $T_p\mathbf{S}^n$ has dimension n we can choose $n-1$ linearly independent parallel fields $w(t)$ along γ satisfying the required conditions. Hence $\gamma(\pi)$ is a conjugate point of multiplicity $n-1$.

Definition 16. The set of (first) conjugate points to the point $p \in M$ for all geodesics that start at p is called the *conjugate locus* of p and is denoted by $C(p)$.

2.1 Conjugate points and the singularities of the exponential map

The following proposition will be an important result relating conjugate points with the singularities of the exponential map:

Proposition 17. *Let $\gamma : [0, a] \to M$ be a geodesic and put $\gamma(0) = p$. The point $q = \gamma(t_0), t_0 \in (0, a]$ is conjugate to p along γ if and only if $v_0 = t_0\gamma'(0)$ is a critical point of $\exp_p$. In addition, the multiplicity of q as a conjugate point of p is equal to the dimension of the kernel of the linear map $(d\exp_p)_{v_0}$.*

Proof. By definition we have that the point $q = \gamma(t_0)$ is a conjugate point of p along γ if and only if there exists a non-zero Jacobi field J along γ with $J(0) = J(t_0) = 0$. Let $v = \gamma'(0)$ and $w = J'(0)$. By Corollary 8 we know that the Jacobi field is of the form

$$J(t) = (d\exp_p)_{tv}(tw), t \in [0, a],$$

in particular we get that J is non-zero if and only if $w \neq 0$. Therefore q is conjugate to p if and only if

$$0 = J(t_0) = (d\exp_p)_{t_0 v}(t_0 w), w \neq 0,$$

that is if and only if $t_0 v$ is a critical point of $\exp_p$ which proves the first assertion.

For the second assertion we know that the multiplicity of q is equal to the number of linearly independent Jacobi fields $J_1, \ldots, J_k$ which are zero at 0 and at t_0. From the construction above and Lemma 13 we get that the multiplicity of q is equal to the dimension of the kernel of $(d\exp_p)_{t_0 v}$. □

2.2 Properties of Jacobi fields

We will now give some more properties of Jacobi fields using the tools introduced before:

Proposition 18. *Let J be a Jacobi field along the geodesic $\gamma : [0, a] \to M$. Then*

$$\langle J(t), \gamma'(t)\rangle = \langle J'(0), \gamma'(0)\rangle t + \langle J(0), \gamma'(0)\rangle.$$

Proof. The Jacobi equation yields

$$\langle J', \gamma'\rangle' = \langle J'', \gamma'\rangle = -\langle R(\gamma', J)\gamma', \gamma'\rangle = 0,$$

therefore we must have $\langle J', \gamma'\rangle = \langle J'(0), \gamma'(0)\rangle$. In addition

$$\langle J, \gamma'\rangle' = \langle J', \gamma'\rangle = \langle J'(0), \gamma'(0)\rangle.$$

We can integrate the last equation in t to obtain

$$\langle J, \gamma'\rangle = \langle J'(0), \gamma'(0)\rangle t + \langle J(0), \gamma'(0)\rangle$$

as desired. □

From the last proposition we can draw two immediate corollaries:

Corollary 19. *If $\langle J, \gamma' \rangle(t_1) = \langle J, \gamma' \rangle(t_2), t_1, t_2 \in [0, a], t_1 \neq t_2$, then $\langle J, \gamma' \rangle$ does not depend on t; in particular, if $J(0) = J(a) = 0$, then $\langle J, \gamma' \rangle(t) \equiv 0$.*

Proof. By Proposition 18 we have $\langle J(t), \gamma'(t) \rangle = \langle J(0), \gamma'(0) \rangle$ for all $t \in [0, a]$ and if $J(0) = J(a) = 0$ we have $\langle J(0), \gamma'(0) \rangle = 0$. □

Corollary 20. *Suppose that $J(0) = 0$. Then $\langle J'(0), \gamma'(0) \rangle = 0$ if and only if $\langle J, \gamma' \rangle(t) \equiv 0$; in particular the space of Jacobi fields J with $J(0) = 0$ and $\langle J, \gamma' \rangle(t) \equiv 0$ has dimension equal to $n - 1$.*

Proof. The first assertion is immediate from Proposition 18. Furthermore in case of $J(0) = 0$ and $\langle J, \gamma' \rangle(t) \equiv 0$ we have $n - 1$ degrees of freedom for $J'(0)$ and hence by applying once again Lemma 13 we get that the dimension of the space of such Jacobi fields is $n - 1$. □

We now come to our last result:

Proposition 21. *Let $\gamma : [0, a] \to M$ be a geodesic. Let $V_1 \in T_{\gamma(0)}M$ and $V_2 \in T_{\gamma(a)}M$. If $\gamma(a)$ is not conjugate to $\gamma(0)$ there exists a unique Jacobi field J along γ with $J(0) = V_1$ and $J(a) = V_2$.*

Proof. Let $\mathcal{J}$ be the space of Jacobi fields with $J(0) = V_1$ and define the mapping

$$\theta : \mathcal{J} \to T_{\gamma(a)}M, \theta(J) = J(a).$$

Since $\gamma(a)$ is not conjugate to $\gamma(0)$ we know that θ is injective. Indeed if $J_1 \neq J_2$ with $J_1(a) = J_2(a)$, then $J_1 - J_2$ would be a non-zero Jacobi field with $(J_1 - J_2)(0) = V_1 - V_1 = 0 = J_1(a) - J_2(a) = (J_1 - J_2)(a)$ which would be a contradiction.

Since θ is a linear injection and we have $\dim \mathcal{J} = \dim T_{\gamma(a)}M$ we see that θ is in fact an isomorphism. Hence there exists $\overline{J} \in \mathcal{J}$ with $\overline{J}(0) = V_1$ and $\overline{J}(a) = V_2$. As θ is an isomorphism the uniqueness is clear as well. □

Corollary 22. *Let $\gamma : [0, a] \to M$ be a geodesic in M and let $\mathcal{J}^\perp$ be the space of Jacobi fields with $J(0) = 0$ and $J'(0) \perp \gamma'(0)$. Let $\{J_1, \ldots, J_{n-1}\}$ be a basis of $\mathcal{J}^\perp$. If $\gamma(t), t \in (0, a]$, is not conjugate to $\gamma(0)$, then $\{J_1(t), \ldots, J_{n-1}(t)\}$ is a basis for the orthogonal complement $\{\gamma'(t)\}^\perp \subset T_{\gamma(t)}M$ of $\gamma'(t)$.*

References

[Car92] Manfredo P.do Carmo, *Riemannian Geometry*, Birkhäuser, 1992.