The figure-eight knot complement

Geometric structures on manifolds

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Contents

In this paper we will examine a crooked Farey tesselation whose tiling group has a surprising connection to the figure eight-knot complement. Crooked Farey tesselations arise from Farey tesselations by suitably bending them into the 3-dimensional hyperbolic space $\mathbf{H}^3$.

The first chapter will be dedicated to introducing crooked Farey tesselations as well as recalling the notion of the 3-dimensional hyperbolic space $\mathbf{H}^3$. We will then introduce the aforementioned crooked Farey tesselation, investigate its properties and illustrate its connection to the figure-eight knot complement.

1 Prerequisites

We start with introducing the 3-dimensional hyperbolic space along with a selection of essential notions and facts.

1.1 The 3-dimensional hyperbolic space $\mathbf{H}^3$

The 3-dimensional hyperbolic space is defined in complete analogy with the hyperbolic plane. We set

$$\mathbf{H}^3 = \{(x, y, u) \in \mathbf{R}^3 \mid u > 0\}$$

and for every piecewise differentiable curve γ in $\mathbf{H}^3$ parametrized by

$$t \mapsto (x(t), y(t), u(t)) \text{ with } a \leq t \leq b$$

we define the *hyerbolic length of* γ to be

$$l_{hyp}(\gamma) = \int_a^b \frac{\sqrt{x'(t)^2 + y'(t)^2 + u'(t)^2}}{u(t)} dt.$$

We then make the following definition:

Definition 1 (3-dimensional hyperbolic space)**.** The 3-*dimensional hyperbolic sapce* is the space $\mathbf{H}^3$ endowed with the metric

$$d_{hyp}(P,Q) = \inf \{l_{hyp}(\gamma) \mid \gamma \text{ piecewise differentiable curve from } P \text{ to } Q\}$$

That $(\mathbf{H}^3, d_{hyp})$ is in fact a metric space can be seen in a manner identical to the 2-dimensional case. As in dimension 2 we set the norm of a vector v at the point $P = (x,y,u) \in \mathbf{H}^3$ to be

$$\|v\|_{hyp} = \frac{1}{u}\|v\|_{euc}$$

For a more complete and comprehensive coverage of the hyperbolic space we refer the reader to [Bon09, Chapter 9]. However, we will be needing the following notions, also defined analogously to the 2-dimensional case:

Definition 2. A *hyperbolic plane* in the hyperbolic space $\mathbf{H}^3$ is the intersection H of $\mathbf{H}^3$ with a euclidean sphere centered on the xy-plane or with a vertical euclidean plane.

Definition 3. A *horosphere centered at* $z \in \mathbf{C}$ is the intersection with $\mathbf{H}^3$ of a euclidean sphere S which is tangent to the xy-plane $\mathbf{C} \subset \mathbf{R}^3$ and lies above the xy-plane. A horosphere centered at ∞ is just a horizontal euclidean plane contained in $\mathbf{H}^3$.

1.2 Crooked Farey tesselations

Given real numbers $s_1, s_3, s_5 \in \mathbf{R}$ with $s_1 + s_3 + s_5 = 0$ we can obtain a tesselation of the hyperbolic plane $\mathbf{H}^2$. We do so by considering the two ideal triangles T^+ and T^- with vertices $0, 1, \infty$ and $-e^{s_5}, 0, \infty$, respectively. Considering the linear maps

$$\varphi_1(z) = \frac{e^{s_5}z + 1}{e^{s_5}z + e^{-s_1} + 1} \text{ and } \varphi_3(z) = e^{-s_5}\frac{z-1}{-z + e^{s_3} + 1}$$

the tesselation then consists of all triangles of the form $\varphi(T^+)$ and $\varphi(T^-)$ as φ ranges over all elements of the transformation group Γ generated by φ_1 and φ_3.

Now let s_1, s_3, s_5 be complex numbers and T^+ and T^- be the ideal triangles in the hyperbolic space $\mathbf{H}^3$ with vertices $0, 1, \infty$ and $-e^{s_5}, 0, \infty$, respectively. The linear fractional maps φ_1 and φ_3 from before also define isometries on the hyperbolic space $(\mathbf{H}^3, d_{hyp})$. With edges labeled form E_1 to E_5 the following figure illustrates this constellation:

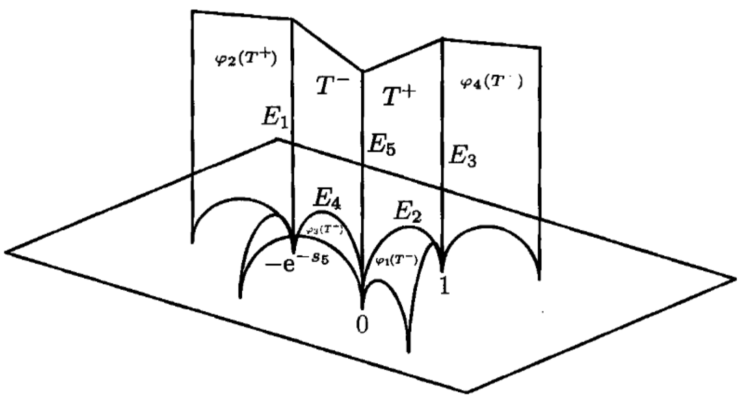

Figure 1: Bending the Farey tesselation in hyperbolic space $\mathbf{H}^3$ (cf. [Bon09, Figure 10.1])

Definition 4. The familty $\mathcal{T}$ of all triangles $\varphi(T^{\pm})$ where φ ranges over all elements of the transformation group Γ generated by φ_1 and φ_3 is called a *crooked Farey tesselation* in $\mathbf{H}^3$.

Before introducing further concepts let us look at some basic examples:

Example 5. The bending of the standard Farey tesselation, i. e. if we set $s_1 = s_3 = s_5 = 0$ looks like

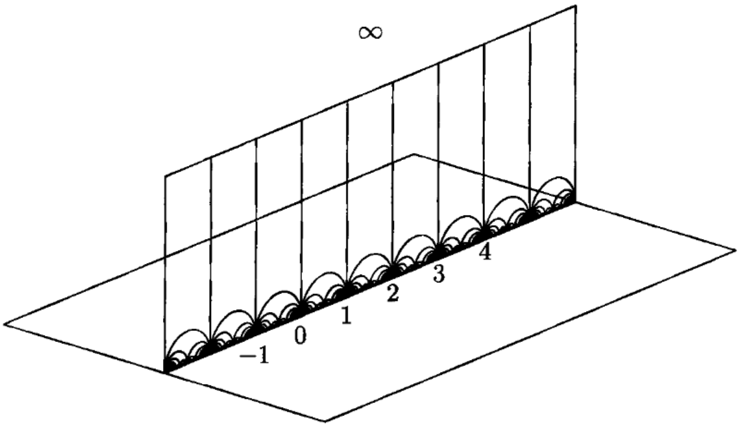

Figure 2: The standard Farey tesselation (cf. [Bon09, Figure 10.2])

If we slightly move the s_i away from 0, e. g. if we set $s_1 \approx 0.19 + 0.55i, s_3 \approx 0.15 + 0.42i$ and $s_5 = 0.04 - 0.97i$ we obtain:

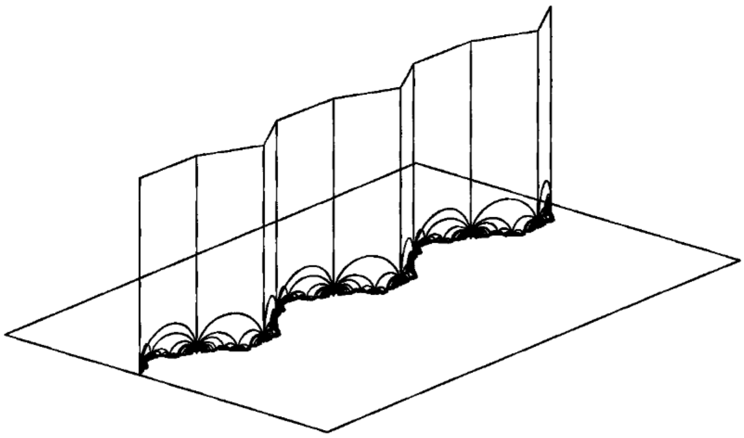

Figure 3: Slighly bending the farey tesselation (cf. [Bon09, Figure 10.3])

To study the action of the tiling groups of such crooked Farey tesselations as well as the "footprints" of these actions on the boundary $\hat{\mathbf{C}}$ of $\mathbf{H}^3$ we introduce the following concepts:

Definition 6. A *kleinian group* is a group Γ of isometries of the hyperbolic space $(\mathbf{H}^3, d_{hyp})$ whose action on $\mathbf{H}^3$ is discontinuous.

Let now P_0 be an arbitrary point in $\mathbf{H}^3$.

Definition 7. A *limit point* of the orbit $\Gamma(P_0)$ is a point $P \in \mathbf{R}^3 \cup \{\infty\}$ for which there exists a sequence $(\gamma_n)_{n\in\mathbf{N}}$ of elements of Γ such that $\gamma_n(P_0) \neq P$ for all $n \in \mathbf{N}$ and

$$P = \lim_{n\to\infty} \gamma_n(P_0).$$

Definition 8. The *limit set* of the kleinian group Γ is the set Λ_Γ of all limit points of the orbit $\Gamma(P_0)$ for $P_0 \in \mathbf{H}^3$.

The definition of a limit set is indeed well-defined as the following lemma shows:

Lemma 9. *The limit set Λ_Γ of a kleinian group Γ is contained in the Riemann sphere $\hat{\mathbf{C}}$ bounding the hyperbolic space $\mathbf{H}^3$ and is independent of the point $P_0 \in \mathbf{H}^3$ chosen.*

Proof. [Bon09, Lemma 10.1]. □

1.3 Poincareé's Polyhedron Theorem

Before introducing the main crooked Farey tesselation we intend to study we have to give the Poincaré Polyhedron Theorem, which will prove an essential part in the investigations to follow. To do so, we first fix some basic terminology:

Definition 10. A *polyhedron* in the hyperbolic space $\mathbf{H}^3$ is a region X in $\mathbf{H}^3$ delimited by finitely many polygons, called its *faces*.

A polygon is thereby a subset F of a hyperbolic plane of $\mathbf{H}^3$ corresponding to a polygon in $\mathbf{H}^2$. The face glueing data is defined in complete analogy with the case of polyongs in $\mathbf{H}^2$. Given gluing maps $\varphi_i : F_i \to F_{i\pm1}$ for the faces, each of them extends to an isometry $\varphi_i : \mathbf{H}^3 \to \mathbf{H}^3$. The group of isometries generated by these extended gluing maps will be called the *tiling group*.

Definition 11. Let X be a polyhedron and Γ a tiling group associated to gluing data for X. The images of X under the elements of Γ form a *tesselation* of $\mathbf{H}^3$ if

- As γ ranges over all elements of Γ the tiles $\gamma(X)$ cover $\mathbf{H}^3$
- The intersection of two distinct tiles $\gamma(X)$ and $\gamma'(X)$ consists only of vertices, edges and faces of $\gamma(X)$
- (Local finiteness) For every $P \in \mathbf{H}^3$, there exists a ball $B_\varepsilon(P)$ which meets only finitely many tiles $\gamma(X)$

Furthermore the bending of the boundary of a hyperbolic polyhedron along an edge is given by its *dihedral angle* defined as in euclidean geometry. We can now state the Poincaré Polyhedron Theorem which is a 3-dimensional version of the Tesselation Theorem (cf. [Bon09, Theorem 6.1]) and of Poincaré's Polygon Theorem (cf. [Bon09, Theorem 6.25]):

Theorem 12 (Poincaré Polyhedron Theorem)**.** *For a connected polyhedron $X \subset \mathbf{H}^3$ with face gluing data as defined above, suppose in addition that the following three conditions hold:*

1. *(Dihedral Angle Condition) For every edge E of X the dihedral angles along the edges that are glued to E add up to $\frac{2\pi}{n_E}$ for some integer $n_E \geq 1$ depending on E.*
2. *(Edge Orientation Condition) The edges of X can be oriented in such a way that whenever a gluing map $\varphi_i : F_i \to F_{i\pm1}$ sends an edge E to an edge E' it sends the orientation of E to the orientation of E'.*
3. *(Horosphere Condition) For every ideal vertex ξ of X we can select a horosphere S_ξ such that whenever the gluing map $\varphi_i : F_i \to F_{i\pm1}$ sends the ideal vertex to the ideal vertex ξ', it also sends S_ξ to $S_{\xi'}$.*

Then, as γ ranges over all the elements of the tiling group Γ generated by the extended gluing maps $\varphi_i : \mathbf{H}^3 \to \mathbf{H}^3$, the tiles $\gamma(X)$ form a tesselation of the hyperbolic space $\mathbf{H}^3$ and X is a fundamental domain for this action.

In addition, the tiling group Γ acts discontinuously on $\mathbf{H}^3$, the two quotient spaces $(\mathbf{H}^3/\Gamma, \overline{d}_{hyp})$ and $(\overline{X}, \overline{d}_X)$ are isometric, and these two metric spaces are complete.

Proof. [Bon09, Theorem 10.9]. □

2 The crooked Farey tesselation $\mathcal{T}_8$

In this section we will be examining the following crooked Farey tesselation:

Definition 13. Let $\mathcal{T}_8$ denote the crooked Farey tessellation corresponding to the parameters $s_1 = \frac{2\pi}{3}i$, $s_3 = -\frac{2\pi}{3}i$ and $s_5 = 0$.

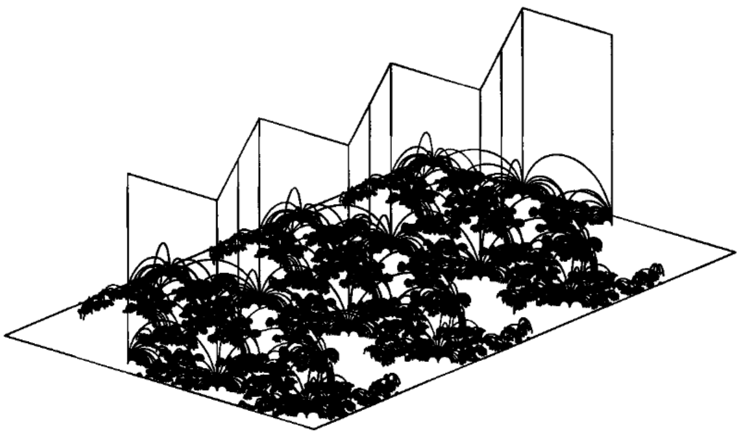

Figure 4: An approximation of $\mathcal{T}_8$ (cf. [Bon09, Figure 11.1])

For the tiling group Γ_8 of $\mathcal{T}_8$ we obtain:

Lemma 14. *Let $\omega = e^{\frac{\pi}{3}i}$. The tiling group Γ_8 of $\mathcal{T}_8$ is the transformation group generated by*

$$\varphi_1(z) = \frac{z+1}{z+\omega^{-1}} \text{ and } \varphi_3(z) = \frac{z-1}{-z+\omega^{-1}}.$$

Proof. To see these equalities first note that $\omega^3 = \left(e^{\frac{\pi}{3}i}\right)^3 = e^{\pi i} = -1$, so in particular we have

$$\omega^2 - \omega + 1 = \frac{\omega^3+1}{\omega+1} = 0$$

and

$$1 + \omega^{-2} = 1 - \omega = -\omega^2 = \omega^{-1}.$$

From the first chapter we know that the tiling group Γ_8 is generated by

$$\varphi_1(z) = \frac{e^{s_5}z+1}{e^{s_5}z + e^{-s_1}+1} \text{ and } \varphi_3(z) = e^{-s_5}\frac{z-1}{-z+e^{s_3}+1}.$$

Using the calculations above we obtain

$$\varphi_1(z) = \frac{e^0 z + 1}{e^0 z + e^{-\frac{2\pi}{3}i} + 1} = \frac{z+1}{z + 1 + e^{-\frac{2\pi}{3}i}} = \frac{z+1}{z+\omega^{-1}}$$

and

$$\varphi_3(z) = e^{-0}\frac{z-1}{-z+e^{-\frac{2\pi}{3}i}+1} = \frac{z-1}{-z+\omega^{-1}}$$

as desired. □

2.1 The tiling group Γ_8

We now want to examine the tiling group Γ_8. In particular, we will show that it acts discontinuously on the hyperbolic space $\mathbf{H}^3$, i. e. that it is a Kleinian group.

In order to understand the tiling group Γ_8 it will prove helpful to study an enlargement of this group, namely the transformation group generated by φ_1, φ_3 and the translation τ defined by

$$\tau(z) = z + \omega.$$

Definition 15. Let $\hat{\Gamma}_8$ denote the transformation group generated by φ_1, φ_3 and τ.

In particular, as we will see later, it is the transformation group $\hat{\Gamma}_8$ which will provide the connection to the figure-eight knot complement.

We want to give a fundamental domain for the action $\hat{\Gamma}_8$. For this let Δ_1 be the tetrahedron with ideal vertices at $0, 1, \infty$ and ω and Δ_2 be the tetrahedron with ideal vertices at $0, 1, \infty$ and $\omega^{-1} = 1-\omega$. Furthermore let Δ be the union of Δ_1 and Δ_2. Then Δ is a polyhedron with five ideal vertices, nine edges and six faces. The following figure offers a top view of Δ_1 and Δ_2:

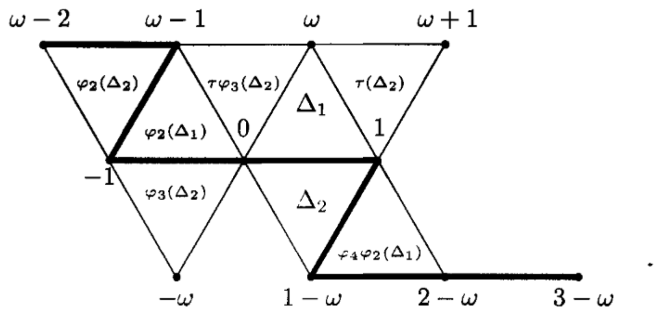

Figure 5: A top view of Δ_1 and Δ_2 (cf. [Bon09, Figure 11.3])

We will consider the following gluing isometries for Δ to show that $\hat{\Gamma}_8$ acts discontinuously on $\mathbf{H}^3$:

$$\psi_1 = \tau, \psi_3 = \varphi_4 \circ \varphi_2 \text{ and } \psi_5 = \tau \circ \varphi_3$$

Note that ψ_1 maps $[0, 1-\omega, \infty]$ to $[\omega, 1, \infty]$, ψ_3 maps $[0, 1, \omega]$ to $[\infty, 1-\omega, 1]$ and ψ_5 maps $[0, 1, 1-\omega]$ to $[0, \omega, \infty]$.

Proposition 16. *The group $\hat{\Gamma}_8$ acts discontinuously and freely on the hyberpbolic space $\mathbf{H}^3$ and the ideal polyhedron Δ is a fundamental domain for this action.*

Proof. Set

$$\psi_2 = \psi_1^{-1}, \psi_4 = \psi_3^{-1} \text{ and } \psi_6 = \psi_5^{-1}$$

and denote by $(\overline{\Delta}, \overline{d}_{hyp})$ the thusly obtained quotient space.

Applying the gluing maps ψ_1, ψ_3 and ψ_5 the edges $[0,1], [\infty, 1-\omega], [\infty, 1]$ and $[0, \omega]$ are glued together as are $[0, \infty], [\omega, \infty], [1, 1-\omega]$ and $[\omega, 1]$. If we orient each of the above edges $[a, b]$ from a to b, the orientations are respected by the gluing maps ψ_i, in particular the Edge Orientation Condition of Poincaré's Polyhedron Theorem holds.

By Figure 5 we see that the dihedral angles of Δ along the edges $[0, \infty], [1, \infty], [\omega, \infty]$ and $[1-\omega, \infty]$ are $\frac{2\pi}{2}, \frac{2\pi}{3}, \frac{\pi}{3}$ and $\frac{\pi}{3}$, respectively.

The inversion across the sphere of radius 1 centered at 0 defines an isometry of $\mathbf{H}^3$ that exchanges 0 and ∞ and fixes $1, \omega$ and $1-\omega$. In particular we get that the dihedral angle Δ along the edge $[0, a]$ is equal to the dihedral angle along $[\infty, a]$ for $a = 1, \omega$ or $1-\omega$.

From these considertaions we conclude that the sum of the dihedral angles of Δ along the edges that are glued to $[0,1]$ is equal to $\frac{2\pi}{3} + \frac{\pi}{3} + \frac{2\pi}{3} + \frac{\pi}{3} = 2\pi$ and similarly one can see that the sum of the dihedral angles along the edges that are glued to $[0, \infty]$ adds up to $\frac{2\pi}{3} + \frac{\pi}{3} + \frac{\pi}{3} + \frac{2\pi}{3} = 2\pi$. Therefore the Dihedral Angle Condition of Poincaré's Polyhedron Theorem is satisfied.

To prove the Horosphere Condition let S_0, S_1, S_ω and $S_{1-\omega}$ denote the horospheres at $0, 1, \omega$ and $1-\omega$, repectively, that have euclidean diameter 1, and let S_∞ be the horizontal plane of equation $u = 1$. For any face $[a, b, c]$ the horospheres S_a, S_b and S_c are tangent to each other. So in particular, if $[a, b, c]$ is glued to $[a', b', c']$ then the gluing map must send S_a, S_b and S_c to horospheres centered at a', b' and c' and tangent to each other.

Following from [Bon09, Lemma 8.4] there are only three such horospheres satisfying this condition, namely $S_{a'}, S_{b'}$ and $S_{c'}$. Therefore, if a gluing map φ sends an ideal vertex a of Δ to $\varphi(a)$, it also sends the horospheres S_a to $S_{\varphi(a)}$.

Hence by the Poincaré Polyhedron Theorem the group $\hat{\Gamma}'_8$ generated by ψ_1, ψ_3 and ψ_5 acts discontinuously on $\mathbf{H}^3$ and admits Δ as a fundamental domain. By [Bon09, Theorem 10.11] we also obtain that $\hat{\Gamma}'_8$ acts freely.

Now every element of $\hat{\Gamma}'_8$ can be expressed as a composition of terms $\psi_i^{\pm 1}$, so in particular as a composition of $\tau^{\pm 1}, \varphi_1^{\pm 1}\ \varphi_3^{\pm 1}$. Conversely we have $\tau = \psi_1, \varphi_3 = \psi_1^{-1} \circ \psi_5$ and $\varphi_1 = \psi_3^{-1} \circ \psi_5^{-1} \circ \psi_1$. This shows $\hat{\Gamma}'_8 = \hat{\Gamma}_8$ and hence our assertion. □

We furthemore have the following result:

Proposition 17. *The quotient space* $(\mathbf{H}^3/\hat{\Gamma}_8, \overline{d}_{hyp})$ *is locally isometric to the hyperbolic space* $(\mathbf{H}^3, d_{hyp})$.

Proof. This is an immediate consequence of [Bon09, Theorem 10.11]. □

Since Γ_8 is contained in $\hat{\Gamma}_8$ an immediate consequence of Proposition 16 is:

Corollary 18. *The tiling group* Γ_8 *acts discontinuously on the hyperbolic space* $\mathbf{H}^3$.

2.2 The limit set of Γ_8

We will now calculate the limit set of the Kleinian group Γ_8. Once again the enlarged group $\hat{\Gamma}_8$ will prove an important tool in this process as we will see that their limit sets coincide.

Lemma 19. *The limit set of the enlarged group $\hat{\Gamma}_8$ is the whole Riemann sphere $\hat{\mathbf{C}}$.*

Proof. The proof is essentially the same as the one of [Bon09, Lemma 10.7]. □

To see that the two groups Γ_8 and $\hat{\Gamma}_8$ have the same limit set we first recall the following algebraic notion:

Definition 20. A *normal subgroup* of a transformation group Γ is a transformation group Γ' contained in Γ and such that for every $\gamma \in \Gamma$ and $\gamma' \in \Gamma'$ the composition $\gamma \circ \gamma' \circ \gamma^{-1}$ is also an element of Γ'.

The reason we are recalling this notion is due to the following observation:

Proposition 21. *If Γ' is a normal subgroup of the Kleinian group Γ and if the limit set of Γ' has at least two elements, then the limit sets Λ_Γ and $\Lambda_{\Gamma'}$ coincide.*

Proof. Let $\xi \in \Lambda_{\Gamma'}$ and $\gamma \in \Gamma$. Fix a base point $P_0 \in \mathbf{H}^3$ and consider its orbit $\Gamma'(P_0)$ under the action of Γ'. We have a sequence $(\gamma'_n)_{n\in\mathbf{N}} \in \Gamma'$ such that $\lim_{n\to\infty} \gamma'_n(P_0) = \xi$ in $\mathbf{R}^3 \cup \{\infty\}$ for the euclidean metric. Furthermore we obtain

$$\gamma(\xi) = \lim_{n\to\infty} \gamma \circ \gamma'_n(P_0) = \lim_{n\to\infty} \eta'_n(P'_0)$$

with $P'_0 = \gamma(P_0)$ and $\eta'_n = \gamma \circ \gamma'_n \circ \gamma^{-1} \in \Gamma'$. From Lemma 9 we know that the limit set is independent of the base point, i. e. $\gamma(\xi)$ must be in $\Lambda_{\Gamma'}$ for every $\xi \in \Lambda_{\Gamma'}$ and $\gamma \in \Gamma$.

By [Bon09, Lemma 10.2] we know that $\Lambda_{\Gamma'}$ is closed in $\hat{\mathbf{C}}$. As $\Lambda_{\Gamma'}$ has at least two elements by hypothesis, [Bon09, Proposition 10.3] yields that $\Lambda_\Gamma \subseteq \Lambda_{\Gamma'}$. As Γ' is contained in Γ we furthermore have $\Lambda_{\Gamma'} \subseteq \Lambda_\Gamma$. □

Due to Proposition 21 it will suffice to prove that Γ_8 is a normal subgroup of $\hat{\Gamma}_8$ in order to show that the limit set of Γ_8 is the whole Riemann sphere. To do so we first prove the following lemma:

Lemma 22. *The following equations hold*

$$\tau^{-1} \circ \varphi_1 \circ \tau = \varphi_3^{-1} \tag{1}$$

$$\tau^{-1} \circ \varphi_3 \circ \tau = \varphi_3^2 \circ \varphi_1 \circ \varphi_3 \tag{2}$$

$$\tau \circ \varphi_1 \circ \tau^{-1} = \varphi_1^2 \circ \varphi_3 \circ \varphi_1 \tag{3}$$

$$\tau \circ \varphi_3 \circ \tau^{-1} = \varphi_1^{-1}. \tag{4}$$

Proof. We will first prove that

$$\psi_5 \circ \psi_1^{-1} \circ \psi_3^{-1} \circ \psi_5^{-1} \circ \psi_1 = \mathrm{Id}_{\mathbf{H}^3} \tag{5}$$

To see this observe that the gluing map ψ_1 sends the edge $[0, \infty]$ of Δ to the edge $[\omega, \infty]$, which is sent by $\psi_6 = \psi_5^{-1}$ to $[1, 1-\omega]$, which is sent by $\psi_4 = \psi_3^{-1}$ to $[\omega, 1]$, which is sent by $\psi_2 = \psi_1^{-1}$

to $[0, 1-\omega]$ which then is sent to back to $[0, \infty]$ by ψ_5.

In particular the mapping $\psi_5 \circ \psi_1^{-1} \circ \psi_3^{-1} \circ \psi_5^{-1} \circ \psi_1$ fixes the point $S_0 \cap [0, \infty]$ provided by the horosphere S_0. As $\hat{\Gamma}_8$ is a free action (Proposition 16) this map must be the identity on all of $\mathbf{H}^3$.

If we substitute $\psi_1 = \tau, \psi_3 = \varphi_4 \circ \varphi_2$ and $\psi_5 = \tau \circ \varphi_3$ in 5 we get

$$\tau \circ \varphi_3 \circ \tau^{-1} \circ \varphi_1 \circ \varphi_3 \circ \varphi_3^{-1} \circ \tau^{-1} \circ \tau = \mathrm{Id}^{\mathbf{H}^3}$$

which yields

$$\tau^{-1} \circ \varphi_1 \circ \tau = \varphi_3^{-1}$$

thus proving 1.

Using the same reasoning as above one finds that

$$\psi_5^{-1} \circ \psi_3^{-1} \circ \psi_1 \circ \psi_3 = \mathrm{Id}_{\mathbf{H}^3} .$$

Substituting again we get

$$\varphi_3^{-1} \circ \tau^{-1} \circ \varphi_1 \circ \varphi_3 \circ \tau \circ \varphi_3^{-1} \circ \varphi_1^{-1} = \mathrm{Id}_{\mathbf{H}^3}$$

which simplifies to

$$\tau^{-1} \circ \varphi_1 \circ \varphi_3 \circ \tau = \varphi_3 \circ \varphi_1 \circ \varphi_3$$

Applying 1 to the left-hand side of this equation we get

$$\tau^{-1} \circ \varphi_3 \circ \tau = \varphi_3^2 \circ \varphi_1 \circ \varphi_3$$

which proves 2. Using 2 we furthermore obtain

$$\begin{aligned} \varphi &= \left(\tau \circ \varphi_3 \circ \tau^{-1}\right) \circ \left(\tau \circ \varphi_3 \circ \tau^{-1}\right) \circ \left(\tau \circ \varphi_1 \circ \tau^{-1}\right) \circ \left(\tau \circ \varphi_3 \circ \tau^{-1}\right) \\ &= \varphi_1^{-2} \circ \left(\tau \circ \varphi_1 \circ \tau^{-1}\right) \circ \varphi_1^{-1} \end{aligned}$$

which shows 3. Finally 4 is an immediate consequence of 1. □

With these calculations done we may now prove the following fact:

Lemma 23. *The group Γ_8 is a normal subgroup of the enlarged group $\hat{\Gamma}_8$.*

Proof. Let $\gamma \in \hat{\Gamma}_8$ and $\gamma' \in \Gamma_8$. If we have $\gamma = \varphi_1^{\pm 1}$ or $\gamma = \varphi_3^{\pm 1}$ we immediately get that $\gamma \circ \gamma' \circ \gamma^{-1} \in \Gamma_8$. In the case of $\gamma = \tau^{\pm 1}$ the calculations from Lemma 22 show that we also have $\gamma \circ \gamma' \circ \gamma^{-1} \in \Gamma_8$ since Γ_8 is generated by φ_1 and φ_3.

As every $\gamma \in \hat{\Gamma}_8$ can be expressed as a composition of terms $\varphi_1^{\pm 1}, \varphi_3^{\pm 1}$ and $\tau^{\pm 1}$ this already proves the general result. □

In particular we obtain the following result:

Corollary 24. *The limit set of the Kleinian group Γ_8 is equal to $\hat{\mathbf{C}}$.*

3 The connection to the figure-eight knot complement

In this chapter we will illustrate that the resulting quotient space of the action of $\hat{\Gamma}_8$ on $\mathbf{H}^3$ is homeomorphic to the complement of the figure-eight knot. By the figure-eight knot we mean the following figure in $\mathbf{R}^3$:

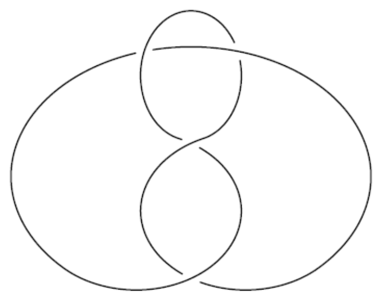

Figure 6: The figure-eight knot (cf. [Rat06, Figure 10.3.4])

Theorem 25. *The complement of the figure-eight knot in* $\hat{\mathbf{R}}^3$ *is homeomorphic to the quotient space* $\mathbf{H}/\hat{\Gamma}_8$.

Proof. We will give a sketch of the proof. The basic idea is to show that the complement of the figure-eight knot can be homeomorphically deformed into the space $\overline{\Delta}$ from chapter 2. Recall that Δ was a fundamental domain for the action of $\hat{\Gamma}_8$. Furthermore we obtained the quotient space $\overline{\Delta}$ by gluing Δ together by the following isometries:

$$\psi_1 = \tau, \psi_3 = \varphi_4 \circ \varphi_2 \text{ and } \psi_5 = \tau \circ \varphi_3$$

with φ_i defined as in chapter 2. By the Poincaré Polyhedron Theorem we know that the spaces $\overline{\Delta}$ and $\mathbf{H}^3/\hat{\Gamma}_8$ are isometric with respect to their induced metrices.

To see that the complement of the figure-eight knot can be homeomorphically deformed into $\overline{\Delta}$ we fill follow [Rat06, §10.3]. First note that Δ_1 corresponds to the left and Δ_2 corresponds to the right tetrahedron in Figure 7 with the edges indicating the gluing patterns to obtain Δ:

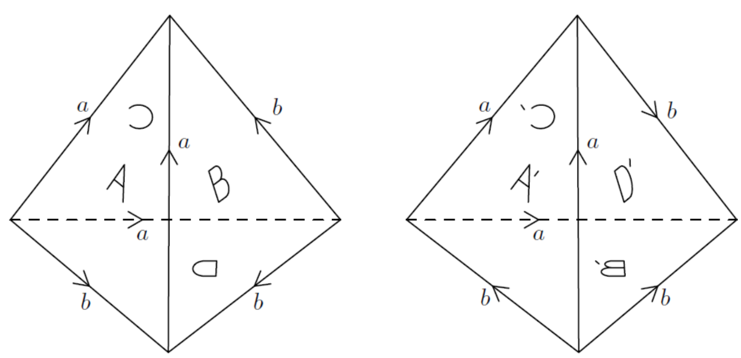

Figure 7: The gluing pattern of the two tetrahedrons Δ_1 and Δ_2 (cf. [Rat06, Figure 10.3.2])

Let K denote the figure-eight knot. We put K on top of Δ_1 and add two directed arcs a and b to the knot which will correspond to the two edges a and b from Figure 7.

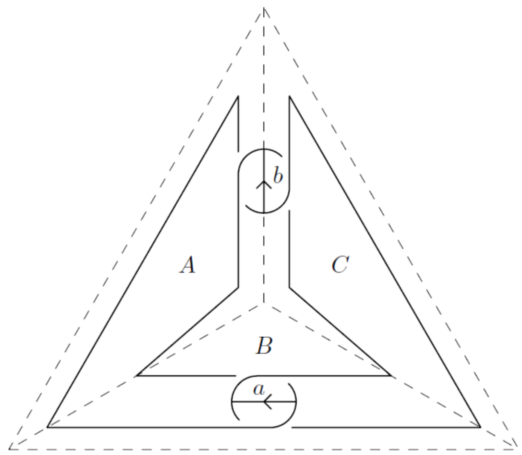

Figure 8: Draping the knot K over Δ_1 (cf. [Rat06, Figure 10.3.5])

Now consider the boundary sides of A, B, C and D from Δ_1. If we glue the side A according to the given pattern, the quotient space is homeomorphic to a closed disk with two points removed. The space is further homeomorphic to a disk with one interior point and part of its boundary removed. The following figure illustrates this procedure:

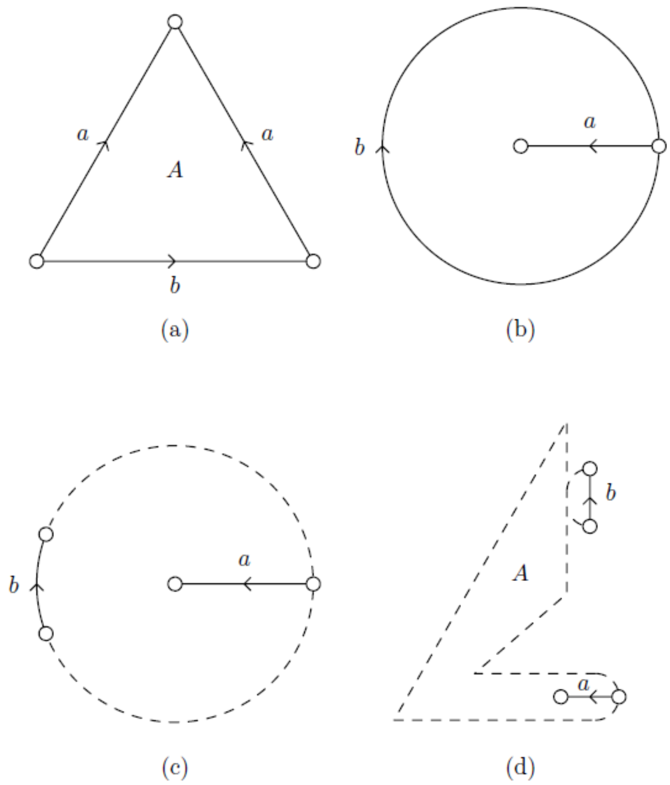

Figure 9: Deforming side A (cf. [Rat06, Figure 10.3.6])

In particular we see that the disk in (d) spans the part of K that follows the contour of side A so we can glue them together. In a similar manner one can see that the sides B, C and D give rise to disks spanning the parts of K on the sides B, C and D, respectively.

Let L denote the result of gluing the thusly obtained disks to the parts of K as indicated above. Each of the arcs a and b meets all of the disks we glued. Collapsing a and b to points we see that L has the homotopy type of a 2-sphere. In particular this yields that $\hat{\mathbf{R}}^3 - L$ is the union of two open 3-balls.

We now cut the complement of the figure-eight knot $\hat{\mathbf{R}}^3 - L$ along the interiors of the cells of L corresponding to the disks and split apart a and b along their interiors. This yields two connected 3-manifolds with boundaries. The boundaries are 2-spheres minus four points and they have the same cell decomoposition as the two tetrahedrons above.

The interiors of these two manifolds with boundary are open 3-balls, so they are in fact closed 3-balls minus four points on their boundaries. In particular there is a function from the disjoint union of these manifolds with boundary to the disjoint union of Δ_1 and Δ_2 which induces a homeomorphism $\hat{\mathbf{R}}^3 - K$ to Δ. $\square$

References

[Bon09] Francis Bonahon, *Low-dimensional geometry : from Euclidean surfaces to hyperbolic knots*, American Mathematical Society, 2009.

[Rat06] John Ratcliffe, *Foundations of Hyperbolic Manifolds*, Springer, 2006.