Seminar "Differential forms and their use"

Differentiable manifolds

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1 Definitions and properties

Motivation 1.
In $\mathbb{R}^3$ we defined a regular surface S as a subset $S \subset \mathbb{R}^3$ such that for every point $p \in S$ there exists a neighbourhood V of p and a map $f_\alpha : U_\alpha \subset \mathbb{R}^2 \longrightarrow V \cap S$ with the property that

(i) f_α is a differentiable homeomorphism.

(ii) the differential $(df_\alpha)_q : T_q(U_\alpha) \longrightarrow \mathbb{R}^3$ is injective for all q in U_α.

The map f_α is called a parametrization of S around p.

A fundamental property of regular surfaces is the fact that a change of parametrizations is differentiable, i.e. for any two parametrizations $f_\alpha : U_\alpha \longrightarrow S$, $f_\beta : U_\beta \longrightarrow S$ with

$$W := f_\alpha(U_\alpha) \cap f_\beta(U_\beta) \neq \emptyset$$

$$f_\beta^{-1} \circ f_\alpha : \quad f_\alpha^{-1}(W) \longrightarrow \mathbb{R}^2$$

is differentiable.

Proof of this fundamental property.
As a composition of homeomorphisms $f^{-1} \circ g$ is again a homeomorphism. The idea is to use the inverse function theorem to proof that $f_\alpha^{-1} \circ f_\beta$ is differentiable. First, we choose a point $x \in f_\beta^{-1}(W)$, then let $(f_\alpha^{-1} \circ f_\beta)(x) = q$. Since the differential df_α is injective for all $q \in U_\alpha$, it has rank 2 and by - if necessary - rearranging the axes, we can assume that

$$det\left(\begin{bmatrix} \frac{\partial f_\alpha^1}{\partial x}|_q & \frac{\partial f_\alpha^1}{\partial y}|_q \\ \frac{\partial f_\alpha^2}{\partial x}|_q & \frac{\partial f_\alpha^2}{\partial y}|_q \end{bmatrix}\right) \neq 0$$

at point q. Now we define an extension of f_α to which we can apply the inverse function theorem.

$$F : U_\alpha \times \mathbb{R} \longrightarrow \mathbb{R}^3$$
$$F(x, y, z) = (f_\alpha^1(x, y), f_\alpha^2(x, y), f_\alpha^3(x, y) + z)$$

We immediately notice that $F(x, y, 0) = f_\alpha(x, y)$ and

$$dF_q = \begin{bmatrix} \frac{\partial f_\alpha^1}{\partial x} & \frac{\partial f_\alpha^1}{\partial y} & \frac{\partial f_\alpha^1}{\partial z} \\ \frac{\partial f_\alpha^2}{\partial x} & \frac{\partial f_\alpha^2}{\partial y} & \frac{\partial f_\alpha^2}{\partial z} \\ \frac{\partial f_\alpha^3 + z}{\partial x} & \frac{\partial f_\alpha^3 + z}{\partial y} & \frac{\partial f_\alpha^3 + z}{\partial z} \end{bmatrix} = \begin{bmatrix} & 0 \\ df_\alpha & 0 \\ & 1 \end{bmatrix}$$

and therefore (Laplace expansion along the last column)

$$det(dF_q) = det\left(\begin{bmatrix} \frac{\partial f_\alpha^1}{\partial x}|_q & \frac{\partial f_\alpha^1}{\partial y}|_q \\ \frac{\partial f_\alpha^2}{\partial x}|_q & \frac{\partial f_\alpha^2}{\partial y}|_q \end{bmatrix}\right) \neq 0$$

The inverse function theorem gives us the existence of a neighbourhood M of $f_\alpha(q)$ such that F^{-1} exists and is differentiable on M. We can now restrict F^{-1} to a neighbourhood $f_\beta(N) \subset M$, where N is a neighbourhood of x in U_β. In this neighbourhood

$$F^{-1} \circ f_\beta|_N = f_\alpha^{-1} \circ f_\beta|_N$$

is differentiable as the composition of differentiable maps. Since x was arbitrary $f_\alpha^{-1} \circ f_\beta$ is differnetiable on $f_\beta^{-1}(W)$. □

We are going to use this property to formulate a definition that is independent of the ambient space.

Definition 1 (**n-dimensional differentiable manifold**).
An *n-dimensional differential manifold* is a set M with a family of injective maps $f_\alpha : U_\alpha \subset \mathbb{R}^n \longrightarrow M$ of open sets $U_\alpha \subset \mathbb{R}^n$ into M, such that

1. $\bigcup_\alpha f_\alpha(U_\alpha) = M$
2. For each pair α, β with $f_\alpha(U_\alpha) \cap f_\beta(U_\beta) = W = \emptyset$, the sets $f_\alpha^{-1}(W)$ and $f_\beta^{-1}(W)$ are open sets in $\mathbb{R}^n$ and the maps $f_\beta^{-1} \circ f_\alpha : f_\alpha^{-1}(W) \longrightarrow U_\beta$ and $f_\alpha^{-1} \circ f_\beta : f_\beta^{-1}(W) \longrightarrow U_\alpha$ are differentiable.
3. The family $\{U_\alpha, f_\alpha\}$ is maximal relative to 1. and 2..

A pair (U_α, f_α) with $p \in f_\alpha(U_\alpha)$ is called a *parametrization* or *coordinate system* of M at p.
A familiy $\{U_\alpha, f_\alpha\}_\alpha$ satisfying 1. and 2. is called a differentiable structure of M.

Example 1 (**The real projective space**).
We identify the real projective space $\mathbb{P}^n_\mathbb{R}$ with the quotient space $\mathbb{R}^{n+1} \setminus \{0\}_{/\sim}$, where $\sim$ is the equivalence relation given by

$$(x_1, \ldots, x_{n+1}) \sim (\lambda x_1, \ldots, \lambda x_{n+1}) \qquad \lambda \in \mathbb{R}, \lambda \neq 0$$

We denote an element in $x \in \mathbb{P}^n_\mathbb{R}$ by $x = [x_1, \ldots, x_{n+1}]$. In order to proof that the real projective space is a differentiable manifold, we need to find a family satisfying 1. *and* 2. *in definition* 1. *Let*

$$V_i = \{[x_1, \ldots, x_{n+1}] | x_i \neq 0\} \subset \mathbb{P}^n_\mathbb{R}$$

and define the functions $f_i : \mathbb{R}^n \longrightarrow V_i$ by $f_i(y_1, \ldots, y_n) = [y_1, \ldots, y_{i-1}, 1, y_i, \ldots, y_n]$.
We claim that $\{V_i, f_i\}_{(i \in \{1,\ldots,n+1\})}$ is a differential structure on M.
First we note that $f_i : \mathbb{R}^n \longrightarrow V_i$ is bijective. Moreover, $f_i^{-1}(V_i) = \mathbb{R}^n$ is open in $\mathbb{R}^n$ and $\bigcup_{i=1}^n f_i(\mathbb{R}^n) = \mathbb{P}^n_\mathbb{R}$.
We have $f_i(\mathbb{R}^n) \cap f_j(\mathbb{R}^n) = V_i \cap V_j \neq \emptyset$. We need to show that $f_i^{-1}(V_i \cap V_j)$ is open in $\mathbb{R}^n$ and that $f_j^{-1} \circ f_i$ is differentiable. Note that

$$V_i \cap V_j = \{[x_1, \ldots, x_{n+1} | x_i \neq 0 \wedge x_j \neq 0\} = \{[\frac{x_1}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, 1, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_n}{x_i}] | x_i \neq 0 \wedge x_j \neq 0\}$$

With that observation we get

$$f_i^{-1}(V_i \cap V_j) = \{\left(\frac{x_1}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_n}{x_i}\right) | x_j \neq 0 \wedge x_i \neq 0\} = \{(x_1, \ldots, x_n) | x_j \neq 0\}$$

which is open in $\mathbb{R}^n$. Moreover, we have (without loss of generality $i > j$)

$$\begin{aligned} f_j^{-1} \circ f_i(x_1, \ldots, x_n) &= f_j^{-1}([x_1, \ldots, x_{i-1}, 1, x_i, \ldots, x_{n+1}]) \\ &= f_j^{-1}([\frac{x_1}{x_j}, \ldots, \frac{x_{j-1}}{x_j}, 1, \frac{x_{j+1}}{x_j}, \ldots, \frac{x_{i-1}}{x_j}, \frac{1}{x_j}, \frac{x_i}{x_j}, \ldots, \frac{x_{n+1}}{x_j}]) \\ &= (\frac{x_1}{x_j}, \ldots, \frac{x_{j-1}}{x_j}, \frac{x_{j+1}}{x_j}, \ldots, \frac{x_{i-1}}{x_j}, \frac{1}{x_j}, \frac{x_i}{x_j}, \ldots, \frac{x_{n+1}}{x_j}) \end{aligned}$$

which is differentiable on $f_i^{-1}(V_i \cap V_j)$.

Definition 2 (differentiable map between manifolds).
Let M_1 and M_2 be n resp. m-dimensional manifolds. A map $\varphi : M_1 \longrightarrow M_2$ is *differentiable* at a point $p \in M_1$ if given a parametrization $g : V \subset \mathbb{R}^m \longrightarrow M_2$ around $\varphi(p)$, there exists a parametrization $f : U \subset \mathbb{R}^n \longrightarrow M_1$ around p such that $\varphi(f(U)) \subset g(V)$ and the map

$$g^{-1} \circ \varphi \circ f : U \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

is differentiable at $f^{-1}(p)$. The map φ is differentiable on an open set of M_1 if it is differentiable at all points of this set.
We call a differentiable map $\varphi_1 : M_1 \longrightarrow M_2 = \mathbb{R}$ a *differentiable function* on M_1 and a differentiable map $\varphi_2 : I \subset \mathbb{R} \longrightarrow M_2$ a *differentiable curve* on M_2.

The differentiability of a map between manifolds is well-defined, i.e. independent of the choice of parametrizations, because the change of parametrizations is by definition differentiable.

Motivation 2.
In $\mathbb{R}^3$ a tangent vector v at a point p to a differentiable curve $\alpha : I \subset \mathbb{R} \longrightarrow S \subset \mathbb{R}^3, \alpha(0) = p$ on a regular surface S is simply defined as $v := \alpha'(0) = [\alpha_1'(0), \alpha_2'(0), \alpha_3'(0)]^t$. Again, since we do not necessarily have the ambient space $\mathbb{R}^n$, we need an alternative characterization. The idea is that a tangent vector is fully characterized by the derivatives of functions along this tangent vector.
Let $\varphi : \mathbb{R}^3 \longrightarrow \mathbb{R}$ be differentiable in a neighbourhood of $p \in S$. We calculate the directional derivative of φ in direction of the tangent vector v:

$$\frac{d}{dt}(\varphi \circ \alpha)|_{t=0} = \sum_{i=1}^{3} \frac{\partial \varphi}{\partial \alpha_i} \frac{\partial \alpha_i}{\partial t}|_{t=0} = \left(\sum_{i=1}^{3} \alpha_i'(0) \frac{\partial}{\partial \alpha_i} \right) \varphi$$

Notice that we can get the coordinates of v by applying the operator $\left(\sum_{i=1}^{3} \alpha_i'(0) \frac{\partial}{\partial \alpha_i} \right)$ to the functions $\varphi_1(x, y, z) = (x, 0, 0), \varphi_2(x, y, z) = (0, y, 0), \varphi_3(x, y, z) = (0, 0, z)$. This observation motivates the following definition:

Definition 3 (tangent vector).
Let $\{0\} \subset I$ be an open subset of $\mathbb{R}$ and $\alpha : I \longrightarrow M$ a differentiable curve on a differentiable manifold M through $p \in M$, $\alpha(0) = p$. Let D be the set of functions on M, which are differentiable at p. The *tangent vector* to the curve α at p is the map

$$\alpha'(0) : D \longrightarrow \mathbb{R}$$

given by

$$\alpha'(0)\varphi = \frac{d}{dt}(\varphi \circ \alpha)|_{t=0}, \qquad \varphi \in D$$

A tangent vector at $p \in M$ is the tangent vector to some differential curve $\alpha : I \longrightarrow M$ with $\alpha(0) = p$.

Proposition 1. *The set T_pM of tangent vectors at a point $p \in M$ of an n-dimensional differentiable manifold M, called the tangent space, is an n-dimensional real vector space.*

Proof. Choose a parametrization $f : U \subset \mathbb{R}^n \longrightarrow M$ around p, without loss of generality $p = f(\vec{0})$, and a curve $\alpha : I \longrightarrow M$ on M through p, $\alpha(0) = p$. Let $\varphi \in D$.
Then by definition $f^{-1} \circ \alpha \circ id \equiv f^{-1} \circ \alpha : I \longrightarrow \mathbb{R}^n$ is differentiable. Therefore, we can write $f^{-1} \circ \alpha = (x_1(t), \dots, x_n(t))$ with differentiable functions $x_i : I \longrightarrow \mathbb{R}$. We also write $\varphi \circ f(q) = \varphi(x_1, \dots, x_n)$, where $q = (x_1, \dots, x_n) \in U$. Then we have

$$\begin{aligned}\alpha'(0)\varphi = \frac{d}{dt}(\varphi \circ \alpha)|_{t=0} &= \frac{d}{dt}(\varphi \circ f \circ f^{-1} \circ \alpha)|_{t=0} \\ &= \frac{d}{dt}\varphi(x_1(t), \dots, x_n(t))|_{t=0} \\ &= (\sum_{i=1}^{n} x_i'(0)(\frac{\partial}{\partial x_i})|_0)\varphi\end{aligned}$$

This observation leads us to the assumption that T_pM, the tangent space to M at p, is equal to $T_f = \langle (\frac{\partial}{\partial x_i})|_0 | i = 1, \dots, n\rangle$. (Notice that $\frac{\partial}{\partial x_i}$ depends on the choice of f!). We have already seen that $T_pM \subset T_f$.

For the other direction, $T_f \subset T_pM$, let $v \in T_f$, i.e. $v = \sum_{i=1}^n \lambda_i(\frac{\partial}{\partial x_i})|_0 \in T_f$. We now have to look for a curve α for which v is the tangent vector at p.
Let $\alpha : I \longrightarrow M$ be given by $f^{-1} \circ \alpha(t) = (\lambda_1 t, \dots, \lambda_n t) = \vec{x}(t)$ or equivalently $\alpha(t) := f(\lambda_1 t, \dots, \lambda_n t)$ where f is the given parametrization of M around p. Then

$$\alpha'(0) = \sum_{i=1}^{n} x_i'(0)\frac{\partial}{\partial x_i}|_0 = \sum_{i=1}^{n} \lambda_i(\frac{\partial}{\partial x_i})|_0$$

Finally, $T_pM = T_f$ is an n-dimensional vector space and the chosen parametrization f determines a basis $\{(\frac{\partial}{\partial x_i})|_0\}_{(i=1,\dots,n)}$ of T_pM.

□

Motivation 3.
We first consider the case of a differentiable map $\varphi : S_1 \longrightarrow S_2$ between two regular surfaces S_1 and S_2 in $\mathbb{R}^3$. At a point $p \in S_1$ the differential $d\varphi_p$ is a map from the tangent space T_pS_1 to the tangent space $T_{\varphi(p)}S_2$. It maps a tangent vector $v = \alpha'(0)$ to a curve α, $\alpha(0) = p$, to the tangent vector $w = (\varphi \circ \alpha)'(0)$ to the curve $\varphi \circ \alpha$, $(\varphi \circ \alpha)(0) = \varphi(p)$.
To accept that this is well-defined we need to check that $d\varphi_p(v)$, $v = \alpha'(0)$ is independent of the choice of the curve α. Moreover, we will proof that $d\varphi_p$ is linear.

Proof. We choose parametrizations (f, U) around p and (g, V) around $\varphi(p)$. Let $\varphi(x, y) = (\varphi^1(x, y), \varphi^2(x, y))$, be the coordinate expression of φ with respect to the chosen parametrizations, let $\alpha(t) = (\alpha_1(t), \alpha_2(t))$, then $\beta(t) = \varphi(\alpha(t)) = \varphi(\alpha_1(t), \alpha_2(t))$. We get

$$d\varphi_p(v) = \beta'(0) = (\varphi \circ \alpha)'(0) = \left(\sum_{i=1}^{2} \frac{\partial \varphi^1}{\partial \alpha_i}\alpha_i'(0), \sum_{i=1}^{2} \frac{\partial \varphi^2}{\partial \alpha_i}\alpha_i'(0)\right)$$

This does not depend on α but only on the coordinates $v = [\alpha_1'(0), \alpha_2'(0)]$ w.r.t the basis that is determined by the choice of parametrization. Moreover the map $d\varphi_p$ can be written as:

$$d\varphi_p(v) = \beta'(0) = \begin{bmatrix} \frac{\partial \varphi^1}{\partial \alpha_1} & \frac{\partial \varphi^1}{\partial \alpha_2} \\ \frac{\partial \varphi^2}{\partial \alpha_1} & \frac{\partial \varphi^2}{\partial \alpha_2} \end{bmatrix} \underbrace{\begin{bmatrix} \alpha_1'(0) \\ \alpha_2'(0) \end{bmatrix}}_{=v}$$

Therefore it is a linear map. □

Definition 4 (**The differential**).
Let M_1,M_2 be n- respectively m-dimensional differentiable manifolds and $\varphi : M_1 \longrightarrow M_2$ a differentiable map. For all $p \in M$ the *differential* of φ at p is the linear map

$$\begin{aligned} d\varphi_p : T_pM_1 &\longrightarrow T_{\varphi(p)}M_2 \\ v &\longmapsto d\varphi_p(v) \end{aligned}$$

such that for all differentiable curves $\alpha : (-\epsilon, \epsilon) \longrightarrow M_1$ with $\alpha(0) = p$, $\alpha'(0) = v$ the following relation holds:

$$d\varphi_p(v) = (\varphi \circ \alpha)'(0) \tag{1}$$

In order for the definition to be well-defined, the differential (1) has to be independent of the choice of α. By taking parametrizations around p and $\varphi(p)$, we reduce the problem to a map between $\mathbb{R}^n$ and $\mathbb{R}^m$, where the claim can be proven analogous to motivation 3.

Definition 5 (**immersion, embedding, submanifold**).
Let M, N be m- respectively n-dimensional differentiable manifolds and $\varphi : M \longrightarrow N$ differentiable.

- φ is called an *immersion*, if $d\varphi_p : T_pM \longrightarrow T_{\phi(p)}M$ is injective for all $p \in M$.
- φ is called an *embedding*, if it is an immersion and a homeomorphism onto $\varphi(M) \subset N$, where $\varphi(M)$ has the topology induced by N.
- If $M \subset N$ and the inclusion $\iota : M \longrightarrow N$ is an embedding, then M is called a *submanifold* of N.

Example 2 (**alternative representation of the projective plane**).
We can model the projective space in two dimensions, $\mathbb{P}^2_{\mathbb{R}}$ *as the quotient space of the sphere* $\mathcal{S}^2 = \{p \in \mathbb{R}^3 | \|p\| = 1\}$ *by the equivalence relation* $p \sim q :\Leftrightarrow q = \pm p$*, i.e.* $\mathbb{P}^2_{\mathbb{R}} = \mathcal{S}^2/_\sim$*. Basically, we identify antipodal points.*
We are now looking for a differentiable structure for this representation of the projective plane. To find this we first look at a well-known differentiable structure on the sphere:

$$\begin{aligned} f_i^+ &: U_i \longrightarrow \mathcal{S}^2 \\ f_i^- &: U_i \longrightarrow \mathcal{S}^2 \qquad i = 1, 2, 3 \end{aligned}$$

$$\begin{aligned} U_1 &= \{(x_1, x_2, x_3) \in \mathbb{R}^3; x_1 = 0, x_2^2 + x_3^2 < 1\} \\ U_2 &= \{(x_1, x_2, x_3) \in \mathbb{R}^3; x_2 = 0, x_1^2 + x_3^2 < 1\} \\ U_3 &= \{(x_1, x_2, x_3) \in \mathbb{R}^3; x_3 = 0, x_1^2 + x_2^2 < 1\} \end{aligned}$$

$$\begin{aligned} f_1^+(\vec{x}) &= f_1^+(0, x_2, x_3) = (\sqrt{1 - (x_2^2 + x_3^2)}, x_2, x_3) \\ f_1^-(\vec{x}) &= f_1^-(0, x_2, x_3) = (-\sqrt{1 - (x_2^2 + x_3^2)}, x_2, x_3) \\ f_2^+(\vec{x}) &= f_2^+(x_1, 0, x_3) = (x_1, \sqrt{1 - (x_1^2 + x_3^2)}, x_3) \\ f_2^-(\vec{x}) &= f_2^+(x_1, 0, x_3) = (x_1, -\sqrt{1 - (x_1^2 + x_3^2)}, x_3) \\ f_3^+(\vec{x}) &= f_3^+(x_1, x_2, 0) = (x_1, x_2, \sqrt{1 - (x_1^2 + x_2^2)}) \\ f_3^-(\vec{x}) &= f_3^-(x_1, x_2, 0) = (x_1, x_2, -\sqrt{1 - (x_1^2 + x_2^2)}) \end{aligned}$$

Now let $\pi : \mathcal{S}^2 \longrightarrow \mathbb{P}^2_{\mathbb{R}}$ be the canonical projection; $\pi(p) = \{p, -p\}$. Then $\pi(f_i^+(U_i)) = \pi(f_i^-(U_i))$ (but not necessarily $\pi(f_i^+(\vec{x})) = \pi(f_i^-(\vec{x}))$. We claim that by defining

$$g_i = \pi \circ f_i^+$$

we obtain a differentiable structure on $\mathbb{P}^2_{\mathbb{R}}$. Condition 1 in definition 1 is obviously fullfilled. For condition 2 we observe that

$$g_i^{-1} \circ g_j = (\pi \circ f_i^+)^{-1} \circ (\pi \circ f_j^+) = (f_i^+)^{-1} \circ \pi^{-1} \circ \pi \circ f_j^+ = (f_i^+)^{-1} f_j^+$$

where we used the fact that g_i is injective in the second step. Hence, $(g_i^{-1} \circ g_j)$ is differentiable and therefore (g_i, U_i) is a differentiable structure.

Example 3 (Immersion of the projective plane in $\mathbb{R}^4$).
We define the map $\varphi : \mathbb{R}^3 \longrightarrow \mathbb{R}^4$ by $\varphi(x, y, z) = (x^2 - y^2, xy, xz, yz)$ for $(x, y, z) \in \mathbb{R}^3$. Let $\mathcal{S}^2 := \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 = 1\}$ be the unit sphere and let π be the canoncial projection of $\mathcal{S}^2$ onto the real projective plane. First, we observe that φ is symmetric, i.e. $\varphi((x, y, z)) = \varphi((-x, -y, -z))$. Now we define the map $\theta : \mathbb{P}^2_{\mathbb{R}} \longrightarrow \mathbb{R}^4$ by $\theta(\{p, -p\}) = \varphi(p)$. This is well-defined because of the previous observation. We claim that θ is an immersion of $\mathbb{P}^2_{\mathbb{R}}$ into $\mathbb{R}^4$. We have already seen that π is a local diffeomorphism. It is therefore enough to show that $\varphi|_{\mathcal{S}^2}$ is an immersion. We will prove this by using the parametrization of $\mathcal{S}^2$ that we found in the previous example. We have

$$f_3^+(\vec{x}) = f_3^+(x_1, x_2, 0) = (x_1, x_2, \sqrt{1 - (x_1^2 + x_2^2)})$$

and composed with φ

$$\varphi \circ f_3^+(\vec{x}) = \varphi \circ f_3^+(x_1, x_2, 0) = (x_1^2 - x_2^2, x_1 x_2, x_1\sqrt{1 - (x_1^2 + x_2^2)}, x_2\sqrt{1 - (x_1^2 + x_2^2)})$$

To simplify the equation we will define $D_{12} = \sqrt{1 - (x_1^2 + x_2^2)}$ and we get

$$\varphi \circ f_3^+(\vec{x}) = (x_1^2 - x_2^2, x_1 x_2, x_1 D_{12}, x_2 D_{12})$$

The differential $d(\varphi \circ f_3^+)$ is injective, since

$$D(\varphi \circ f_3^+)(x, y) = \begin{bmatrix} 2x & y & D_{12} + x\frac{\partial}{\partial x} D_{12} & y\frac{\partial}{\partial x} D_{12} \\ -2y & x & x\frac{\partial}{\partial y} D_{12} & D_{12} + \frac{\partial}{\partial y} D_{12} \end{bmatrix}$$

has full row rank, i.e. has rank 2. An analogous argument for the other parametrizations shows that $\varphi|_{\mathcal{S}^2}$ is an immersion and therefore θ is also an immersion from $\mathbb{P}^2_{\mathbb{R}}$ into $\mathbb{R}^4$.

Remark 1. *The immersion in the previous example is actually an embedding. Therefore the projective plane can be embedded in $\mathbb{R}^4$, it can be shown, however, that it cannot be embedded in $\mathbb{R}^3$.*

2 Differential forms on manifolds

We want now to extend the notion of differential forms for differentiable manifolds.

Definition 6. Let M^n an n-dimensional differentiable manifold. An *exterior k-form w* in M is the choice, for every $p \in M$, of an element $w(p)$ of the space $\Lambda^k(T_pM)^*$ of alternate k-linear forms of the tangent T_pM.

Definition 7. Given an exterior k-form w and $f_\alpha : U_\alpha \to M^n$ a parametrization, around $p \in f_\alpha(U_\alpha)$ we define the *representation* of w in this parametrization as the exterior k-form in $U_\alpha \subset \mathbb{R}^n$ given by

$$w_\alpha(v_1, \dots, v_n) = w(df_\alpha(v_1), \dots, df_\alpha(v_n)), \qquad \text{with} \qquad v_1, \dots, v_n \in \mathbb{R}^n$$

If we change coordinates to a different $f_\beta : U_\beta \to M^n$ we obtain

$$\begin{aligned}(f_\beta^{-1} \circ f_\alpha)^* w_\beta(v_1, \dots, v_n) =& w(d(f_\beta^{-1} \circ f_\alpha)(v_1), \dots, d(f_\beta^{-1} \circ f_\alpha)(v_k)) \\ =& w((df_\beta \circ d(f_\beta^{-1} \circ f_\alpha))(v_1), \dots, (df_\beta \circ d(f_\beta^{-1} \circ f_\alpha))(v_k)) \\ =& w_\alpha(v_1, \dots, v_n)\end{aligned}$$

i.e. $(f_\beta^{-1} \circ f_\alpha)^* w_\beta = w_\alpha$.

Definition 8. A *differential form of order k* or *differential k-form* in a differentiable manifold M^n is an exterior k-form such that, in some coordinate system, its representation is differentiable.

From all this, it follows that a differential k-form is the choice, for each parametrization (U_α, f_α) of a differential form w_α in U_α in such a way that for a different parametrization (U_β, f_β) with $f_\alpha(U_\alpha) \cap f_\beta(U_\beta) \neq \emptyset$, applies that $w_\alpha = (f_\beta^{-1} \circ f_\alpha)^* w_\beta$.

A remarkable and important fact is that all operations defined for differential forms in $\mathbb{R}^n$ can naturally be extended to differential forms on manifolds through their local representations. Indeed, given a differential form w in M, dw is the differential form whose local representation is dw_α. Since

$$dw_\alpha = d(f_\beta^{-1} \circ f_\alpha)^* w_\beta = (f_\beta^{-1} \circ f_\alpha)^* dw_\beta$$

it follows that dw is well defined as differential form on M.

3 Vector fields

The concept of vector field is closely associated with differential forms. Let us first define what it is.

Definition 9. A *vector field* on a differentiable manifold M is a correspondence that associates to each point $p \in M$ a vector $X(p) \in T_pM$. The vector field X is said to be *differentiable* if $X\varphi$ is differentiable for each differentiable $\varphi : M \to \mathbb{R}$.

Let $f_\alpha : U_\alpha \to M^n$ be a parametrization for M and $X_i = \frac{\partial}{\partial x_i}$, $i = 1, \dots, n$, the basis associated to the parametrization. Then a vector field, for each point p can be described in the parametrization through its *local expression*

$$X(p) = \sum_{i=1}^{n} a_i(p) X_i$$

where each a_i is a C^∞ function.

We can now denote with $\mathfrak{X}(M)$ the set of all vector fields on M and observe that given $X, Y \in \mathfrak{X}(M)$ and $f \in C^\infty(M)$, we can define the sum $(X+Y)(p) = X(p) + Y(p)$ and a product $(fX)(p) = fX(p)$, that give $\mathfrak{X}(M)$ a structure of vector space.

Lemma 1. *Let X and Y be differentiable vector fields on a differentiable manifold. Then there exists a unique $Z \in \mathfrak{X}(M)$ such that, for every differentiable function φ, $Z\varphi = (XY - YX)\varphi$.*

The vector field so defined is called bracket *and it is denoted as $[X, Y]\varphi = (XY - YX)\varphi$.*

Proof. To prove that if such Z exists then it is unique, let $f : U \to M$ be a parametrization and

$$X = \sum_i a_i \frac{\partial}{\partial x_i} \quad \text{and} \quad Y = \sum_i b_i \frac{\partial}{\partial x_i}$$

be the expressions of X and Y in the parametrization f. Then

$$XY\varphi = X\left(\sum_j b_j \frac{\partial \varphi}{\partial x_j}\right) = \sum_{ij} a_i \frac{\partial b_j}{\partial x_i} \frac{\partial \varphi}{\partial x_j} + \sum_{ij} a_i b_j \frac{\partial^2 \varphi}{\partial x_i \partial x_j}$$

$$YX\varphi = Y\left(\sum_i a_i \frac{\partial \varphi}{\partial x_i}\right) = \sum_{ij} b_j \frac{\partial a_i}{\partial x_j} \frac{\partial \varphi}{\partial x_i} + \sum_{ij} a_i b_j \frac{\partial^2 \varphi}{\partial x_i \partial x_j}$$

So

$$(XY - YX)\varphi = \sum_i \left(\sum_j \left(a_i \frac{\partial b_j}{\partial x_i} - b_i \frac{\partial a_j}{\partial x_i}\right) \frac{\partial}{\partial x_j}\right)\varphi$$

It follows that any Z with the required properties must be expressed in this form in any coordinate system, hence it is unique.

To see that it always exists, it is enough to define Z_α in each coordinate neighborhood by the expression just given. By uniqueness $Z_\alpha = Z_\beta$ in $f_\alpha(U_\alpha) \cap f_\beta(U_\beta)$, thus Z is well defined on all M. □

Let's now see some of the properties of this bracket.

Proposition 2. *Given $X, Y, Z \in \mathfrak{X}(M)$, $a, b \in \mathbb{R}$ and $f, g \in C^\infty(M)$, then:*

1. $[X, Y] = -[Y, X]$
2. $[aX + bY, Z] = a[X, Z] + b[Y, Z]$
3. $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$
4. $[fX, gY] = fg[X, Y] + fX(g)Y - gY(f)X$

There exist an interesting correlation between exterior differentiation of differential forms and the operation of bracket. For the case of 1-forms, it applies the following.

Proposition 3. *Let ω be a differential 1-form on a differentiable manifold M and $X, Y \in \mathfrak{X}(M)$. Then*

$$d\omega(X, Y) = X\omega(Y) - Y\omega(X) - \omega([X, Y])$$

Proof. As usual, let $f : U \to M$ be a parametrization and

$$X = \sum_i a_i \frac{\partial}{\partial x_i} \quad \text{and} \quad Y = \sum_i b_i \frac{\partial}{\partial x_i}$$

be the expressions of X and Y in the parametrization f. First of all, we notice that if our claim holds for X_i and Y_j, then holds for $\sum X_i$ and $\sum Y_j$. Next, we claim that if it holds for X and Y, it also does for θX and φY, when θ, φ are differentiable functions. First of all, we notice that

$$d\omega(\theta X, \varphi Y) = \theta\varphi d\omega(X, Y) = \theta\varphi\{X\omega(Y) - Y\omega(X) - \omega([X, Y])\}$$

by definition. By the properties of the bracket

$$\begin{aligned}
(\theta X)\omega(\varphi Y) - (\varphi Y)\omega(\theta X) - \omega([\theta X, \varphi Y]) = \\
=&\theta X(\varphi)\omega(Y) + (\theta\varphi X)\omega(Y) - \varphi Y(\theta)\omega(X) - (\theta\varphi Y)\omega(X) \\
&- \theta\varphi\omega([X, Y]) - \theta X(\varphi)\omega(Y) + \varphi Y(\theta)\omega(X) = \\
=&\theta\varphi\{X\omega(Y) - Y\omega(X) - \omega([X, Y])\} = \\
=&\theta\varphi d\omega([X, Y])
\end{aligned}$$

that proves our claim. It follows that we just have to prove our statement for vectors $\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}$, and since their bracket is 0, it suffices to prove that

$$d\omega\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = \frac{\partial}{\partial x_i}\omega\left(\frac{\partial}{\partial x_j}\right) - \frac{\partial}{\partial x_j}\omega\left(\frac{\partial}{\partial x_i}\right)$$

Notice that if the property above applies to two 1-forms, then it applies to their sum. Thus we can again restrict and prove that

$$d(\alpha dx_k)\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = \frac{\partial}{\partial x_i}\alpha dx_k\left(\frac{\partial}{\partial x_j}\right) - \frac{\partial}{\partial x_j}\alpha dx_k\left(\frac{\partial}{\partial x_i}\right)$$

where α is a differentiable function. The above reduces to

$$(d\alpha \wedge dx_k)\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = \delta_{kj}\frac{\partial \alpha}{\partial x_i} - \delta_{ki}\frac{\partial \alpha}{\partial x_j}$$

that is true by definition of exterior product. □

We can also observe that, with basically the same proof, one can generalize the last proposition for a differentiable k-form ω:

$$\begin{aligned} d\omega(X_1, \dots, X_{k+1}) = &\sum_{i=1}^{k+1} (-1)^{i+1} X_i \omega(X_1, \dots, \hat{X}_i, \dots, X_{k+1}) \\ &+ \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_1, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots X_{k+1}) \end{aligned}$$

where each X_i is a differentiable vector field and $\hat{X}_i$ means that X_i is missing.

4 Orientation

As for surfaces we might need to know "which side" we are walking on. Of course it is not always possible to determine, but we can give a generalization of the concept of orientation for manifolds.

Definition 10. A differentiable manifold M is *orientable* if it has a differentiable structure $\{(U_\alpha, f_\alpha)\}$ such that for each pair α, β with $f_\alpha(U_\alpha) \cap f_\beta(U_\beta) \neq \emptyset$, the differential of the change of coordinates $f_\beta^{-1} \circ f_\alpha$ has positive determinant. Otherwise M is called *non-orientable.*

Since on manifolfds we can not generally use arrows, we can let ordered basis of the tangent space be the needle of our compass. Indeed two different basis can be brought one into the other by a regular linear transformation T of the tangent space at a point, so we will define *equivalent* two basis such that the determinant of T is positive. This is of course an equivalence relation between ordered basis of T_pM and an equivalence class is called *orientation at p.* Let's take now a closer look to what happens when we consider the orientation at two points: obviously it is not sure that, a priori, they have the same orientation. If for each point p the orientation at p specifies also an orientation for an arbitrary point in a neighborhood, the orientation is said to be *coherent.*

Thinking about this you will realize that this is a very intuitive definition, and it corresponds to the idea we had previously that a surface is orientable if any closed path leads back to the starting point on the same side of the surface. This gives us the input for an equivalent definition of orientation for manifolds.

Definition 11. A differentiable manifold M is *orientable* if we can assign an orientation to each point of M such that the orientations at any two sufficiently near points in M are coherent. If the orientation is specified, M is called *oriented manifold.*

Remark 2. *If M is orientable, there are always exactly two orientations, that can be* positive *or* negative *in according with the orientation they acquire when seen as local ordered basis at a point. Moreover, if two orientations are the same at a point, then they are equal.*

Let's now go back to differential forms and take a look at how orientation is connected to them.

Proposition 4. *Any non vanishing n-form ω on a differentiable n-dimensional manifold M determines a unique orientation for which ω is positively oriented at each point. Conversely, if M is given an orientation, there is a smooth nonvanishing n-form that is positively oriented at each point.*

Because of this proposition, if M is a smooth n-dimensional differentiable manifold, any nonvanishing n-form on M is called an *orientation form.* If M is oriented and ω is an orientation form determining the given orientation, we also say that ω is *(positively) oriented.* It is easy to check that if ω and $\tilde{\omega}$ are two positively oriented smooth forms on M, then $\tilde{\omega} = f\omega$ for some strictly positive smooth real-valued function f.

Proposition 5. *Suppose M and N are differentiable manifolds of dimension n. If $F : M \to N$ is a local diffeomorphism and N is oriented, then M has a unique orientation, called the* pullback orientation induced by *F, such that F sends positively oriented basis of T_pM into positively oriented basis in $T_{F(p)}N$. Such diffeomorphisms are called* orientation-preserving.

Proof. For each $p \in M$ there is a unique orientation on T_pM that makes the isomorphism $dF_p : T_pM \to T_{F(p)}N$ orientation-preserving. This defines a pointwise orientation on M and provided it is continuous, it is the unique orientation on M with respect to which F is orientation-preserving. To see that it is continuous, just choose an orientation form ω for N and notice that $F^*\omega$ is an orientation form for M. □

We want to use these results to have some more detailed information on the manifolds.

Proposition 6. *Any open subset $U \subset M$ of a differentiable orientable manifold is orientable.*

Proof. It is enough to restrict an orientation form of M to U. □

For the viceversa it is less trivial.

Proposition 7. *Let $V = (V_i)_{i \in I}$ be an open covering for a differentiable manifold M. Let's suppose each V_i to be oriented and the restriction of the orientation forms from V_i and V_j to $V_i \cap V_j$ to be equal for $i \neq j$. Then there exists a unique orientation for M with the given restriction on each V_i.*

Proof. Let us denote ω_i the orientation form for V_i for each i, ω_p the orientation form with respect to p and consider a partition of unity ϕ_i. We can define ω as $\omega = \sum_{i \in I} \phi_i \omega_i$, with $\phi_i \omega_i$ extended to an n-form on the whole M (just setting it to 0 in $M \setminus Supp_M(\phi_i)$). This is still an orientation form, since given $p \in V_i \subset M$ and $b_1, ..., b_n$ basis for T_pM, with $\omega_{i,p}(b_1, ..., b_n) > 0$, we have $\omega_{j,p}(b_1, ..., b_n) > 0$ for each other j with $p \in V_j$, and in the following

$$\omega_p(b_1, ..., b_n) = \sum_i \phi_i(p)\omega_{i,p}(b_1, ..., b_n)$$

all terms are greater or equal than 0. So ω is an orientation form on M and $b_1, ..., b_n$ are positively oriented with respect to ω, and this orientation satisfies the properties we desired. If τ is an other orientation form giving M an orientation that satisfies the same properties, then $\tau = f\omega$ and $\tau_p(b_1, ..., b_n) = f(p)\omega_p(b_1, ..., b_n)$. Since both τ_p and ω_p are positive on $b_1, ..., b_n$ then $f(p) > 0$, and so ω and τ determine the same orientation. □

Proposition 8. *Let M be a differentiable manifold and V_1, V_2 coordinate neighborhoods such that $V_1 \cup V_2 = M$ and $V_1 \cap V_2 := W$ is connected. Then M is orientable.*

Let's conclude making an idea of how all of it works with a couple exercises.

Exercise 1. *Show that the sphere $\mathcal{S}^n$ is orientable.*

Carry out. Let's consider the stereographic projection $\pi : \mathcal{S}^n \to \mathbb{R}^n$. We can use it to cover $\mathcal{S}^n$ with two open sets such that their intersection is connected. $\square$

Exercise 2. *Prove that the projective space $\mathbb{P}^n_{\mathbb{R}}$ is orientable for n odd and non-orientable for n even.*

Carry out. Let's see $\mathbb{P}^n_{\mathbb{R}}$ as $\mathcal{S}^n/\{\pm 1\}$. The function $f : \mathcal{S}^n \to \mathcal{S}^n$ defined by $f(p) = -p$ is an orientation-preserving diffeomorphism if n is odd and orientation-reversing if n is even, because if we extend f to a diffeomorphism $\tilde{f}$ of the whole $\mathbb{R}^{n+1}$ by the same formula, $d\tilde{f}\big(\frac{\partial}{\partial x_i}\big) = -\big(\frac{\partial}{\partial x_i}\big)$. On the other hand, $\tilde{f}$ carries the outward normal vector to the outward normal vector, so that the orientation stays coherent for n odd but not if n is even. $\square$

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