Differential Geometry of Surfaces

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This article is based on [Car94, pp. 77-96].

1 The Structure Equations of $\mathbb{R}^n$

Definition 1. *A* Riemannian manifold *is a differentiable manifold M and a choice, for each point $p \in M$, of a positive definite inner product $\langle\ ,\ \rangle_p$ in T_pM which varies differentiably with p. I.e. if X and Y are differentiable vector fields in M, then $p \to \langle X, Y\rangle_p$ is differentiable in M. The inner product is called a* Riemannian metric *on M.*

Definition 2. *A diffeomorphism $\varphi : M \to M'$ between two Riemannian manifolds M and M' is called an* isometry *if*

$$\langle x, y\rangle_p = \langle \mathrm{d}\varphi_p(x), \mathrm{d}\varphi_p(y)\rangle_{\varphi(p)}$$

holds for all $p \in M$ and all pairs $x,\ y \in T_pM$.

Definition 3. *Let $U \subset \mathbb{R}^n$ be an open set. A set $e_1, \dots, e_n$ of n differentiable vector fields such that for each $p \in U$: $\langle e_i, e_j\rangle_p = \delta_{ij}$, where δ_{ij} is the Kronecker delta, is called* orthonormal moving frame. *Given an orthonormal moving frame $\{e_i\}$, the set of differential 1-forms $\{\omega_i\}$ given by $\omega_i(e_j) = \delta_{ij}$, $i, j \in \{1, \dots, n\}$ at all p is called the* coframe associated to $\{e_i\}$.

In the following, the terminus "orthonormal" will be implicit. Next, we construct the *connection forms* on $\mathbb{R}^n$: By differentiability of the $e_i : U \to \mathbb{R}^n$ they give for any $p \in U$ the linear map $(\mathrm{d}e_i)_p : \mathbb{R}^n \to \mathbb{R}^n$. We can therefore write

$$(\mathrm{d}e_i)_p(v) = \sum_j (\omega_{ij})_p(v) e_j,$$

for any $v \in \mathbb{R}^n = T_p\mathbb{R}^n$. Each $(\omega_{ij})_p(v)$ is linear in v, i.e. $(\omega_{ij})_p$ is a linear form in $\mathbb{R}^n$ $\forall i, j \in \{1, \dots, n\}$. Likewise we denote the corresponding differential 1-forms in U with ω_{ij} and call them *connection forms* of $\mathbb{R}^n$. Then $\mathrm{d}e_i = \sum_j \omega_{ij} e_j$ holds at any p. Note that the connection forms are antisymmetric:

$$\langle e_i, e_j\rangle = \delta_{ij} \Rightarrow 0 = \langle \mathrm{d}e_i, e_j\rangle + \langle e_i, \mathrm{d}e_j\rangle = \omega_{ij} + \omega_{ji}$$

Proposition 4 (The structure equations of $\mathbb{R}^n$)**.** *Let $\{e_i\}$ be a moving frame in an open set $U \subset \mathbb{R}^n$. Let $\{\omega_i\}$ be the coframe associated to $\{e_i\}$ and $\{\omega_{ij}\}$ the connection forms of U in the frame $\{e_i\}$. Then*

$$\mathrm{d}\omega_i = \sum_k \omega_k \wedge \omega_{ki}, \tag{1}$$

$$\mathrm{d}\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} \tag{2}$$

for $i, j, k = 1, \ldots, n$.

Proof. The proof is not complicated and found in [Car94] Chapter 5.1, Prop. 1. $\square$

To clarify what this means and where the names come from consider the following: We denote by $x : U \hookrightarrow \mathbb{R}^n$ the inclusion map. Then to say that the ω_i are dual to the frame $\{e_i\}$ is equivalent to the statement $dx = \sum_i \omega_i e_i$. A curve $\alpha(s)$ in U can be described by the inclusion x together with a moving frame $\{e_i\}$. Their differentials describe how the point and the frame in this point are varying:

$$\mathrm{d}x = \sum_i \omega_i e_i, \qquad \mathrm{d}e_i = \sum_j \omega_{ij} e_j$$

The structure equations then follow from the above and that $\mathrm{d}^2 x = 0$ and $\mathrm{d}^2 e_i = 0$

Lemma 5 (Cartan's lemma)**.** *Let* V *be a real vector space of dimension* n*, and let* $\omega_1, \ldots, \omega_r : V \to \mathbb{R}$, $r \leq n$*, be linear forms in* V *that are linearly independent. Assume that there exist 1-forms* $\theta_1, \ldots, \theta_r : V \to \mathbb{R}$*, such that* $\sum_{i=1}^r \omega_i \wedge \theta_i = 0$*. Then*

$$\theta_i = \sum_j a_{ij} \omega_j, \quad \textit{with } a_{ij} = a_{ji}.$$

Proof. We complete the forms ω_i into a basis $\omega_1, \ldots, \omega_r, \omega_{r+1}, \ldots, \omega_n$ of V^*. Then for some coefficients a_{ij}, b_{il}

$$\theta_i = \sum_{j=1}^r a_{ij} \omega_j + \sum_{l=r+1}^n b_{il} \omega_l.$$

Now from the hypothesis

$$\begin{aligned} 0 = \sum_{i=1}^r \omega_i \wedge \theta_i &= \sum_{i=1}^r \left[\sum_{j=1}^r a_{ij} \omega_i \wedge \omega_j + \sum_{l=r+1}^n b_{il} \omega_i \wedge \omega_l \right] \\ &= \sum_{i=1}^r \left[\sum_{j>i}^r (a_{ij} - a_{ji}) \omega_i \wedge \omega_j + \sum_{l=r+1}^n b_{il} \omega_i \wedge \omega_l \right]. \end{aligned}$$

From the fact that the $\omega_i \wedge \omega_k$ are linearly independent for $i < k$, $i, k = 0, \ldots, n$ necessarily $b_{il} = 0$ and $a_{ij} = a_{ji}$. $\square$

Lemma 6. *Let* $U \subset \mathbb{R}^n$ *be open and let* $\omega_1, \ldots, \omega_n$ *be linearly independent differential 1-forms in* U*. Assume that there exists a set of differential 1-forms* $\{\omega_{ij}\}$, $i, j \in \{1, \ldots, n\}$ *satisfying*

$$\omega_{ij} = -\omega_{ij}, \qquad \mathrm{d}\omega_i = \sum_k \omega_k \wedge \omega_{ki}.$$

Then such a set is unique.

Proof. Suppose there exists another such set denoted by $\bar{\omega}_{ij}$. Then

$$\sum_k \omega_k \wedge (\bar{\omega}_{kj} - \omega_{kj}) = 0.$$

By Cartan's Lemma 5, we can write the appearing 1-forms above as

$$\bar{\omega}_{kj} - \omega_{kj} = \sum_i B^j_{ki} \omega_i, \quad B^j_{ki} = B^j_{ik}.$$

Using additionally that the antisymmetry of $\bar{\omega}_{kj} - \omega_{kj}$ in k, j one finds that $B^j_{ki} = -B^k_{ji}$. With these:

$$B^k_{ji} = -B^j_{ki} = -B^j_{ik} = B^i_{jk} = B^i_{kj} = -B^k_{ij} = -B^k_{ji} = 0, \quad \text{i.e. } \bar{\omega}_{kj} = \omega_{kj}$$

$\square$

2 Surfaces in $\mathbb{R}^3$

To make use of the moving frames for describing the geometry of surfaces in $\mathbb{R}^3$, we will need one further definition. It relies on the fact that any immersion $x : M \to \mathbb{R}^{n+k}$ of a n-dimensional differentiable manifold M into euclidean space R^{n+k} is *locally* an embedding. This means that for any $p \in M$ there exists a neighbourhood $U \subset M$ of p such that the restriction $x\big|_U : U \to \mathbb{R}^{n+k}$ is an embedding (c.f.[Car94] Chapter 3, Exercise 4).

Definition 7. *Let M, n, k and x as above and $V \subset \mathbb{R}^{n+k}$ be a neighbourhood of $x(p)$ in $\mathbb{R}^{n+k}$ such that $V \cap x(M) = x(U)$. Assume that V is such that there exists a moving frame $\{e_1, \dots, e_{n+k}\}$ in V with the property that, when restricted to $x(U)$, the vectors $e_1, \dots, e_n$ are tangent to $x(U)$. Such a moving frame is then called an* adapted frame.

Let in the following M, M' be two-dimensional differentiable manifolds with an immersion $x : M \to \mathbb{R}^3$. We define an inner product $\langle\ ,\ \rangle_p$ in T_pM as

$$\langle v_1, v_2\rangle_p = \langle \mathrm{d}x_p(v_1), \mathrm{d}x_p(v_2)\rangle_{x(p)},$$

where the bracket on the RHS is the canonical inner product on $\mathbb{R}^3$. Clearly it is differentiable and positive definite (since the canonical inner product is), i.e. a Riemannian metric induced by the immersion x.
Let us

- fix a point $p \in M$ and study a neighbourhood $U \subset M$ such that the restriction $x\big|_U$ is an embedding,
- choose $V \subset \mathbb{R}^3$ as in Definition 7 such that $V \cap x(M) = x(U)$,
- choose an adapted moving frame e_1, e_2, e_3 in V such that, when restricted to $x(U)$, e_1 and e_2 are tangent to $x(U)$.

Then we from now on denote by $\{\sigma_i\}$, $\{\sigma_{ij}\}$, $i, j \in \{1, 2, 3\}$ the coframe of the adapted frame in and the connection forms in $\mathbb{R}^3$ respectively while omegas are reserved for M. Then we have six structure equations

$$\begin{aligned}
\mathrm{d}\sigma_1 &= \sigma_2 \wedge \sigma_{21} + \sigma_3 \wedge \sigma_{31}, \\
\mathrm{d}\sigma_2 &= \sigma_1 \wedge \sigma_{12} + \sigma_3 \wedge \sigma_{32}, \\
\mathrm{d}\sigma_3 &= \sigma_1 \wedge \omega_{13} + \sigma_2 \wedge \sigma_{23}, \\
\mathrm{d}\sigma_{12} &= \sigma_{13} \wedge \sigma_{32}, \\
\mathrm{d}\sigma_{13} &= \sigma_{12} \wedge \sigma_{23}, \\
\mathrm{d}\sigma_{23} &= \sigma_{21} \wedge \sigma_{13}.
\end{aligned} \tag{3}$$

The immersion x induces the pullback x^* of differential forms, in particular $x^*(\sigma_i)$ and $x^*(\sigma_{ij})$ in U preserving the structure equations. This justifies writing for all indices $x^*(\sigma_i) = \omega_i$ and $x^*(\sigma_{ij}) = \omega_{ij}$. Note that $\omega_3 = x^*(\sigma_3) = 0$, since

$$x^*(\sigma_3)(v) = \sigma_3(\mathrm{d}x(v)) = \sigma_3(v) = 0 \quad \forall q \in U, v \in T_qM$$

So we can read from the structure equations (3):

$$\begin{aligned}
\mathrm{d}\omega_3 &= \omega_1 \wedge \omega_{13} + \omega_2 \wedge \omega_{23} = 0 \\
\Rightarrow \omega_{13} &= h_{11}\omega_1 + h_{12}\omega_2 \\
\omega_{23} &= h_{21}\omega_1 + h_{22}\omega_2,
\end{aligned} \tag{4}$$

with $h_{ij} = h_{ji}$ differentiable functions in U, using Cartan's lemma.

Definition 8. *Fix the orientation of U and $\mathbb{R}^3$ and choose an adapted frame such that $\{e_1, e_2\}$ is in the orientation of U and $\{e_1, e_2, e_3\}$ is in the orientation of $\mathbb{R}^3$. Then the* Gauss map *is*

$$e_3 : U \to S^2 \subset \mathbb{R}^3,$$

and assigns to any vector $v \in U$ the unique unit vector e_3 perpendicular to v and lying in the orientation of $\mathbb{R}^3$.

Remark 9. *1. The Gauss map is independent of the choice of the frame.*

2. If M is orientable, the Gauss map can be defined globally on M.

3. $\mathrm{d}e_3 = \omega_{31}e_1 + \omega_{32}e_2$ *implies that we can write* $\forall q \in U$ *and* $\forall v = a_1e_1 + a_2e_2 \in T_qM$

$$\mathrm{d}e_3(v) = -\begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix},$$

i.e. $-h_{ij}$ *are the components of the matrix of the differential of the Gauss map in the basis* $\{e_1, e_2\}$.

We can further use the spectral theorem from linear algebra: (h_{ij}) is symmetric, so $\mathrm{d}e_3 : TM \to TS^2$ is a self-adjoint linear map and can as such be diagonalised with orthogonal eigenvectors and eigenvalues $-\lambda_1$, $-\lambda_2$.

Definition 10. *In the above setup, one defines the* Gaussian curvature K of M in p *by*

$$K = \det(\mathrm{d}e_3)_p = \lambda_1\lambda_2 = h_{11}h_{22} - h_{12}^2$$

and the mean curvature H of M at p *by*

$$H = -\frac{1}{2}\left(\mathrm{tr}(\mathrm{d}e_3)\right)_p = \frac{\lambda_1 + \lambda_2}{2} = \frac{h_{11} + h_{22}}{2},$$

where all functions are computed at p.

We conclude:

- Since e_3 does not depend on the choice of frame, neither do K and H
- H changes sign under change of orientation, while K is invariant
- The structure equations (3) and our previous computation (4) show:

$$\mathrm{d}\omega_{12} = \omega_{13} \wedge \omega_{32} = -(h_{11}h_{22} - h_{12}^2)\omega_1 \wedge \omega_2 = -K\omega_1 \wedge \omega_2,$$
$$\omega_{12} \wedge \omega_2 + \omega_1 \wedge \omega_{23} = (h_{11} + h_{22})\omega_1 \wedge \omega_2 = 2H\omega_1 \wedge \omega_2.$$

Theorem 11 (Gauss)**.** *K only depends on the induced metric of M; that is, if $x, x' : M \to \mathbb{R}^3$ are two immersions with the same induced metrics, then $K(p) = K'(p)$, where K and K' are the Gaussian curvatures of the immersions x and x' respectively.*

Proof. Let $U \subset M$ be a neighbourhood of p and choose a moving frame $\{e_1, e_2\}$ in U, orthonormal in the induced metric. For both immersions we can extend the frames to adapted frames $\{\mathrm{d}x(e_1), \mathrm{d}x(e_2), e_3\}$ in $V \supset x(U)$ and $\{\mathrm{d}x'(e_1), \mathrm{d}x'(e_2), e_3'\}$ in $V' \supset x'(U)$. The coframes $\{\sigma_1, \sigma_2, \sigma_3\}$ and $\{\sigma_1', \sigma_2', \sigma_3'\}$ are pulled back onto U, where

$$\omega_i(e_j) = x^*(\sigma_i)(e_j) = \sigma_i(\mathrm{d}x(e_j)) = \delta_{ij},$$
$$\omega_i'(e_j) = x'^*(\sigma_i')(e_j) = \sigma_i'(\mathrm{d}x'(e_j)) = \delta_{ij} \quad \text{for } i, j \in \{1, 2\}.$$

That is, by duality $\omega_1 = \omega_1'$, $\omega_2 = \omega_2'$. Then, by uniqueness, Lemma 6, $\omega_{12} = \omega_{12}'$ and thus

$$\mathrm{d}(\omega_{12}) = \mathrm{d}(\omega_{12}') = -K\omega_1 \wedge \omega_2 = -K'\omega_1 \wedge \omega_2.$$

So $K = K'$. □

Definition 12. *Fix a point $p \in M$ and an immersion $x : M \to \mathbb{R}^3$. The* first fundamental form[1] $I_p : T_pM \times T_pM \to \mathbb{R}$ *is the quadratic form associated to the induced metric:*

$$I_p(v,w) = \langle v, w \rangle_p \qquad \forall v, w \in T_pM$$

The second fundamental form[2] $II_p : T_pM \times T_pM \to \mathbb{R}$ *is defined by*

$$II_p(v,w) = -\langle \mathrm{d}e_3(v), w \rangle_p \qquad \forall v, w \in T_pM,$$

where e_3 is the Gauss map. In both definitions products of co-frame forms are symmetric products. One also finds the notation $I(v) = I(v,v)$ and $II(v) = II(v,v)$.

For any $p \in M$, in an adapted moving frame they take the forms

$$\begin{aligned} I_p(v,w) &= \omega_1^2(v,w) + \omega_2^2(v,w), \\ II_p(v,w) &= (\omega_{13}\omega_1 + \omega_{23}\omega_2)(v,w) = \sum_{ij} h_{ij}\omega_i(v)\omega_j(w), \quad i,j \in \{1,2\}. \end{aligned}$$

Theorem 13. *Let M and M' be additionally connected submanifolds in $\mathbb{R}^3$. Assume that there exist adapted frames $\{e_i\}$ in M, $\{e_i'\}$ in M', $i \in \{1,2,3\}$ and a diffeomorphism $f : M \to M'$ such that*

$$f^*\omega_i' = \omega_i, \quad f^*\omega_{ij}' = \omega_{ij}, \quad i,j \in \{1,2,3\}$$

Then there exists a rigid motion $\rho : \mathbb{R}^3 \to \mathbb{R}^3$ such that the restriction $\rho\big|_M = f$.

Proof. The proof is found in [Car94] Chapter 5.2, Theorem 2. □

Corollary 14. *Let M and M' as above. Assume there exists a diffeomorphism $f : M \to M'$ that preserves the first and the second fundamental form, i.e.*

$$I_p(v,v) = I'_{f(p)}(\mathrm{d}f(v), \mathrm{d}f(v)), \quad II_p(v,v) = II'_{f(p)}(\mathrm{d}f(v), \mathrm{d}f(v))$$

for all $p \in M$ and all $v \in T_pM$. Then there exists a rigid motion $\rho : \mathbb{R}^3 \to \mathbb{R}^3$ such that $\rho\big|_M = f$.

Proof. Consider an adapted frame $\{e_i\}$ in M and define in M' a frame $\{e_i'\} = \{\mathrm{d}f(e_i)\}$. Since f preserves inner products, this is again an adapted frame, and $f^*\omega_i' = \omega_i$. Because the second fundamental forms are preserved, one also has $f^*\omega_{13}' = \omega_{13}$ and $f^*\omega_{23}' = \omega_{23}$. Then one can apply again the uniqueness lemma 6 to see that $f^*\omega_{12} = \omega_{12}$ and the statement follows with Theorem 13. □

We see that the local geometry of a surface in $\mathbb{R}^3$ is completely determined by the two fundamental forms. The first gives the metric. But how can we interpret the second form? Let $\alpha : (-\varepsilon, \varepsilon) \to M$ be a curve parametrised by s with $\alpha(0) = p$ and $\alpha'(0) = v \in T_pM$. Let us abbreviate the expressions $x(\alpha(s))$ with $x(s)$ and $e_i(\alpha(s))$ with $e_i(s)$. Then it is easy to show that

$$\left\langle \frac{\mathrm{d}^2 x}{\mathrm{d}s^2}(0), e_3(0) \right\rangle = II_p(v).$$

With $k(s)$ the curvature of $\alpha(s)$ and $n(s)$ the principal normal of $\alpha(s)$, we can write

$$\left\langle \frac{\mathrm{d}^2 x}{\mathrm{d}s^2}(0), e_3(0) \right\rangle = k(0) \langle n(0), e_3(0) \rangle.$$

Definition 15. *The expression $k \langle n, e_3 \rangle$ is called the* normal curvature $k_{\mathrm{N}}(v)$ of the surface in the direction $v = \alpha'(0)$ at the point p *which only depends on the tangent vector v at p, since*

$$II_p(v) = -\langle \mathrm{d}e_3(v), v \rangle_p = k_{\mathrm{N}}(v) \qquad \forall p \in M, \; v \in T_pM.$$

The maximum and minimum of $II_p(v)$ for vectors v on the unit circle $S_1 \subset T_pM$ are the eigenvalues λ_1 and λ_2 of $(-\mathrm{d}e_3)$, called the principal curvatures. *The corresponding vectors generate the eigenspaces of $(-\mathrm{d}e_3)$ and give the* principal directions *at p.*

[1] Sometimes called *first quadratic form*, e.g. in [Car94].

[2] Sometimes called *second quadratic form*.

3 Intrinsic Geometry of Surfaces

The aim of this section is to develop intrinsic geometric properties of a two-dimensional submanifold of $\mathbb{R}^3$. Our starting point is:

- A Two-dimensional Riemannian manifold M together with metric $\langle , \rangle$.
- For each point $p \in M$ choose a neighbourhood $U \subset M$ such that one can define orthonormal vector fields e_1 and e_2 on U.
- The corresponding coframe is $\{\omega_1, \omega_2\}$.
- We have to analyse the behaviour of geometric entities under change of basis. Therefore denote be $\{\bar{e}_1, \bar{e}_2\}$ another moving frame related to $\{e_1, e_2\}$ via Equation (5).

Lemma 16 (Theorem of Levi-Civita)**.** *Let M be a two-dimensional Riemannian manifold. Let $U \subset M$ be an open set where a moving orthonormal frame $\{e_1, e_2\}$ is defined, and let $\{\omega_1, \omega_2\}$ be the associated coframe. Then there exists a unique 1-form $\omega_{12} = -\omega_{21}$ such that*

$$\mathrm{d}\omega_1 = \omega_{12} \wedge \omega_2 \quad \textit{and} \quad \mathrm{d}\omega_2 = \omega_{21} \wedge \omega_1.$$

Proof. Uniqueness: Lemma 6.
Existence: Define

$$\begin{aligned} \omega(e_1) &= \mathrm{d}\omega_1(e_1, e_2), \\ \omega(e_2) &= \mathrm{d}\omega_2(e_1, e_2). \end{aligned}$$

This is possible since $\omega_1 = \omega_1(e_1)$ and respectively ω_2. The above choice has the desired properties:

$$\mathrm{d}\omega_1(e_1, e_2) \equiv \omega_{12}(e_1) = \omega_{12}(e_1)\omega_2(e_2) - \omega_{12}(e_2)\underbrace{\omega_2(e_1)}_{=0} \equiv (\omega_{12} \wedge \omega_2)(e_1, e_2).$$

The other calculation works similarly. □

Now, we wish to define geometric entities. This means that we wish to combine the forms ω_1, ω_2 and ω_{12} into something which does not depend on the choice of coordinates. So lets investigate the behaviour of ω_{12} under a change of frame.
First, we have to perform a change of frame of our moving frame.

Lemma 17. *Let $\{e_1, e_2\}$ and $\{\bar{e}_1, \bar{e}_2\}$ be moving frames in U. If $\{\bar{e}_1, \bar{e}_2\}$ has the same orientation like $\{e_1, e_2\}$, we obtain:*

$$\begin{pmatrix} \bar{e}_1 \\ \bar{e}_2 \end{pmatrix} = \begin{pmatrix} f & g \\ -g & f \end{pmatrix} \cdot \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}, \tag{5}$$

where f, g are differentiable functions on U satisfying $f^2 + g^2 = 1$. If the orientation is opposite, we obtain:

$$\begin{pmatrix} \bar{e}_1 \\ \bar{e}_2 \end{pmatrix} = \begin{pmatrix} f & g \\ g & -f \end{pmatrix} \cdot \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}.$$

Proof. Consider the first case. The change of frame shall be orientation preserving, i.e. the matrix of the transformation has determinant 1. Besides the system $\{\bar{e}_1, \bar{e}_2\}$ shall be orthonormal, i.e. $\langle \bar{e}_1, \bar{e}_2 \rangle = 0$ and $\langle \bar{e}_1, \bar{e}_1 \rangle = 1 = \langle \bar{e}_2, \bar{e}_2 \rangle$. These conditions fix the matrix of the transformation to the above form. f and g are by definition differentiable.
The second case is analogous except for the determinant which shall be -1 here. □

Now, we proceed with the transformation law of ω_{12}.

Lemma 18. *Let $\{\bar{e}_1, \bar{e}_2\}$ and $\{e_1, e_2\}$ be moving frames being related to each other as stated in Lemma 17. If both have the same orientation, then*

$$\omega_{12} = \bar{\omega}_{12} - \tau,$$

where $\tau = f\mathrm{d}g - g\mathrm{d}f$. If the above orientations are opposite,

$$\omega_{12} = -\bar{\omega}_{12} - \tau.$$

Proof. In this proof we will make have use of the first structure equation of $\mathbb{R}^n$ ($\mathrm{d}\omega_i = \sum_k \omega_k \wedge \omega_{ki}$, (1)) in combination with the fact that the connection form is unique (Lemma 6).
Transformation (5) implies:

$$\begin{pmatrix} \bar{\omega}_1 \\ \bar{\omega}_2 \end{pmatrix} = \begin{pmatrix} f & g \\ -g & f \end{pmatrix} \cdot \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} \tag{6}$$

which can be checked explicitly. Therefore the inverse transformation is:

$$\begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \underbrace{\frac{1}{f^2 + g^2}}_{=1} \begin{pmatrix} f & -g \\ g & f \end{pmatrix} \cdot \begin{pmatrix} \bar{\omega}_1 \\ \bar{\omega}_2 \end{pmatrix}.$$

Differentiating the transformation law of ω_1 we can conclude:

$$\begin{aligned}
\mathrm{d}\omega_1 &= \mathrm{d}f \wedge \bar{\omega}_1 + f\mathrm{d}\bar{\omega}_1 - \mathrm{d}g \wedge \bar{\omega}_2 - g\,\mathrm{d}\bar{\omega}_2 \\
&\overset{\text{Structure Eq.}}{=} \mathrm{d}f \wedge \bar{\omega}_1 + f \cdot \sum_k \bar{\omega}_k \wedge \bar{\omega}_{k1} - \mathrm{d}g \wedge \bar{\omega}_2 - g \cdot \sum_k \bar{\omega}_k \wedge \bar{\omega}_{k2} \\
&= \mathrm{d}f \wedge \bar{\omega}_1 + f \cdot (\bar{\omega}_1 \wedge \underbrace{\bar{\omega}_{11}}_{=0} + \bar{\omega}_2 \wedge \underbrace{\bar{\omega}_{21}}_{=-\bar{\omega}_{12}}) - \mathrm{d}g \wedge \bar{\omega}_2 - g \cdot (\bar{\omega}_1 \wedge \bar{\omega}_{12} + \bar{\omega}_2 \wedge \underbrace{\bar{\omega}_{22}}_{=0}) \\
&= \mathrm{d}f \wedge \bar{\omega}_1 - \mathrm{d}g \wedge \bar{\omega}_2 - (f\bar{\omega}_2 + g\bar{\omega}_1) \wedge \bar{\omega}_{12} \\
&\overset{\text{Transformation Law}}{=} \mathrm{d}f \wedge (f\omega_1 + g\omega_2) - \mathrm{d}g \wedge (-g\omega_1 + f\omega_2) - \omega_2 \wedge \bar{\omega}_{12} \\
&= (f\mathrm{d}f + g\mathrm{d}g) \wedge \omega_1 + (g\mathrm{d}f - f\mathrm{d}g) \wedge \omega_2 + \bar{\omega}_{12} \wedge \omega_2.
\end{aligned}$$

The first term vanishes since $f^2 + g^2 = 1$ implies $\mathrm{d}(f^2 + g^2) = \mathrm{d}(1)$ which gives $2f\mathrm{d}f + 2g\mathrm{d}g = 0$. So we are left with:

$$\mathrm{d}\omega_1 = [\bar{\omega}_{12} - (f\mathrm{d}g - g\mathrm{d}f)] \wedge \omega_2 \equiv (\bar{\omega}_{12} - \tau) \wedge \omega_2.$$

Similarly one shows:

$$\mathrm{d}\omega_2 = -(\bar{\omega}_{12} - \tau) \wedge \omega_1.$$

This is exactly the form of the structure equations for ω_1 and ω_2 with ω_{12} replaced by $\bar{\omega}_{12}$. Since the connection forms are unique (Lemma 6) we conclude:

$$\omega_{12} = \bar{\omega}_{12} - \tau. \tag{7}$$

□

Let us continue by developing some geometric intuition for the form τ.

Lemma 19. *Let $p \in U \subset M$ be a point and let $\gamma : I \to U$ be a curve such that $\gamma(t_0) = p$. Let $\phi_0 =$ angle$(e_1(p), \bar{e}_1(p))$. Then*

$$\phi(t) = \int_{t_0}^{t} \left(f\frac{dg}{dt} - g\frac{df}{dt} \right) dt + \phi_0$$

is a differentiable function such that

$$\cos\phi(t) = f(t), \quad \sin\phi(t) = g(t), \quad \phi(t_0) = \phi_0, \quad \mathrm{d}\phi = \gamma^*\tau.$$

Proof. First we show that

$$f(t)\cos\phi(t) + g(t)\sin\phi(t) = 1 \quad \text{for all } t \in I. \tag{8}$$

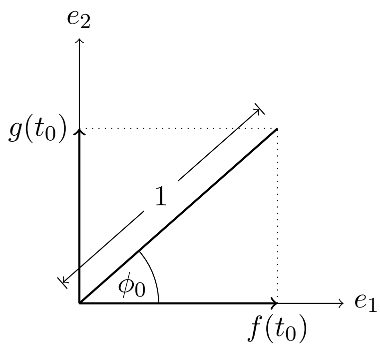
By definition of ϕ we know $\phi' = fg' - f'g$. Thus,

$$\begin{aligned}(f\cos\phi + g\sin\phi)' &= f'\cos\phi - f\phi'\sin\phi + g'\sin\phi + g\phi'\cos\phi \\ &= (g' + fgf' - f^2 g)\sin\phi + (f' + gfg' - g^2 f')\cos\phi. \end{aligned} \tag{9}$$

Again we use $ff' + gg' = 0$ (since $f^2 + g^2 = 1$). The coefficients of (9) therefore read:

$$\begin{aligned} g' + fgf' - f^2 g &= g' - g^2 g' - f^2 g' = g' - g' \cdot (g^2 + f^2) = g' - g' = 0, \\ f' + gfg' - g^2 f' &= f' - g^2 f' - f^2 f' = f' - f' \cdot (g^2 + f^2) = f' - f' = 0. \end{aligned}$$

So $(f\cos\phi + g\sin\phi)'$ vanishes for all $t \in I$ and $(f\cos\phi + g\sin\phi)$ is constant which can be evaluated at $t = t_0$.



We realise that $\cos(t_0) = f(t_0)$ and $\sin(t_0) = g(t_0)$ (results from $f^2 + g^2 = 1$). This implies:

$$f(t_0)\cos\phi(t_0) + g(t_0)\sin\phi(t_0) = (f^2 + g^2)(t_0) = 1.$$

Altogether we showed (8). The lemma follows immediately:

$$(f - \cos\phi)^2 + (g - \sin\phi)^2 = \underbrace{f^2 + g^2}_{=1} \underbrace{-2f\cos\phi - 2g\sin\phi}_{\overset{(8)}{=} -2} + 1 = 0$$

showing $\cos\phi(t) = f(t)$ and $\sin\phi(t_0) = g(t)$ for all $t \in I$. $\square$

Now we are able to define our first intrinsic geometrical object.

Proposition 20. *In an oriented surface the 2-form $\omega_1 \wedge \omega_2 \equiv \sigma$ does not depend on the choice of frames and is called* area element of M.

Proof. σ does not depend on the choice of frame since

$$\omega_1 \wedge \omega_2 = (f\bar\omega_1 - g\bar\omega_2) \wedge (g\bar\omega_1 + f\bar\omega_2) = f^2 \bar\omega_1 \wedge \bar\omega_2 - g^2 \bar\omega_2 \wedge \bar\omega_1 = (f^2 + g^2)\,\bar\omega_1 \wedge \bar\omega_2 = \bar\omega_1 \wedge \bar\omega_2.$$

where we used (6) and the fact that $\omega_i \wedge \omega_j = -\omega_j \wedge \omega_i$ including $\omega_i \wedge \omega_i = 0$.
The interpretation as area element is obtained as follows. Let $v_i = a_{i1}e_1 + a_{i2}e_2, i \in \{1, 2\}$ be two linearly independent vectors at a point $p \in M$. Then

$$\begin{aligned}\sigma(v_1, v_2) &= (\omega_1 \wedge \omega_2)(v_1, v_2) \\ &= \omega_1(v_1)\omega_2(v_2) - \omega_2(v_1)\omega_1(v_2) = a_{11}a_{22} - a_{21}a_{12} \equiv \det(a_{ij}) \equiv \text{area}(v_1, v_2).\end{aligned}$$ $\square$

Remark 21. *Since σ does not depend on the choice of frames it is globally defined.*

Now, we are ready for our second geometric entity.

Proposition 22. *Let M be a Riemannian manifold of dimension two. For each $p \in M$, we define a number $K(p)$ by choosing a moving frame $\{e_1, e_2\}$ around p and setting*

$$\mathrm{d}\omega_{12}(p) = -K(p)(\omega_1 \wedge \omega_2)(p). \tag{10}$$

Then $K(p)$ does not depend on the choice of frames, and it is called the Gaussian curvature of M at p.

Proof. Let $\{\bar{e}_1, \bar{e}_2\}$ another moving frame around p. We have two cases.

1. Suppose the orientations are the same: We have $\omega_{12} = \bar{\omega}_{12} - \tau$ with $\tau = f\mathrm{d}g - g\mathrm{d}f$, therefore $\mathrm{d}\tau = 0$ and we conclude $\mathrm{d}\omega_{12} = \mathrm{d}\bar{\omega}_{12}$. It follows that:

$$-K\omega_1 \wedge \omega_2 = \mathrm{d}\omega_{12} = \mathrm{d}\bar{\omega}_{12} = -\bar{K}\bar{\omega}_1 \wedge \bar{\omega}_2 = -\bar{K}\omega_1 \wedge \omega_2.$$

   This shows the proposition.

2. Suppose the orientations are not the same. Here we obtain $\mathrm{d}\omega_{12} = \mathrm{d}(-\bar{\omega}_{12} - \tau) = -\mathrm{d}\bar{\omega}_{12}$ but additionally $\omega_1 \wedge \omega_2 = -\bar{\omega}_1 \wedge \omega_2$. The two signs cancel in the above computation.

□

The next frame-independent quantity is the covariant derivative of vectors.

Definition 23. *Let M be a Riemannian manifold and let Y be a differentiable vector field on M. Let $p \in M, x \in T_pM$, and consider a curve $\alpha : (-\epsilon, \epsilon) \to M$ with $\alpha(0) = p$ and $\alpha'(0) = x$. To define the* covariant derivative $(\nabla_x Y)(p)$ of Y relative to x in p, *we choose a moving frame $\{e_i\}$ around p, express $Y(\alpha(t))$ in this frame*

$$Y(\alpha(t)) = \sum y_i(t)e_i, \quad i \in \{1, 2\},$$

and set

$$(\nabla_x Y)(p) = \sum_{i=1}^{2} \left(\left.\frac{dy_i}{dt}\right|_{t=0} + \sum_{j=1}^{2} \omega_{ij}(x)y_j(0) \right) e_i,$$

where the convention is made that $\omega_{ii} = 0$.

Lemma 24. *The covariant derivative does not depend on the choice of frames.*

Proof. Let $\{e_1, e_2\}$ and $\{\bar{e}_1, \bar{e}_2\}$ be two orthonormal frames around p. Assume that they have the same orientation. Then $Y(\alpha(t)) = \sum y_i(t)e_i = \sum \bar{y}_i(t)\bar{e}_i$ with

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} f & -g \\ g & f \end{pmatrix} \cdot \begin{pmatrix} \bar{y}_1 \\ \bar{y}_2 \end{pmatrix}, \quad \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} f & -g \\ g & f \end{pmatrix} \cdot \begin{pmatrix} \bar{e}_1 \\ \bar{e}_2 \end{pmatrix}, \tag{11}$$

f, g differentiable functions with $f^2 + g^2 = 1$. By definition,

$$\nabla_x Y = \left(\frac{dy_1}{dt} + \omega_{21}(x)y_2 \right) e_1 + \left(\frac{dy_2}{dt} + \omega_{12}(x)y_1 \right) e_2$$

where the functions are taken at $t = 0$. Using (11), $\omega_{12} = \bar{\omega}_{12} - \tau$ and $ff' + gg' = 0$, we arrive after a long and painful computation at

$$\nabla_x Y = \left(\frac{d\bar{y}_1}{dt} + \bar{\omega}_{21}(x)\bar{y}_2 \right) \bar{e}_1 + \left(\frac{d\bar{y}_2}{dt} + \bar{\omega}_{12}(x)\bar{y}_1 \right) \bar{e}_2.$$

Because it's fun. Let us take a look at this computation.

$$
\begin{aligned}
\nabla_x Y &= \left(\frac{dy_1}{dt} + \omega_{21}(x) y_2\right) e_1 + \left(\frac{dy_2}{dt} + \omega_{12}(x) y_1\right) e_2 \\
&= \left(\frac{d(f\bar{y}_1 - g\bar{y}_2)}{dt} - (\bar{\omega}_{12} - \tau)(x)(g\bar{y}_1 + f\bar{y}_2)\right)(f\bar{e}_1 - g\bar{e}_2) + \ldots \\
&\qquad \ldots + \left(\frac{d(g\bar{y}_1 + f\bar{y}_2)}{dt} + (\bar{\omega}_{12} - \tau)(x)(f\bar{y}_1 - g\bar{y}_2)\right)(g\bar{e}_1 + f\bar{e}_2) \\
&= A + B + C
\end{aligned}
$$

with

$$
\begin{aligned}
A &:= -\bar{\omega}_{12}(x)(g\bar{y}_1 + f\bar{y}_2)(f\bar{e}_1 - g\bar{e}_2) + \bar{\omega}_{12}(x)(f\bar{y}_1 - g\bar{y}_2)(g\bar{e}_1 + f\bar{e}_2) \\
B &:= \left(\frac{d(f\bar{y}_1)}{dt} - \frac{d(g\bar{y}_2)}{dt}\right)(f\bar{e}_1 - g\bar{e}_2) + \left(\frac{d(g\bar{y}_1)}{dt} + \frac{d(f\bar{y}_2)}{dt}\right)(g\bar{e}_1 + f\bar{e}_2) \\
C &:= \tau(x)(g\bar{y}_1 + f\bar{y}_2)(f\bar{e}_1 - g\bar{e}_2) - \tau(x)(f\bar{y}_1 - g\bar{y}_2)(g\bar{e}_1 + f\bar{e}_2)
\end{aligned}
$$

First of all take a look at A.

$$
\begin{aligned}
A &= \bar{\omega}_{12}(x)\left[(g\bar{y}_1 + f\bar{y}_2)(-f\bar{e}_1 + g\bar{e}_2) + (f\bar{y}_1 - g\bar{y}_2)(g\bar{e}_1 + f\bar{e}_2)\right] \\
&= \bar{\omega}_{12}(x)\left[(f^2 + g^2)\bar{y}_1\bar{e}_2 + (-f^2 - g^2)\bar{y}_2\bar{e}_1 + fg(\bar{y}_2\bar{e}_2 + \bar{y}_1\bar{e}_1 - \bar{y}_2\bar{e}_2 - \bar{y}_1\bar{e}_1)\right] \\
&= \bar{\omega}_{12}(x)\left[\bar{y}_1\bar{e}_2 - \bar{y}_2\bar{e}_1\right] \\
&= \bar{\omega}_{12}(x)\bar{y}_1\bar{e}_2 + \bar{\omega}_{21}(x)\bar{y}_2\bar{e}_1.
\end{aligned}
$$

So all terms involving ω_{12} are invariant under change of frames. A similar calculation shows that B and C is invariant as well.
When the orientations of the frames are opposite, the proof is similar. □

The covariant derivative can be used to give a geometric interpretation of the connection form ω_{12}. We obtain $\nabla_x e_1 = \omega_{12}(x) e_2$ or

$$
\omega_{12}(x) = \langle \nabla_x e_1, e_2 \rangle . \tag{12}
$$

Remark 25. *For the induced metric of surfaces $M \subset \mathbb{R}^3$ it can be shown that the covariant derivative is just the projection of the usual derivative in $\mathbb{R}^3$ onto the tangent plane of M.*
Note that the covariant derivative is $\mathbb{R}$-bilinear, tensorial in the lower argument and a derivation and additive in the main argument:

$$
\begin{aligned}
\nabla_{fX} Y &= f \nabla_X Y, \\
\nabla_X (fY) &= df(X) \cdot Y + f \nabla_X Y, \\
\nabla_X (Y_1 + Y_2) &= \nabla_X Y_1 + \nabla_X Y_2.
\end{aligned}
$$

The covariant derivative makes it possible to define a variety of geometric entities (parallel transport, geodesics, geodesic curvature, etc.).

Definition 26. *A vector field Y along a curve $\alpha : I \to M$ is said to be* parallel along α *if $\nabla_{\alpha'(t)} Y = 0$ for all $t \in I$.*

Remark 27. *For a general manifold it is not trivial what is meant by $\nabla_{\alpha'(t)}$ since $\alpha'(t) \notin T_{\alpha(t)} M$ but rather in $T_t I$. In this case one pushes the tangential vector $\alpha'(t)$ to I to our main manifold M by using the differential map $d\alpha : T_x I \to T_\alpha(x) M, v \mapsto d\alpha(v)$ such that $d\alpha(v)(\phi) = v(\phi \circ \alpha)$.*

Definition 28. *A curve $\alpha : I \to M$ is a* geodesic *if $\alpha'(t)$ is a parallel field along α, i.e. $\nabla_{\alpha'(t)} \alpha'(t) = 0$ for all $t \in I$.*

Definition 29. *Assume that M is oriented, and let $\alpha : I \to M$ be a differentiable curve parametrized by arc length s with $\alpha'(s) \neq 0$ for all $s \in I$.*[3] *In a neighbourhood of a point $\alpha(s) \in M$, consider a moving frame $\{e_1, e_2\}$ in the orientation of M such that, restricted to α, $e_1(\alpha) = \alpha'(s)$. The* geodesic curvature k_g of α in M *is defined by*

$$k_g = (\alpha^* \omega_{12}) \left(\tfrac{d}{ds}\right)$$

where $\frac{d}{ds}$ is the canonical basis of $\mathbb{R}$.

A nice way to think about geodesic curvature is the following. The geodesic curvature measures the deviation of a curve of being a geodesic. A differentiable curve is a geodesic if and only if its geodesic curvature vanishes. To show this we state the even more general following proposition.

Proposition 30. *Let $\alpha : I \to M$ and $\{e_1, e_2\}$ be as in Definition 29 (here we do not need to assume that M is orientable). Then e_1 is parallel along α if and only if $\alpha^* \omega_{12} = 0$.*

Proof. By definition e_1 is parallel along α if and only if $\nabla_{e_1} e_1 = 0$. Thus, $\langle \nabla_{e_1} e_1, e_1 \rangle = 0$ and it follows that

$$\nabla_{e_1} e_1 = 0 \quad \Leftrightarrow \quad 0 = \langle \nabla_{e_1} e_1, e_2 \rangle \stackrel{(12)}{=} \omega_{12}(e_1).$$

And $\omega_{12}(e_1) = 0$ if and only if $\alpha^* \omega_{12} = 0$. $\square$

Corollary 31. *A differentiable curve $\alpha : I \to M$ is a geodesic if and only if its geodesic curvature vanishes everywhere.*

Proposition 32. *Let M be oriented and let $\alpha : I \to M$ be a differentiable curve parametrized by arc length s with $\alpha'(s) \neq 0, s \in I$. Let V be a parallel vector field along α and let $\phi(s) =$ angle$(V, \alpha'(s))$ where the angle is measured in the given orientation. Then*

$$k_g(s) = \frac{d\phi}{ds}.$$

Proof. Let us choose frames $\{e_1, e_2\}$ and $\{\bar{e}_1, \bar{e}_2\}$ around $\alpha(s)$ such that $e_1 = V/|V|$ and $\bar{e}_1 = \alpha'(s)$. Let e_2 and $\bar{e}_2$ be normal in positive direction to e_1 and $\bar{e}_1$ respectively and ω_{12} and $\bar{\omega}_{12}$ the respective connection forms. Then $\phi = \text{angle}(e_1, \bar{e}_1)$.
By Lemma 19 and equation (7) we know $d\phi$:

$$d\phi = \alpha^* \bar{\omega}_{12} - \alpha^* \omega_{12}. \tag{13}$$

Since e_1 is parallel along α it follows with the help of Proposition 30 $\alpha^* \omega_{12} = 0$. We are left with

$$k_g \equiv (\alpha^* \omega_{12}) \left(\frac{d}{ds}\right) = d\phi \left(\frac{d}{ds}\right) = \frac{d\phi}{ds}.$$

$\square$

Let us see how this enables us to understand the geodesic curvature descriptively.

Remark 33. *Let $p \in M$ and $D \subset M$ an open neighbourhood of p homeomorphic to a disk with smooth boundary ∂D. Parametrize the boundary ∂D by α (by arc length).*
Let $q \in \partial D$ and $V_0 \in T_q M, |V_0| = 1$. Let $V(s)$ be the parallel transport of V_0 along α. Generally, there will be a non-vanishing angle between V_0 and the vector at q obtained by parallel transport.
Use the frames $\{e_1(s) = \alpha'(s), e_2\}$, $\{\bar{e}_1(s) = V(s), \bar{e}_2\}$[4] *we obtain:*

$$- \int_{\partial D} \alpha^*(\omega_{12}) \stackrel{(13)}{=} \int_{\partial D} d\phi = \phi.$$

[3] A differential curve is called *parametrized by arc length* if its velocity has unit norm at every point.
[4] Note that these frames are different from those used in the proof of Proposition 32. Therefore the other term in (13) vanishes.

The integral on the left-hand side can be evaluated as well by using Stoke's theorem:

$$\phi = -\int_{\partial D} \alpha^*(\omega_{12}) = -\int_D \mathrm{d}\omega_{12} \stackrel{(10)}{=} \int_D K\sigma.$$

By elementary calculus we interpret the Gaussian curvature as:

$$K(p) = \lim_{D\to p} \frac{\phi}{areaD}$$

where we take the limit in the sense that area(D) $\to 0$ and p is element of all D. Thus, the Gaussian curvature measures how different from identity is parallel transport along small circles about p.

List of intrinsic geometrical objects

- Area element $\sigma \equiv \omega_1 \wedge \omega_2$ (only for orientable manifolds globally defined).
- Gaussian curvature: $\mathrm{d}\omega_{12}(p) = -K(p)(\omega_1 \wedge \omega_2)(p)$.
- Covariant derivative of vectors
  - Parallel transport ($\nabla_{\alpha'(t)} X = 0$)
  - Geodesics ($\nabla_{\alpha'(t)} \alpha'(t) = 0$)
- Geodesic curvature

Remark 34 (General remarks on notation)**.** *In differential geometry there exists another broadly known formulation. It starts with differentiable manifolds. Those are equipped by additional structure: the covariant derivative. This is completely independent of a potentially existent metric.*
With the help of the covariant derivative it is possible to define entities like torsion, Riemann curvature, Ricci curvature, scalar curvature, parallel transport, geodesics, etc. The notion of covariant derivative can be generalized (`https://en.wikipedia.org/wiki/Ehresmann_connection`).
If we consider a pseudo-Riemannian manifold (a manifold endowed with a metric tensor which does not need to be positive-definite but only non-degenerate) it can be shown that there exists a unique (torsion-free) covariant derivative known as Levi-Civita connection *which is in some sense compatible with the covariant derivative.*
To embed our discussion into another formulation I give the corresponding equations but do no derive them. Note that all signs depend on conventions. τ is the torsion 2-form, R_{klij} is the Riemann tensor and [,] is the Lie bracket.

First structure equation	Second structure equation
$\mathrm{d}\omega_i = \sum_k \omega_k \wedge \omega_{ki}$	$\mathrm{d}\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj}$
$\mathrm{d}\omega_i = \sum_k \omega_k \wedge \omega_{ki} + \tau$	$\mathrm{d}\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} + \frac{1}{2}\sum_{j,l} R_{klij}\, \omega_k \wedge \omega_l$
$\nabla_X Y - \nabla_Y X - [X,Y] = \tau(X,Y)$	$[\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z = R(X,Y)Z$

One can nicely observe that the existence of the structure equations express the fact that $\mathbb{R}^n$ has a Levi-Civita connection (which is torsion-free) and flat (vanishing Riemann tensor).

References

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