1 Preface

In this presentation we state and prove the Closed Subgroup Theorem, also referred to as Cartan's Theorem. In this, we will follow [1]. The enumeration of the Propositions and Theorems follows the enumeration in [1] too - given in the form (no. x.y). Because it is impossible to give rigorous proofs of every Proposition and Theorem in the given time, the oral presentation is focused on giving an imagination of the used concepts and proving the Closed Subgroup Theorem.

First, we will introduce the concepts of slice charts and integral curves. Here slice charts are just given in the written presentation. This will be followed by the introduction of one-parameter subgroups and their connection to integral curves. Using the concept of one-parameter subgroups, we introduce the exponential map and give some properties of this map - of which two will be proven in the presentation and the others are given in the written presentation. After this, we start proving the closed subgroup theorem starting with two propositions which are used in the proof but will (maybe) omitted in the oral presentation.

To prevent an exaggeration in the written part, the theorems in the Preliminaries are not proven. Please refer to [1] for proofs of those theorems.

2 Preliminaries

The following theorem will be used in the proof of the closed subgroup theorem. More accurate, we are going to show that something is an embedded submanifold.

Theorem. *(no. 7.11) Let G be a Lie group, and suppose $H \subset G$ is a subgroup that is also an embedded submanifold. Then H is a Lie group.*

At one point we are using slice charts for embedded submanifolds. We recall their definition.

Definition. Let M be a smooth n-manifold. Let $S \subset M$ and $k \in \mathbb{N}$. We say S satisfies the local k-slice condition if each point of S is contained in the domain of a smooth chart (U, ϕ) for M such that $S \cap U$ is a single k-slice in U. Here a k-slice S of U is any subset of the form:

$$S = \left\{ \left(x^1, \dots, x^k, x^{k+1}, \dots, x^n\right) \in U \,:\, x^{k+1} = c^{k+1}, \dots, x^n = c^n \right\}$$

for some constants $c^{k+1}, \dots, c^n$.

At one point we are going to use the following theorem:

Theorem. *(Local Slice Criterion for Embedded Submanifolds, no. 5.8) Let M ba a smooth n-manifold. If $S \subset M$ is an embedded k-dimensional submanifold, then S satisfies the local k-slice condition. Conversely, is $S \subset M$ is a subset that satisfies the local k-slice condition, then with the subspace topology, S is a topological manifold of dimension k, and it has a smooth structure making it into a k-dimensional embedded submanifold of M.*

Remark. For proofs of the theorems please refer to the given enumeration in [1].

3 The Exponential Map

3.1 Integral Curves

Suppose G is a manifold and $J \subset \mathbb{R}$, J an intervall, define: $\gamma : J \to G$ is a smooth curve, then for each $t \in J$, the velocity vector $\gamma'(t)$ is a vector in $T_{\gamma(t)}G$. We want to look at a way to work backwards:

If V is a vector field on G, an **integral curve of** V is a differentiable curve $\gamma : J \to G$ whose velocity at each point is equal to the value of V at that point:

$$\gamma'(t) = V_{\gamma(t)} \, \forall t \in J.$$

If $0 \in J$, the point $\gamma(0)$ is called the **starting point of** γ.
We use the term of an **maximal integral curve** for an integral curve that cannot be extended on any open interval. At last, we want to give an example for an integral curve, which shows why the name integral curve is used.

Example. (no. 9.1) Let $W = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$ on $\mathbb{R}^2$. If we have a smooth $\gamma : \mathbb{R} \to \mathbb{R}^2$ curve in standard coordinates $\gamma(t) = (x(t), y(t))$, then the condition $\gamma' = W_{\gamma(t)}$ for γ to be an integral curve is:

$$x'(t) \frac{\partial}{\partial x}\bigg|_{\gamma(t)} + y'(t) \frac{\partial}{\partial y}\bigg|_{\gamma(t)} = x(t) \frac{\partial}{\partial y}\bigg|_{\gamma(t)} - y(t) \frac{\partial}{\partial x}\bigg|_{\gamma(t)}$$

One may see that this gives a system of differential equations

$$x'(t) = -y(t),$$
$$y'(t) = x(t).$$

With the solutions

$$x(t) = a\,cos(t) - b\,sin(t),$$
$$y(t) = a\,sin(t) + b\,cos(t).$$

for arbitrary a and b. And thus the name integral curve because we get them as solutions to differential equations.

To conclude the discussion of integral curves we state the following theorem.

Theorem. ***(no. 9.2)*** *Let V be a smooth vector field on a smooth manifold M. For each point $p \in M$, there exists $\epsilon > 0$ and a smooth curve $\gamma : (-\epsilon, \epsilon) \to M$ that is an integral curve of V starting at p.*

3.2 One-Parameter-Subgroups

Definition. A **one-parameter subgroup of G** is defined to be a Lie group homomorphism $\gamma : \mathbb{R} \to G$, with $\mathbb{R}$ considered as a Lie group under addition.

Theorem. ***(Characterization of One-Parameter Subgroups, no. 20.1)*** *Let G be a Lie group. The one-parameter subgroups of G are precisely the maximal integral curves of left-invariant vector fields starting at the identity.*

Proof. *Please refer to [1] because in the proof results from chapter 9 are needed.*

Proposition. ***(20.3)*** *Suppose G is a Lie group and $H \subset G$ is a Lie subgroup. The one-parameter subgroups of H are precisely those one-parameter subgroups of G whose initial velocities lie in T_eH.*

Proof. *Let $\gamma : \mathbb{R} \to H$ be a one-parameter subgroub. Then the composite map*

$$\mathbb{R} \xrightarrow{\gamma} H \hookrightarrow G$$

is a Lie group homomorphism and thus a one-parameter subgroup of G, which clearly satisfies $\gamma'(0) \in T_eH$.

Conversely, suppose $\gamma : \mathbb{R} \to G$ is a one-parameter subgroup whose initial velocity lies in T_eH. Let $\hat{\gamma} : \mathbb{R} \to H$ be the one-parameter subgroup of H with the same initial velocity $\hat{\gamma}'(0) = \gamma'(0) \in T_eH subgroup T_eG$. Using the inclusion map, we can consider $\hat{\gamma}$ as a one-parameter subgroup of G. Since γ and $\hat{\gamma}$ are one-parameter subgroups with the same initial velocity, they must be equal.

3.3 The Exponential Map

To prove the Closed Subgroup Theorem we introduce the *exponential map of G*, as follows:

Definition. For a given Lie group G with Lie algebra $\mathfrak{g}$, we define a map $exp : \mathfrak{g} \to G$ called the **exponential map of** G by:

$$\forall X \in \mathfrak{g} \text{ define: } exp(X) = \gamma(1)$$

where γ is the one-parameter subgroup generated by X.

Remark. The exponential map is well defined because using the identification of one-parameter subgroups as maximal integral curves we do have two integral curves being equal if they have the same starting velocities.

Proposition. ***(no. 20.5)*** *Let G be a Lie group. For any $X \in Lie(G)$, $\gamma(s) = exp(sX)$ is the one-parameter subgroup of G generated by X.*

Proof: *Needs one-parameter subgroup as integral curve of X starting at e. and the rescaling lemma 9.3.*

Example. (no. 20.7) If V is a finite-dimensional real vector space, a choice of basis for V yields isomorphisms $GL(V) \cong GL(n, \mathbb{R})$ and $\mathfrak{gl}(V) \cong \mathfrak{gl}(n, \mathbb{R})$. Tha analysis of the $GL(n, \mathbb{R})$ case shows that the exponential map can be written in the form:

$$exp(A) = \sum_{k=0}^{\infty} \frac{1}{k!} A^k,$$

where $A \in \mathfrak{gl}(V)$ is a linear map from V to itself and $A^k = A \circ \cdots \circ A$ is the k-fold composition of A with itself.

Remark. In the following proposition some properties of the exponential map are proven. As one can see some of those properties hold for the exponential function $exp : \mathbb{R} \to \mathbb{R}^+$ too. In fact, the exponential map is a useful generalization. We get the exponential function as a special case of the exponential map if we identify $G = (\mathbb{R}^+, \cdot)$ and $\mathfrak{g} = (\mathbb{R}, +)$.

Proposition. ***(Properties of the Exponential Map, no. 20.8)*** *Let G be a Lie group and let $\mathfrak{g}$ be its Lie algebra.*

(a) The exponential map is smooth (from $\mathfrak{g}$ to G).

(b) $\forall X \in \mathfrak{g},\ s,t \in \mathbb{R}:\ exp(s+t)X = exp(sX) + exp(tX)$.

(c) $\forall X \in \mathfrak{g}, (exp(X))^{-1} = exp(-X)$.

(d) $\forall X \in \mathfrak{g}, n \in \mathbb{Z}:\ (exp(X))^n = exp(nX)$.

(e) The differential $(d\,exp)_0 : T_0\mathfrak{g} \to T_eG$ is the identity map, under the canonical identifications of both $T_0\mathfrak{g}$ and T_eG with $\mathfrak{g}$ itself.

(f) The exponential map restricts to a diffeomorphism from some neighborhood of 0 in $\mathfrak{g}$ to a neighborhood of e in G.

(g) If H is another Lie group, $\mathfrak{h}$ is its Lie algebra, and $\Phi : G \to H$ is a Lie group of homomorphism, the following diagram commutes:

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\Phi_*"'] \arrow[d, "exp"'] & \mathfrak{h} \arrow[d, "exp"] \\ G \arrow[r, "\Phi"'] & H \end{tikzcd}$$

(h) The flow θ of a left-invariant vector field X is given by $\theta_t = R_{exp(tX)}$ (right multiplication by $exp(tX)$).

Proof. *The proof is given for completeness but discussion would be beyond the scope of this written discussion.*
In this proof, for any $X \in \mathfrak{g}$ we let $\theta_{(X)}$ denote the flow of X.

(a) We need to show that the expression $\theta^{(e)}_{(X)}(1)$ depends smoothly on X, which amounts to showing that the flow varies smoothly as the vector filed varies. Define a vector field Z on the product manifold $G \times \mathfrak{g}$ by

$$\mathrm{Z}_{(g,X)} = \left(X_g, 0\right) \in T_gG \oplus T_X\mathfrak{g} \cong T_{(g,X)}\left(G \times \mathfrak{g}\right).$$

To see that Z is a smooth vector field, chhose any basis $\left(X_1, \dots, X_k\right)$ for $\mathfrak{g}$, and let $\left(x^i\right)$ be the corresponding global coordinates for $\mathfrak{g}$, defined by $\left(x^i\right) \leftrightarrow x^iX_i$. Let $\left(w^i\right)$ be any smooth local coordinates for G. If $f \in C^\infty\left(G \times \mathfrak{g}\right)$ is arbitrary, then locally we can write

$$\mathrm{Z}f\left(w^i, x^i\right) = x^jX_jf\left(w^i, x^i\right),$$

where each vector field X_j defferentiates f only in the w^i-directions. Since this depends smoothly on $\left(w^i, x^i\right)$, it follows from Prop 8.14 in [1] that Z is smooth. The flow of Θ of Z is given by

$$\Theta_t(g, X) = \left(\theta_{(X)}(t, g), X\right).$$

Then, using the fundamental theorem of flows, Θ is smooth. Since $exp(X) = \pi_G\left(\Theta_1(e, X)\right)$, where π_G is the projection, it follows that exp is smooth.

(b),(c) Follow directly from Proposition 20.5, because $t \mapsto exp(tX)$ is a group homomorphism from $\mathbb{R}$ to G.

(d) Follows from (b) for nonnegative n and from (c) for negative n.

(e) Let $X \in \mathfrak{g}$ be arbitrary, and let $\sigma : \mathbb{R} \to \mathfrak{g}$ be the curve $\sigma(t) = tX$. Then $\sigma'(0) = X$, and Proposition 20.5 implies

$$(d\,exp)_0(X) = (d\,exp)_0\left(\sigma'(0)\right) = (exp \circ \sigma)'(0) = \left.\frac{d}{dt}\right|_{t=0} exp(tX) = X.$$

(f) Follows from (e) using the inverse function theorem.

(g) We have to show: $exp\left(\Phi_*X\right) = \Phi(exp(X))$ for every $X \in \mathfrak{g}$. We will show that $\forall t \in \mathbb{R}$ it holds: $exp\left(t\Phi_*X\right) = \Phi(exp(tX))$. The left-hand side is, by Proposition 20.5, the one-parameter subgroup generated by Φ_*X. Thus, if we put $\sigma(t) = \Phi(exp(tX))$, it suffices to show that σ is a Lie group homomorphism satisfying $\sigma'(0) = \left(\Phi_*X\right)_e$. It is a Lie group homomorphism because it is a composition of homomorphisms. And the initial velocity is given by:

$$\sigma'(0) = \left.\frac{d}{dt}\right|_{t=0} \Phi(exp(tX)) = d\Phi_0\left(\left.\frac{d}{dt}\right|_{t=0} exp(tX)\right) = d\Phi_0\left(X_e\right) = \left(\Phi_*X\right)_e.$$

(h) Use the fact that for any $g \in G$ the map L_g takes integral curves of X to integral curves of X. Thus the map $t \mapsto L_g(exp(tX))$ is the integral curve starting at g, which means it is equal to $\theta^{(g)}_{(X)}(t)$. It follows that

$$R_{exp(tX)}(g) = g\, exp(tX) = L_g(exp(tX)) = \theta^{(g)}_{(X)}(t) = \left(\theta_{(X)}\right)_t(g).$$

Proposition. ***(no. 20.9)*** *Let G be a Lie group, and let $H \subset G$ be a Lie supgroup. With $Lie(H)$ conidered as a subalgebra of $Lie(G)$ in the usual way, the exponential map of H is the restriction to $Lie(H)$ of the exponential map of G, and*

$$Lie(H) = \{X \in Lie(G) : exp(tX) \in H\, \forall t \in \mathbb{R}\}$$

Proof. *The fact that the exponential map of H is the restriction of that of G is an immediate consequence of Proposition 20.3. To prove the second assertion, by the way we have identified $Lie(H)$ as a subalgebra of $Lie(G)$, we need to establish the following equivalence for every $X \in Lie(G)$:*

$$exp(tX) \in H\, \forall t \in \mathbb{R} \Leftrightarrow X_e \in T_e H.$$

Assume first that $exp(tX) \in H\, \forall t$. Since H is weakly embedded in G by Theorem 19.25, it follows that the curve $t \mapsto exp(tX)$ is smooth as a mup into H, and thus $X_e = \gamma'(0) \in T_e H$. Conversely, if $X_e \in T_e H$, then Proposition 20.3 implies that $exp(tX) \in H$ for all t.

4 The Closed Subgroup Theorem

Proposition. ***(no. 20.10)*** *Let G be a Lie group and let $\mathfrak{g}$ be its Lie algebra. For any $X, Y \in \mathfrak{g}$, there is a smooth function $Z : (\epsilon, \epsilon) \to \mathfrak{g}$ for some $\epsilon > 0$ such that the following identity holds $\forall t \in (\epsilon, \epsilon)$:*

$$(exp(tX))(exp(tY)) = exp\left(t(X+Y) + t^2 Z(t)\right).$$

Proof. *Since the exponential map is a diffeomorphism os some neighborhood of the origin in $\mathfrak{g}$, there is some $\epsilon > 0$ such that the map $\phi : (-\epsilon, \epsilon) \to \mathfrak{g}$ defined by*

$$\phi(t) = exp^{-1}(exp(tX)\, exp(tY))$$

is smooth. It obviously satisfies $\phi(0) = 0$ and

$$exp(tX)\, exp(tY) = exp(\phi(t)).$$

Observe that we can write ϕ as the composition

$$\mathbb{R} \xrightarrow{e_X \times e_Y} G \times G \xrightarrow{m} G \xrightarrow{exp^{-1}} \mathfrak{g},$$

where $e_X(t) = exp(tX)$ and $e_Y(t) = exp(tY)$. The result of Problem 7-2 shows that $dm_{(e,e)}(X,Y) = X+Y$ for $X, Y \in T_e G$, which implies

$$\phi'(0) = \left((d\, exp)_0\right)^{-1}\left(e'_X(0) + e'_Y(0)\right) = X + Y.$$

Therefore, Taylortheorem yields

$$\phi(t) = t(X+Y) + t^2 Z(t)$$

for some smooth function Z.

Corollary. ***(no. 20.11)*** *Under the hypotheses of the proceeding proposition,*

$$\lim_{n\to\infty}\left[exp\left(\frac{t}{n}X\right) exp\left(\frac{t}{n}Y\right)\right]^n = exp(t(X+Y)).$$

Proof. *The conclusion of the preceeding lemma gives that for any $t \in \mathbb{R}$ and any sufficently large $n \in \mathbb{Z}$,*

$$\left(exp\left(\frac{t}{n}X\right)\right)\left(exp\left(\frac{t}{n}Y\right)\right) = exp\left(\frac{t}{n}(X+Y) + \frac{t^2}{n^2} Z\left(\frac{t}{n}\right)\right),$$

and then Proposition 20.8(d) yields

$$\left(\left(exp\left(\frac{t}{n}X\right)\right)\left(exp\left(\frac{t}{n}Y\right)\right)\right)^n = \left(exp\left(\frac{t}{n}(X+Y)+\frac{t^2}{n^2}Z\left(\frac{t}{n}\right)\right)\right)^n$$
$$= exp\left(t(X+Y)+\frac{t^2}{n^2}Z\left(\frac{t}{n}\right)\right).$$

Theorem. ***(Closed Subgroup Theorem, no. 20.12)*** *Suppose G is a Lie group and $H \subset G$ is a subgroup that is also a closed subset of G. Then H is an embedded Lie subgroup.*

Proof. *By Proposition 7.11 it suffices to show that H is an embedded submanifold of G. We begin by identifying a subspace $Lie(G)$ that will turn out to be the Lie algebra of H.*

Let $\mathfrak{g} = Lie(G)$, and define a subset $\mathfrak{h} \subset \mathfrak{g}$by

$$\mathfrak{h} = \left\{X \in \mathfrak{g} : exp(tX) \in H, \forall t \in \mathbb{R}\right\}.$$

We need to show that $\mathfrak{h}$ is a linear subspace of $\mathfrak{g}$. It is obvious from the definiton that $\mathfrak{h}$ is closend under scalar multiplication: if $X \in \mathfrak{h}$, and let $t \in \mathbb{R}$ be arbitrary. Then $exp\left(\frac{t}{n}X\right)$ and $exp\left(\frac{t}{n}Y\right)$ are in H and because H is a closed subgroup of G, (20.5) implies that $exp(t(X+Y)) \in H$. Thus $X+Y \in \mathfrak{h}$, so $\mathfrak{h}$ is a subspace.

Next we show that there is a neighborhood U of the origin in $\mathfrak{g}$ on which the exponential map of G is a diffeomorphism, and which has the property that

$$exp\left(U \cap \mathfrak{h}\right) = (exp(U)) \cap H. \tag{1}$$

This will enable us to construct a slice chart for H in a neighborhood of the identity, and we will then use left translation to get a slice chart in a neighborhood of any point of H.

If $U \subset \mathfrak{g}$ is any neighborhood of 0 on which exp is a diffeomorphism, then $exp\left(U \cap \mathfrak{h}\right) \subset (exp(U)) \cap H$ by definiton of $\mathfrak{h}$. So to find a neighborhood satisfying (?), all we need to do is to show that U can be chosen small enough that $(exp(U)) \cap H \subset exp\left(U \cap \mathfrak{h}\right)$. Assume this is not possible.

Choose a linear subspace $\mathfrak{b} \subset \mathfrak{g}$ that is complementary to $\mathfrak{h}$, so that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{b}$ as vector spaces. Then the map $\Phi : \mathfrak{h} \oplus \mathfrak{b} \to G$ given by $\Phi(X,Y) = exp(X)\,exp(Y)$ is a diffeomorphism in some neighborhood of $(0,0)$. Choose neighborhoods U_0 of 0 in $\mathfrak{g}$ and $\hat{U}_0$ of $(0,0)$ in $\mathfrak{h} \oplus \mathfrak{b}$ such that both $exp|_{U_0}$ and $\Phi|_{\hat{U}_0}$ are diffeomorphisms on their images. Let $\left\{U_i\right\}$ be a countable neighborhood basis for $\mathfrak{g}$ at 0 - for example a countable sequence of coordinate balls whose radii approach zero. If we set $V_i = exp\left(U_i\right)$ and $\hat{U}_i = \Phi^{-1}\left(V_i\right)$, then $\left\{V_i\right\}$ and $\left\{\hat{U}_i\right\}$ are neighborhood bases of G at e and $\mathfrak{h} \oplus \mathfrak{b}$ at $(0,0)$, respectively. By discarding finitely many terms at the beginning of the sequence, we may assume tthat $U_i \subset U_0$ and $\hat{U}_i \subset \hat{U}_0$ for all i.

Our assumption implies that for each i, there exists $h_i \in \left(exp\left(U_i\right)\right) \cap H$ such that $h_i \notin exp\left(U_i \cap \mathfrak{h}\right)$. This means $h_i = exp\left(Z_i\right)$ for some $Z_i \in U_i$. Because $exp\left(U_i\right) = \Phi\left(\hat{H}_i\right)$, we can also write

$$h_i = exp\left(X_i\right) exp\left(Y_i\right)$$

for some $\left(X_i, Y_i\right) \in \hat{U}_i$. If Y_i were zero, then we would have $exp\left(Z_i\right) = exp\left(X_i\right) \in exp\left(\mathfrak{h}\right)$. But because exp is injective on U_0, this implies $X_i = Z_i \in U_i \cap \mathfrak{h}$, which contradicts our assumption that $h_i \notin exp\left(U_i \cap \mathfrak{h}\right)$. Since $\left\{\hat{U}_i\right\}$ is a neighborhood basis, $Y_i \to 0$ for $i \to \infty$. Observe that $exp\left(X_i\right) \in H$ by definition of $\mathfrak{h}$, so it follows that $exp\left(Y_i\right) = \left(exp\left(X_i\right)\right)^{-1} h_i \in H$ as well.

Choose an inner product on $\mathfrak{b}$ and let $|\cdot|$ denote the norm associated with this inner product. If we define $c_i = |Y_i|$, than we have $c_i \to 0$ as $i \to \infty$. The sequence $\left(c_i^{-1}Y_i\right)$ lies on the unit sphere in $\mathfrak{b}$, so replacing it by a sequence we may assume that $c_i^{-1}Y_i \to Y \in \mathfrak{b}$, with $|Y| = 1$ by continuty. In particular, $Y \neq 0$. We will show that $exp(tY) \in H, \forall t \in \mathbb{R}$, which implies that $Y \in \mathfrak{h}$. Since $\mathfrak{h} \cap \mathfrak{b} = \{0\}$, this is a contradiction.

Let $t \in \mathbb{R}$ be arbitrary, and for each i, let n_i be the greatest integer less than of equal to $\frac{t}{c_i}$. Then

$$|n_i - \frac{t}{c_i}| \leq 1,$$

which implies

$$|n_i c_i - t| \leq c_i \to 0,$$

so $n_i c_i \to t$. *Thus,*

$$n_i Y_i = \left(n_i c_i\right)\left(c_i^{-1} Y_i\right) \to tY,$$

which implies $exp\left(n_i Y_i\right) \to exp\left(tY\right)$ *by continuity. But* $exp\left(n_i Y_i\right) = \left(exp\left(Y_i\right)\right)^{n_i} \in H$, *so the fact that* H *is closed implies* $exp\left(tY\right) \in H$. *This completes the proof of the existence of* U *satisfying (20.6).*

Choose any linear isomorphism $E : \mathfrak{g} \to \mathbb{R}^m$ *that sends* $\mathfrak{h}$ *to* $\mathbb{R}^k$. *The composite map* $\phi \circ exp^{-1} : exp(U) \to \mathbb{R}^m$ *is then a smooth chart for* G, *and* $\phi\left(\left(exp(U)\right) \cap H\right) = E\left(U \cap \mathfrak{h}\right)$ *is the slice obtaned by setting the last* $m - k$ *coordinates equal to zero. Moreover, if* $h \in H$ *is arbitrary, the left translation map* L_h *is a diffeomorphism from* $exp(U)$ *to a neighborhood of* h. *Since* H *is a subgroup,* $L_h\left(H\right) = H$, *and so*

$$L_h\left(\left(exp(U)\right) \cap H\right) = L_h\left(exp(U)\right) \cap H,$$

and $\phi \circ L_h^{-1}$ *is a slice chart for* H *in a neighborhood of* h. *Thus* H *is an embedded submanifold of* G, *hence a Lie subgroup.*

Corollary. ***(no. 20.13)*** *If* G *is a Lie group and* H *is any subgroup of* G, *the following are equivalent:*

(a) H *is closed in* G.

(b) H *is an embedded submanifold of* G.

(c) H *is an embedded Lie subgroup of* G.

References

[1] John M. Lee. *Introduction to Smooth Manifolds.* Springer, 2nd edition, 2013.