Crystallographic Groups

Markus Schreiber
Seminar on the Geometry of Lie groups
Heidelberg University

July 12, 2016

1 Conventions

In the following handout, $\mathrm{Isom}(X)$ denotes the group of isometries of a metric space (X, d).
For $n \in \mathbb{N}$, E^n denotes the euclidean space of dimension n. For a group G, e_G denotes the identity in G.
For $\Gamma \subset \mathrm{Isom}(X)$, $\mathrm{Trans}(\Gamma)$ will denote the set of all translations in Γ, that is: $\mathrm{Trans}(\Gamma) := \{g \in \Gamma : \exists c \in X \text{ such that } g(x) = x + c \ \forall x \in X\}$.
For $A \in \mathrm{Mat}_{n\times n}(\mathbb{R})$, we define the operator norm $\|A\|_{op} = \sup\{|Ax| : x \in S^{n-1}\}$. $\|\cdot\|_{op}$ determines a metric $d_{op}(A, B) = \|A - B\|_{op}$ on $O(n)$.

2 Foundations

Before starting with the initial topic as expected, I find it practical to mention / review a few basic notions of fundamental regions and discrete groups. These following foundations will be necessary in the succeeding section in order to understand crystallographic groups.
The proofs of some of the following theorems will use lemmas and theorems from [1], to which I will simply refer instead of mentionning their content explicitly.

Definition 1. A topological group is a group G equipped with a topology such that $\iota : G \longrightarrow G, \quad \iota(g) = g^{-1}$ and $m : G \times G \longrightarrow G, \quad m(g, h) = gh$ are continuous.
A discrete group is a topological group G where $\{g\}$ is open for all g in G.

Lemma 1. *Let Γ be a topological group. Γ is discrete if and only if $\{e_\Gamma\}$ is open in Γ.*

Proof. "$\Longrightarrow$":
$\{e_\Gamma\}$ is open in Γ by definition.
"$\Longleftarrow$":
Let $g \in \Gamma$ and define $\tau_g : \Gamma \longrightarrow \Gamma, \ \tau_g(h) = gh$ is a homeomorphism, so $\tau_g(\{e_\Gamma\}) = \{g\}$ is open. This holds for all $g \in \Gamma$. □

Theorem 1. *A subgroup Γ of $\mathbb{R}^n$ is a discrete group if and only if Γ is generated by a set of linearly independent vectors.*

In the talk, we will just outline the idea of the proof.

Proof. <u>"$\Longleftarrow$":</u>
First, we assume that $\Gamma = \{0\}$. It holds that Γ is discrete, for Γ is open in itself by definition of a topology.
Now let Γ be nontrivial and let it be generated by linearly independent vectors $\{v_1, ..., v_m\},\ m \leq n$ as a group. $\Gamma = \{x \in \mathbb{R}^n :\ \exists \varepsilon_i \in \mathbb{Z}$, such that $x = \sum_{i=1}^m \varepsilon_i v_i\} = \oplus_{i=1}^m \mathbb{Z} v_i$. We may even choose $v_i = e_i$ for all i by applying an injective linear transformation.
Lemma 1 implies the discreteness of Γ, because for $B(0,1) = \{x \in \mathbb{R}^n :\ \|x\| < 1\}$, $\Gamma \cap B(0,1) = \{0\}$ holds.
<u>"$\Longrightarrow$":</u>
Let Γ be discrete. We use induction on n.
<u>$n = 1$:</u>
Choose $r > 0$, such that there exists $0 \neq g \in B(0,r) \cap \Gamma$. It follows, that for $C(0,r) = \{x \in \mathbb{R}^n :\ \|x\| \leq r\}$ ("closed ball") $C(0,r) \cap \Gamma$ is a closed subset of $C(0,r)$, since every discrete subgroup Γ of a topological group equipped with a metric topology (here: $\mathbb{R}^n$) is closed (this can be proved by contradiction - see [1], 5.3. Lemma 3).
Therefore, $C(0,r) \cap \Gamma$ is a compact discrete space and as such, it is finite. Let $0 < u \in \Gamma$ nearest to 0, $v \in \Gamma$. Let $k \in \mathbb{Z}$, such that $v \in [ku, (k+1)u) = [ku, ku + u)$ and so $v - ku \in [0, u)$. Since Γ is a $\mathbb{Z}$-module, $v - ku \in \Gamma$. Thus $v - ku \in \Gamma \cap [0, u) = \{0\}$ (because u was chosen to be nearest to 0 in Γ). It therefore holds that $v = ku$. Since v is arbitrary, Γ is generated by u.
<u>$n > 1$:</u>
There is $0 < u \in \Gamma$ with absolute value nearest to 0, such that $\Gamma \cap \mathbb{R}u = \mathbb{Z}u$. Let $\{u_1, ..., u_{n-1}, u = u_n\}$ be a basis of $\mathbb{R}^n$. We define a continuous linear map $\varphi : \mathbb{R}^n \longrightarrow \mathbb{R}^{n-1},\ \varphi(u_i) = e_i,\ i = 1, ..., n-1$ and $u \in \ker(\varphi)$.
Let $x \in \mathbb{R}^n,\ x = \sum_{i=1}^n x_i u_i$. $\varphi(x) = \sum_{i=1}^{n-1} x_i e_i$. Since for all $\lambda \in \mathbb{R},\ \varphi(\lambda u) = 0$, it holds that $\varphi^{-1}(\varphi(x)) = x + \mathbb{R}u$. We find $\psi : \mathbb{R}^{n-1} \longrightarrow \mathbb{R}^n,\ \psi(e_i) = u_i,\ i = 1, ..., n-1$ to be a continuous right inverse of φ. According to Theorem 5.1.5 [1], $\overline{\varphi} : \mathbb{R}^n \diagup \mathbb{R}u \longrightarrow \mathbb{R}^{n-1},\ x + \mathbb{R}u \longmapsto \varphi(x)$ is an isomorphism of topological groups.
For the canonical projection $\pi : \mathbb{R}^n \longrightarrow \mathbb{R}^n \diagup \mathbb{R}u$, we claim that $\pi(\Gamma)$ is discrete. Let (v_i) be a sequence in Γ satisfying $\pi(v_i) \longrightarrow 0 \implies \overline{\varphi}((\pi(v_i)) \longrightarrow 0 \implies \varphi(v_i) \longrightarrow 0 \implies \psi(\varphi(v_i)) \longrightarrow 0$ in $\mathbb{R}^n$. It follows that vor $r_i \in \mathbb{R},\ v_i - r_i u_i \longrightarrow 0$ in $\mathbb{R}^n$. Since $v_i \longrightarrow 0 \pmod{\mathbb{R}u}$, we can choose $k \in \mathbb{Z}$ with $|k| = 1$ and $ku + v_i \longrightarrow 0 \pmod{\mathbb{R}u}$.
As a consequence, there are $r_i \in \mathbb{R}$, so that $v_i - r_i u \longrightarrow 0$ and $v_i + u(k - r_i) \longrightarrow 0$ in $\mathbb{R}^n$. We may therefore assume, that $|r_i| \leq \frac{1}{2}$.
Using this upper bound, we get $|v_i - r_i u| < \frac{|u|}{2}$ for large enough i (Sidenote: Define without loss of generality $\widehat{r}_i := r_i + \epsilon_i$ for i large enough, with ϵ_i infinitesimally small; we shall have $v_i = \widehat{r}_i u \implies |v_i - r_i u| = |(\widehat{r}_i - r_i)u| < \frac{|u|}{2}$), which implies $|v_i| < |u|$.
Because of our choice of u and the fact that $v_i \in \Gamma$, we conclude that $v_i = 0$ for sufficiently large i. Therefore, for every convergent sequence (v_i) in $\pi(\Gamma)$, there is an $N \in \mathbb{N}$, such that $v_n = \textit{const}\ \forall n \geq N$. With Lemma 2 in 5.3. [1], we know that $\pi(\Gamma)$ is discrete. According to the induction hypothesis, there are $w_1, ..., w_m \in \Gamma$, such that $\pi(w_i)$ are linearly independent and generators of $\pi(\Gamma)$. Using these vectors, we get a linearly independent set $\{u, w_1, ..., w_m\}$ in $\mathbb{R}^n$ generating Γ. $\square$

Lemma 2. *Let $\Gamma \subset Isom(E^n)$ be a subgroup. For a point a in E^n, a matrix A in $O(n)$ and $r > 0$, let Γ_r be the subgroup of Γ generated by all $\phi = a + A$ with $\|A - I_n\|_{op} < r$. $k_n(r)$ shall denote the maximum number of matrices in $O(n)$ with mutual distances $d_{op}(A, B)$ at least r. Then Γ_r is* *a normal subgroup with* $[\Gamma : \Gamma_r] \leq k_n(r)$.

Proof. Let $A, B \in O(n),\ a, b \in E^n$.
For $\|A - I_n\|_{op}$, we define $\Gamma_r \ni \phi = a + A$. We will furthermore need a second map: $\Gamma \ni \psi = b + B$. We then have: $\psi\phi\psi^{-1} = c + BAB^{-1},\ c \in E^n$.

Remark: For $D \in Mat_{n\times n}(\mathbb{R})$ and $Q \in O(n)$, it holds that $\|DQ\|_{op} = \|D\|_{op}$.

According to that remark, we get $\|BAB^{-1} - I_n\|_{op} = \|BAB^{-1} - BB^{-1}\|_{op} = \|(BA - B)B^{-1}\|_{op} = \|BA - B\|_{op} = \|B(A - I)\|_{op} = \|A - I\|_{op} < r$.
Thus $\psi\phi\psi^{-1} \in \Gamma_r$ and Γ_r is a normal subgroup.
Let $\psi_i = b_i + B_i,\ i = 1, ..., m \leq k_n(r)$ be a maximal sequence in Γ with mutual distances between the matrices B_i at least r. Choose an arbitrary element $\Gamma \ni \varphi = b + B$.
After having chosen the sequence ψ_i to be maximal, there is a $j \in \{1, .., m\}$, such that

$$\|B - B_j\|_{op} < r \implies \|BB_j^{-1} - I_n\|_{op} < r$$

. For $d \in E^n$, we have $\varphi\psi_j^{-1} = d + BB_j^{-1} \ \in \Gamma_r$. Now let $\sigma \in \Gamma_r$, such that

$$\varphi\psi_j^{-1} = \sigma \iff \varphi = \sigma\psi_j$$

. Hence φ is in the coset $\Gamma_r\psi_j$.
Since φ was chosen arbitrarily, it holds that

$$\Gamma = \Gamma_r\psi_1 \cup ... \cup \Gamma_r\psi_m$$

and $[\Gamma : \Gamma_r] \leq m \leq k_n(r)$. □

Remark: The following Theorem will be used in our final proof.

Theorem 2. *Let* $\Gamma \subset Isom(E^n)$ *be a discrete subgoup. Then* Γ *has an abelian normal subgroup* N *with Trans*$(\Gamma) \subset N$ *and there is an integer* $k(n) \in \mathbb{N}$ *with* $[\Gamma : N] \leq k(n) < \infty$.

Proof. We apply Lemma 2 with $N = \Gamma_{\frac{1}{2}}$. We then have $k(n) = k_n(\frac{1}{2})$. N is abelian by lemmas 4 and 5 (5.4 [1]). These lemmas also explain the choice of $\Gamma_{\frac{1}{2}}$.
Let $\tau \in$ Trans(Γ). τ can be written as $\tau = g + I_n,\ g \in E^n$. Clearly, I_n is orthogonal with $\|I_n - I_n\|_{op} = 0 < \frac{1}{2}$, which is why τ is an element of $\Gamma_{\frac{1}{2}} = N$. □

Remark: The following Theorem will be used in our final proof.

Theorem 3. *Let* $\Gamma \subset Isom(E^n)$ *be an abelian discrete subgroup. Then there are subgroups* H *and* K *of* Γ *and an* $m-$*Plane* P *of* E^n*, so that the following features hold:*

1. $\Gamma = H \oplus K$*;*

2. $|K| < \infty$ *and every element of* K *acts trivially on* P*;*

3. H *is a free abelian group of rank* m *and the only element of* H *acting trivially on* P *is* $e_H = e_\Gamma$*;*

Remark: H *acts on* P *as a discrete group of translations.*

Proof. The following proof uses induction on n.

<u>$n = 0$:</u>

$E^n = 0,\ \Gamma = \{e_\Gamma\}$ is the trivial case.

<u>$n > 0$:</u>

Let $\Gamma \ni \phi = a + A$ with $\dim(V := \{v \in E^n :\ Av = v\})$ minimal. If $V = E^n$, $\Gamma = \mathrm{Trans}(\Gamma)$ holds and we can finish the proof in this case by applying Theorem 1 on $\Gamma = H$ (therefore, $K = \{e_\Gamma\}$) and $P = \mathrm{span}(\{\gamma \cdot 0 :\ \gamma \in \Gamma\})$.

Now on to $\dim(V) < n$. For $[\phi, \psi] = \phi\psi\phi^{-1}\psi^{-1}$, $\phi,\ \psi \in \Gamma$ defined as $\phi = a + A,\ \psi = b + B$, we get:

$$[\phi, \psi] = (A - I)b + (I - B)a + I$$

Since Γ is abelian, we also get $[\phi, \psi] = I$ and so, we get

$$(A - I)b + (I - B)a = 0 \iff (A - I)b = (B - I)a \tag{1}$$

For $v \in V$, we have $ABv = BAv = Bv$. Therefore, $B(V) \subset V$ and since $B \in O(n)$, we have that $B(V) = V$. Consequently, $(B - I)(V)$ is a subspace of V.

If $b \in V,\ (A - I)(b) = 0$. For $b \in V^\perp,\ (A - I)(b) \in V^\perp$ would be the case.

Combining this information with the equation 1, we conclude that

$$(B - I)a \in V \cap V^\perp = \{0\}$$

so $Ba = a$ and $Ab = b$, with b being in V, for it is fixed by A. Since $\psi = b + B$ is an arbitrary element in Γ with $\psi(x) \in V$ for all $x \in V$, Γ leaves V invariant.

We may assume that for $k < n$, $E^k = V$. By restricting all elements of Γ to E^k, we obtain a discrete group $\mathrm{Res}(\Gamma) \subset \mathrm{Isom}(E^k)$ and a restriction homomorphism $\rho : \Gamma \longrightarrow \mathrm{Res}(\Gamma)$.

Since $k < n$, we can apply the indutction hypothesis: There are subgroups $H', K' \subset \mathrm{Res}(\Gamma)$ and an m-plane P of E^k such that $\mathrm{Res}(\Gamma) = H' \oplus K'$, where K' is a finite group an H' is a free abelian group of rank m. $e_{H'}$ is the only element of H' acting trivially on P. H' acts generally as a discrete group of translations.

$K := \rho^{-1}(K')$ defines a finite subgroup of Γ acting trivially on P.

The sequence

$$\{e_K\} \to K \hookrightarrow \Gamma \twoheadrightarrow H' \to \{e_{H'}\}$$

is exact and since H' is free abelian, it splits. Therefore, $\Gamma = K \oplus H$ holds for a subgroup H of Γ, that is mapped isomorphically onto H' by ρ. This isomorphism equips H with all the necessary features mentionned above. $\square$

Definition 2. Let X be a metric space and $\Gamma \subset \mathrm{Isom}(X)$ a subgroup.

1. $R \subset X$ is a fundamental region for Γ if and only if:
   (a) R is open in X
   (b) $gR \cap hR = \emptyset$ for $g, h \in \Gamma$, $g \neq h$
   (c) $X = \cup_{g \in \Gamma} g\overline{R}$
2. Γ has a fundamental region, therefore Γ is a discrete subgroup
3. $D \subset X$ is a fundamental domain for Γ if and only if D is a connected fundamental region for Γ

Definition 3. Let R be a fundamental region of $\Gamma \subset \mathrm{Isom}(X)$. R is called locally finite if and only if the family $\{g\overline{R}:\ g \in \Gamma\}$ is locally finite, meaning that for every $x \in X$, there is a neighbourhood U of x that intersects $g\overline{R}$ for finitely many g.

Definition 4. A polyhedron P in E^n is called a fundamental polyhedron, if its interior is a locally finite fundamental domain of Γ.

Definition 5. Let $\Gamma \subset \mathrm{Isom}(X)$ act discontinuously on X and let $g \in \Gamma$.
For $a \in X$ with trivial stabilizer Γ_a, we define $H_g(a) := \{x \in X:\ d(x,a) < d(x,ga)\}$. The Dirichlet domain $D(a)$ for Γ is either X if $\Gamma = \{e_\Gamma\}$, or if Γ is not trivial, we have $D(a) = \cap\{H_g(a):\ g \neq e_\Gamma\}$
The so called Dirichlet polyhedron for Γ with center a is $D(a)$ defined as above, if it is a convex fundamental polyhedron.

3 Crystallographic Groups

This section is supposed to serve with several equivalent definitions of crystallographic groups working our way to the first theorem by Bieberbach presented in this seminar.

Definition 6. A crystallographic group of dimension n is a discrete group $\Gamma \subset \mathrm{Isom}(E^n)$ such that $E^n \diagup \Gamma$ is compact.

Lemma 3. *Let R be a locally finite fundamental region of X and let x in X be a boundary point of R. Then the following properties hold:*

1. $\mid \partial R \cap \Gamma x \mid < \infty$
2. $\exists r > 0$, *such that for* $N(\overline{R}, r) = \cup\{B(x,r):\ x \in \overline{R}\}$, *we get* $N(\overline{R}, r) \cap \Gamma x = \partial R \cap \Gamma x$

Proof. R is locally finite, so for $r > 0$, $B(x,r)$ meets $g_i^{-1}\overline{R}$ for $i \in \{1, ..., m\}$. Assume that $x \in g_i^{-1}\overline{R}$ for all i (after shrinking r, for example).
Now let $g \in \Gamma$ and suppose that $gx \in \partial R$. Then there is an $i \in \{1, ..., m\}$, such that $g = g_i$ and we get that $x \in g_i^{-1}\overline{R}$. Since $gx \in \Gamma x$, we would then get:

$$\partial R \cap \Gamma x \subset \{g_i x:\ i = 1, ..., m\}$$

Conversely, let's have a look at $g_i x$ for some $i \in \{1, ..., m\}$. We get that $g_i x \in \partial R$ for all i. Therefore, we get

$$\partial R \cap \Gamma x \supset \{g_i x:\ i = 1, ..., m\}$$

and equality holds.
Now, let $y \in \overline{R}$ and suppose that $d(gx, y) < r$. Then we get $d(x, g^{-1}y) < r \Longrightarrow g^{-1}y \in B(x,r) \subset N(\overline{R}, r)$. Since $y \in \overline{R}$, It follows, that there is an $i \in \{1, ..., m\}$, such that $g = g_i$. Furthermore, $gx \in \partial R$. We conclude, that

$$N(\overline{R}, r) \cap \Gamma x \subset \partial R \cap \Gamma x$$

is a subset. Since $\partial R \subset N(\overline{R}, r)$, equality holds. □

Theorem 4. *Let $\Gamma \subset Isom(X)$ be discontinuous (discrete) and X be locally compact, such that $X \diagup \Gamma$ is compact. Furthermore, let R be a fundamental region for Γ. If R is locally finite, then $\overline{R}$ is compact.*

Remark: Equivalence also holds, but for this talk, this implication suffices.

Proof. Suppose that $\overline{R}$ is not compact. Then there is a sequence (x_i) in $\overline{R}$, that does not have a convergent subsequence. Let $\pi : X \longrightarrow X\diagup\Gamma$ be the canonical projection. Since $X\diagup\Gamma$ is compact, the sequence $(\pi(x_i))$ has a convergent subsequence in $X\diagup\Gamma$. We therefore may assume, that $(\pi(x_i))$ converges. Hence there is an $x \in \overline{R}$, such that $\pi(x_i) \longrightarrow \pi(x)$ for $i \longrightarrow \infty$.
π maps R homeomorphically onto $\pi(R)$, so $x \in \partial R$ must be the case, for R is a fundamental region and as such, it is an open set. Then, Lemma 3 implies, that there is an $r > 0$, such that

$$N(\overline{R}, r) \cap \Gamma x = \partial R \cap \Gamma x = \{g_i x : \ i = 1, ..., m\}$$

We may assume that $C(g_i x, r) = \{v \in X : \ d(g_i x, v) \leq r\}$ is compact for each i (possibly after shrinking r). Since $(\pi(x_i))$ is a convergent sequence in $X\diagup\Gamma$, there is an $N \in \mathbb{N}$, such that $d_\Gamma(\Gamma x_i, \Gamma x) < r$ for all $i \geq N$, d_Γ being a distance function on $X\diagup\Gamma$.
We then get for our sequence in R and for each $i \geq N$ an $h_i \in \Gamma$, such that

$$d(x_i, h_i x) < r$$

with $h_i x \in \Gamma x \cap N(\overline{R}, r) = \{g_i x : \ i = 1, ..., m\}$. As a consequence, we find a $j \in \{1, ..., m\}$ for each $i \in \{1, ..., m\}$, such that $h_i x = g_j x$. Because of $d(x_i, g_j x) < r$, we find that $x_i \in \cup_{l=1}^m C(g_l x, r)$ for all $i \geq N$, which is a compact set.
In this case, we get a contradiction, for we have found a convergent subsequence of (x_i). ☐

Theorem 5. *Let $\Gamma \subset Isom(E^n)$ be a discrete group. Then the following are equivalent:*

1. *Γ is a crystallographic group*
2. *Every convex fundamental polyhedron of Γ is compact*
3. *Γ has a compact Dirichlet polyhedron*

Proof. We will proof this theorem in the following order: $2 \Longrightarrow 3$, $3 \Longrightarrow 1$, $1 \Longrightarrow 2$.
$2 \Longrightarrow 3$:
According to (2), we only need to show the existence of a Dirichlet polyhedron: This ist the case, fot Γ acts discontinuously, for it is discrete.
$3 \Longrightarrow 1$:
Let D be a compact Dirichlet polyhedron of Γ. According to definition 2 and using the projection map π, $\pi(D) = E^n\diagup\Gamma$ holds. Since D is compact and π is continuos, it follows, that $\pi(D) = E^n\diagup\Gamma$ is compact as well.
$1 \Longrightarrow 2$:
This is the case by Theorem 4. ☐

For an $n-$dimensional grystallographic group Γ with a convex fundamental polyhedron P, we know now that P is compact, so it has finitely many sides. P serves as a model for an n-dimensional crystal and we can cover E^n with copies of P; we get a so called tesselation: $\{gP : \ g \in \Gamma\}$.

In the following lemma, X will stand for the euclidean space E^n or for the hyperbolic space $\mathbb{H}^n$.

Remark: The following Lemma will be used in our final proof.

Lemma 4. *For a discrete group $\Gamma \subset Isom(X)$, let $H \subset \Gamma$ be a subgroup of finite index, that is: $[\Gamma : H] < \infty$. Then $X \diagup \Gamma$ is compact if and only if $X \diagup H$ is compact.*

Proof. $\underline{\text{"}\Longrightarrow\text{"}:}$
First, let $X \diagup \Gamma$ be compact. Let D be a Dirichlet domain for Γ. D therefore is a locally finite fundamental domain. By Theorem 4, $\overline{D}$ is compact. Since H is of finite index, we can list the cosets $g_1 H, ..., g_m H$ of H in Γ and define a compact subset K of X as follows:

$$K = g_1^{-1}\overline{D} \cup ... \cup g_m^{-1}\overline{D}.$$

For $x \in X$, there is a $g \in \Gamma$ such that $gx \in \overline{D}$. Since $\Gamma = \cup_i g_i H$, there is an $i \in \{1, ..., m\}$, such that $g = g_i h$ for $h \in H$.
Let $\eta : X \longrightarrow X \diagup H$ be the quotient map.
$gx = g_i hx \in \overline{D} \iff hx \in g_i^{-1}\overline{D}$
Since hx is in $g_i^{-1}\overline{D}$, Hx lies in $\eta(K)$. After having chosen x arbitrarily, $\eta(K) = X \diagup H$ and so $X \diagup H$ is compact.
$\underline{\text{"}\Longleftarrow\text{"}:}$
Now, let $X \diagup H$ be compact. Define

$$\phi : X \diagup H \longrightarrow X \diagup \Gamma$$

by $\phi(Hx) = \Gamma x$ and let $\pi : X \longrightarrow X \diagup \Gamma$ be the quotient map. η shall be defined as above.
It holds that $\pi = \phi\eta$. Since π and η are continuous, ϕ is continuous as well. And by definition, ϕ is surjective. Hence $X \diagup \Gamma$ is compact. $\square$

Definition 7. A lattice subgroup $\Gamma \subset \text{Isom}(E^n)$ is a group generated by n linearly independent translations.
Γ (as above) is a lattice subgroup if and only if Γ is discrete and free abelian of rank n.

We will now continue with the first theorem among Bieberbach's Theorems. In this talk, we will only look at one implication: A statement on all crystallographic groups. It is to be noted that equivalence holds in the following theorem. The proof of the conversion will not be stated here, but that part of the proof uses Theorem 1 from this handout.

Theorem 6. *Let $\Gamma \subset Isom(E^n)$ be a discrete group. If Γ is crystallographic, then Trans(Γ) is of finite index with rank n.*

Proof. According to Theorem 2, Γ has an abelian subgroup H with $[\Gamma : H] < \infty$ and Trans$(\Gamma) \subset H$. By Lemma 4, H is crystallographic as well. By Theorem 3, there is an m-plane P in E^n with H acting on it by translation. Since H is a group of isometries, points at distance d from P stay at distance d under the action of H. Therefore, for $m < n$, $E^n \diagup H$ is unbounded. Since $E^n \diagup H$ is compact, this cannot be the case: $m = n$ as a consequence.
Hence H is a lattice subgroup and $H = \text{Trans}(\Gamma)$. $\square$

For an n-dimensional crystallographic group Γ, Trans(Γ) is a free abelian group of rank an with finite index by Theorem 6. Trans(Γ) is the unique maximal subgroup of this sort, which follows from Theorem 3.
Since all elements of Trans(Γ) can be written as $\tau = x + I_n$ for some x in E^n and I_n being the identity matrix of rank n, Trans(Γ) is the kernel of the natural projection $\rho : \Gamma \longrightarrow O(n)$ defined

by $\rho(a + A) = A$.
We call the image of $\mathrm{Im}(\rho) = \Pi$ the point group of Γ. The exact sequence

$$\{e_\Gamma\} \to \mathrm{Trans}(\Gamma) \hookrightarrow \Gamma \twoheadrightarrow \Pi \to \{I_n\}$$

shows that $\mathrm{Trans}(\Gamma)$ is a normal subgroup (for it is the kernel of a group homomorphism) and that Π is a finite group.
Now, let $\mathbb{Z}^n \cong L(\Gamma) \subset \mathbb{R}^n$ denote the lattice subgroup corresponding to $\mathrm{Trans}(\Gamma)$. For $a + A = \phi$ in Γ and for a point b in L, we have

$$(a + A)(b + I)(a + A)^{-1} = Ab + I.$$

We now see that Π acts on $L(\Gamma)$ by left matrix multiplication.
We then get an injective representation $\Pi \longrightarrow \mathrm{Aut}(L(\Gamma))$ with $A \mapsto [x \mapsto Ax]$.
For a finite subgroup $Q \subset \mathrm{GL}(n, \mathbb{Z})$, we get an exact sequence

$$\{0\} \to \mathbb{Z}^n \hookrightarrow \Gamma \twoheadrightarrow Q \to \{I\}.$$

This forms the foundation of an approach to proving that there are only finitely many isomorphism classes of n-dimensional crystallographic groups.

References

[1] John G. Ratcliffe. *Foundations of Hyperbolic Manifolds - Second Edition.* S.Axler, K.A. Ribet, Springer Science+Business Media, LLC, New York, 2006.