The Lie Correspondence

1 Motivation

Reminder 1 For a Lie group G the Lie algebra containing all (smooth) left-invariant vector fields on G is called *the Lie algebra of G*.

Notation: $\mathrm{Lie}(G)$.
We identify $\mathrm{Lie}(G)$ with T_eG since they are isomorphic (see the report of the second talk).

Lemma 2 Two isomorphic Lie groups have isomorphic Lie algebras.

Proof: See the report of the second talk, p.5. □

Remark 3 The converse of Lemma 2 is wrong.

Example 4 Consider the two Lie groups $\mathbb{T}^n$ and $\mathbb{R}^n$.

(i) It holds that $\mathrm{Lie}(\mathbb{R}^n) \cong \mathbb{R}^n$:
For any $a \in \mathbb{R}^n$ the left translation is given by $L_a(x) = a + x$. Thus the differential of any left-invariant vector field has constant maps as coefficients. The Lie bracket of two vectorfields with constant coefficients is always zero, therefore we get $\mathrm{Lie}(\mathbb{R}^n) \cong \mathbb{R}^n$.

(ii) The n-Torus $\mathbb{T}^n$ is a Lie group with $\mathrm{Lie}(\mathbb{T}^n) \cong \mathbb{R}^n$:
It holds that $\mathbb{T}^n \cong \mathbb{S}^1 \times \ldots \times \mathbb{S}^1$, where $\mathbb{S}^1$ denotes the circle group of dimension 1. From the second talk we know that the dimension of $\mathrm{Lie}(\mathbb{S}^1)$ has to be equal to the dimension of $\mathbb{S}^1$. Therefore $\mathrm{Lie}(\mathbb{S}^1) \cong \mathbb{R}$. As we will see later on it holds that $\mathrm{Lie}(\mathbb{T}^n) \cong \mathrm{Lie}(\mathbb{S}^1 \times \ldots \times \mathbb{S}^1) \cong \mathbb{R}^n$.

From (i) and (ii) we get that the Lie groups of the n-Torus and $\mathbb{R}^n$ are isomorphic even though $\mathbb{T}^n$ and $\mathbb{R}^n$ are not isomorphic.

Definition 5 Two Lie groups G and H are called *locally isomorphic* if there exist neighborhoods of the respective identitiy element $U \subset G$ and $V \subset H$ and a diffeomorphism $F : U \to V$ such that $F(gg') = F(g)F(g')$ for all $g, g' \in U$ with $gg' \in U$.

Remark 6 Two Lie groups are locally isomorphic if and only if they have isomorphic Lie algebras (*'Fundamental Theorem of Sophus Lie'*).

Example 7 Indeed $\mathbb{T}^n$ and $\mathbb{R}^n$ are locally isomorphic as Lie groups.

2 The Lie Correspondence

Definition 8 Let X and Y be topological spaces.

(i) X is called *simply connected* if it is path-connected and any loop in X is homotopic to a constant path.

(ii) X is called *path-connected* if for any two points in X there exists a path that joins them.

(iii) Let $f, g : X \to Y$ be continuous maps. f and g are called *homotopic* if there exists a continuous map $H : X \times [0, 1] \to Y$ such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ for all $x \in X$.

Remark 9 The n-Torus $\mathbb{T}^n$ is not simply connected.

Theorem 10 There is a one-to-one correspondence between isomorphism classes of simply connected Lie groups and isomorphism classes of finite dimensional Lie algebras given by the map $\mathrm{Lie}(\cdot)$.

Theorem 11 Suppose G is a Lie group, $\mathfrak{g}$ its Lie algebra and $\mathfrak{h} \subseteq \mathfrak{g}$ a Lie subalgebra.
Then there exists a unique connected Lie subgroup $H \subset G$ such that $\mathrm{Lie}(H) = \mathfrak{h}$.

Proof: Consider $\exp_{\mathfrak{g}} : \mathfrak{g} \to G$. Let H be the smallest Lie subgroup of G containing $\mathrm{im}(\exp_{\mathfrak{g}}|_{\mathfrak{h}})$. Our Aim is to show: $\mathrm{Lie}(H) = \mathfrak{h}$.
Let $\mathfrak{h}'$ be the Lie group of H. Therefore it holds that $\mathfrak{h} \subset \mathfrak{h}'$. It remains to show $\mathfrak{h}' \subset \mathfrak{h}$.

It is possible to write $\mathfrak{g}$ as a decomposition into the vector subspace $\mathfrak{h}$ and its complement $\mathfrak{h}^{\perp}$:

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^{\perp}.$$

Since the vector space $\mathfrak{g}$ has a basis you can find the subset which generates the subspace $\mathfrak{h}$ and take the remaining elements of the basis of $\mathfrak{g}$ to generate $\mathfrak{h}^{\perp}$. Thus you can always find such a decomposition.
Also we know that there exist open neighborhoods $U \subset \mathfrak{g}$ with $0_{\mathfrak{g}} \in U$ and $V \subset G$ with $0_G \in V$ sucht that $\exp_{\mathfrak{g}}|_U : U \to V$ is a diffeomorphism. Since $U \subset \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^{\perp}$ you can also write $U = W \oplus W'$ with $0_{\mathfrak{h}} \in W \subset \mathfrak{h}$ and $0_{\mathfrak{h}^{\perp}} \in W' \subset \mathfrak{h}^{\perp}$.

Let $Z \in \mathfrak{h}'$. It follows that $\exp tZ \in H$ for all $t \in \mathbb{R}$. From the last talk we know that for small enough t it holds that

$$\exp tZ = \exp t(X+Y) = \exp tX \exp tY,$$

where $X \in W \subset \mathfrak{h}$ and $Y \in W' \subset \mathfrak{h}^{\perp}$.
Since H is a group and $\mathrm{im}(\exp_{\mathfrak{g}}|_{\mathfrak{h}}) \subset H$ by definition it also holds that $\exp tX \in H$. This implies that $\exp tY \in H$ for all sufficiently small $t \in \mathbb{R}$. With Lemma 12 we get that Y has to be identical to the zero map and thus $Z = X + Y = X \in \mathfrak{h}$ since $\exp_{\mathfrak{g}}|_U$ is a diffeomorphism.
We conclude that $\mathfrak{h} = \mathfrak{h}' = \mathrm{Lie}(H)$. □

Lemma 12 With the notation from the proof of Theorem 11 it holds that

$$\{Y \in W' : \exp Y \in H\}$$

is at most countable.

Proof: For the proof of Lemma 12 see *'Lie groups, Lie algebras, And Representations An Elementry Introduction'* by *Brian C. Hall* (Springer, 2004), Lemma 5.21. □

Proof of Theorem 10: Let G be a simply connected Lie group and $\mathfrak{g} := \mathrm{Lie}(G)$ its Lie algebra. We have to show that $\mathrm{Lie} : G \mapsto \mathfrak{g}$ is both injective and surjective up to isomorphism.

Injectivity:
Let G and H be simply connected Lie groups with isomorphic Lie algebras $\mathfrak{g} := \mathrm{Lie}(G)$ and $\mathfrak{h} := \mathrm{Lie}(H)$. Thus there exists a Lie algebra isomorphism $\varphi : \mathfrak{g} \to \mathfrak{h}$.

It holds that $\mathrm{Lie}(G \times H) \cong \mathrm{Lie}(G) \times \mathrm{Lie}(H)$:

We already know that $G \times H$ is a Lie group. Its Lie algebra $\mathfrak{k} := \mathrm{Lie}(G \times H)$ is defined by a Lie bracket $[\cdot,\cdot]_{\mathfrak{k}}$ such that for all vectorfields $X, \bar{X} \in \mathfrak{g}$ and $Y, \bar{Y} \in \mathfrak{h}$ it holds that

$$[(X, Y), (\bar{X}, \bar{Y})]_{\mathfrak{k}} = ([X, \bar{X}]_{\mathfrak{g}}, [Y, \bar{Y}]_{\mathfrak{h}}).$$

It it obvious that this bracket defines a Lie algebra on $G \times H$ and even the Lie algebra on $G \times H$ since the left translation of a product of Lie groups is also defined componentwise.

Let $\mathfrak{l} \subset \mathfrak{g} \times \mathfrak{h}$ be the graph of φ:

$$\mathfrak{l} := \{(X, \varphi(X)) : X \in \mathfrak{g}\}.$$

It holds that $\mathfrak{l}$ is a vector subspace of $\mathfrak{g} \times \mathfrak{h}$ because φ is linear as a Lie group homomorphism and even a Lie subalgebra:

$$[(X, \varphi(X)), (\bar{X}, \varphi(\bar{X}))]_{\mathfrak{k}} = ([X, \bar{X}]_{\mathfrak{g}}, [\varphi(X), \varphi(\bar{X})]_{\mathfrak{h}}) = ([X, \bar{X}]_{\mathfrak{g}}, \varphi([X, \bar{X}]_{\mathfrak{g}})) \in \mathfrak{l}$$

With Theorem 11 it holds that there exists a unique connected Lie subgroup $L \subset G \times H$ such that $\mathrm{Lie}(L) = \mathfrak{l}$.

It is clear that the following projetions are Lie group homomorphisms:

$$\pi_1 : G \times H \to G,\ \pi_2 : G \times H \to H.$$

It follows that the restrictions of π_1 and π_2 to L are also homomorphisms. In the following we want to show that $\pi_1|_L$ is bijective. Then we could define a Lie group homomorphism in the following way:

$$\Phi = \pi_2|_L \circ (\pi_1|_L)^{-1} : G \to H.$$

Put $\Pi = \pi_1|_L : L \to G$ and consider its induced Lie algebra homomorphism Π_*. It is given by the projection:

$$\Pi_* : \mathfrak{l} \to \mathfrak{g}, (X, \varphi(X)) \mapsto X.$$

Thus Π_* is a Lie algebra isomorphism which gives us that Π itself is also a Lie algebra isomorphism.

With the previous definition we get a Lie group homomophism $\Phi : G \to H$ with $\pi_2|_L = \Phi \circ \pi_1|_L$.

Define $\Pi_2 : \mathfrak{g} \times \mathfrak{h} \to \mathfrak{h}$ as the induced Lie group homomorphism of the projection on the second argument which is the projection onto the second argument on an algebra level. It holds that:

$$\Pi_2|_{\mathfrak{l}} = \Phi_{*,e} \circ \Pi : \mathfrak{l} \to \mathfrak{h}.$$

For any $X \in \mathfrak{g}$ this gives us that

$$\varphi(X) = \Pi_2|_{\mathfrak{l}}(X, \varphi(X)) = \Phi_{*,e} \circ \Pi(X, \varphi(X)) = \Phi_{*,e}(X).$$

Since φ is an isomorphism we get analogously a Lie algebra homomorphism $\Psi : H \to G$ with $\varphi^{-1} = \Psi_{*,e}$. By the construction of the induced homomorphisms $\Phi_{*,e}$ and $\Psi_{*,e}$ it is clear that they are unique with this property. For both the identity map and the composition $\Psi \circ \Phi$ it holds that the induced homomorphism is equal to the identiy map on $\mathfrak{g}$.

With the uniqueness of Ψ and Φ it follows that $\Psi \circ \Phi = \mathrm{id}_G$ and analogously $\Phi \circ \Psi = \mathrm{id}_H$ which makes Ψ a Lie group isomorphism with the inverse map Ψ.

Surjectivity:
Consider a finite dimensional Lie algebra $\mathfrak{g}$. For some vectorfield V it holds that $\mathfrak{g}$ is isomorphic to a subalgebra of $\mathrm{Lie}(\mathrm{GL}(V))$ (see the report of the second talk, p. 2). From Theorem 11 we can get a connected Lie subgroup G of $\mathrm{GL}(V)$ such that $\mathrm{Lie}(G) \overset{\sim}{=} \mathfrak{g}$. By Theorem 13 there exists a simply connected Lie group with $\mathrm{Lie}(\tilde{G}) \overset{\sim}{=} \mathrm{Lie}(G)$. □

Theorem 13 Let G be a connected Lie group. Then there exists a simply connected Lie group $\tilde{G}$ and a smooth covering map $G \to \tilde{G}$ that is also a Lie group homomorphism.

Proof: For the proof see *'Introduction to Smooth Manifolds'* by *John M. Lee* (1st ed., Springer 2002), Theorem 2.13. □

3 Normal Lie Subgroups

Definition 14 Let G be a group. A subgroup $H \subset G$ is called *normal* if $ghg^{-1} \in H$ for all $g \in G$ and $h \in H$.

Definition 15 Let G be a Lie group and $\mathfrak{g} = \mathrm{Lie}(G)$ its Lie algebra. For $g \in G$ the Lie group homomorphism

$$C_g : G \to G, \;\; h \mapsto ghg^{-1}$$

is called the *conjugation map*.
Thus we get for every $g \in G$ the (induced) Lie algebra homomorphism

$$\mathrm{Ad}_g := (C_g)_{*,e} : \mathfrak{g} \to \mathfrak{g}.$$

The following map is called the *adjoint representation of G*:

$$\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g}), \;\; g \mapsto \mathrm{Ad}_g.$$

Reminder 16 The *adjoint representation of a Lie algebra* $\mathfrak{g}$ is the map

$$\mathrm{ad} : \mathfrak{g} \to \mathrm{Lie}(\mathrm{GL}(\mathfrak{g})), \;\; X \mapsto \mathrm{ad}_X,$$

where ad_X denotes the map $\mathrm{ad}_X : \mathfrak{g} \to \mathfrak{g}, \;\; Y \mapsto [X, Y]_{\mathfrak{g}}$.

Theorem 17 Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. The Lie algebra homomorphism induced by the adjoint representation Ad of G is given by $\mathrm{Ad}_{*,e} = ad$.

Proof: The map $t \mapsto \mathrm{exp}tX$ is a smooth curve with tangent vector X at $t = 0$. Thus we can compute $\mathrm{Ad}_{*,e}$ by

$$\mathrm{Ad}_{*,e}(X) = \frac{d}{dt}\bigg|_{t=0} \mathrm{Ad}(\mathrm{exp}tX)$$

For any $Y \in \mathfrak{g}$ we get

$$\mathrm{Ad}_{*,e}(X)Y = \left(\frac{d}{dt}\bigg|_{t=0} \mathrm{Ad}_{\mathrm{exp}tX}\right)Y = \frac{d}{dt}\bigg|_{t=0} \mathrm{Ad}_{\mathrm{exp}tX}Y.$$

Consider $\mathrm{Ad}_{\exp tX}Y$ as a left-invariant vector field on G. It is determined by its value at the identity element $e \in G$:

$$\begin{aligned}(\mathrm{Ad}_{\exp tX}Y)_e &= (C_{\exp tX})_{*,e}Y_e = (R_{\exp(-tX)})_{*,e}(L_{\exp tX})_{*,e}Y_e \\ &= (R_{\exp(-tX)})_{*,e}Y_{\exp tX} = (\theta_{-t})_{*,e}Y_{\theta_t(e)},\end{aligned}$$

where $\theta_t = R_{\exp tX}$ denotes the flow of X. Therefore we get that

$$(\mathrm{Ad}_{*,e}(X)Y) = \left.\frac{d}{dt}\right|_{t=0}(\theta_{-t})_{*,e}Y_{\theta_t(e)} = (\mathcal{L}_XY)_e = [X,Y]_e,$$

where $(\mathcal{L}_XY)_e$ denotes the *Lie derivate* of Y with respect to X which is equal to the Lie bracket at e and is used as a geometric interpretation for the Lie bracket. In the equality above we first used the definition and then the equivalence to the Lie bracket. For the proof of the second step see *'Introduction to Smooth Manifolds'* by *John M. Lee* (1st ed., Springer 2002), Theorem 18.2. □

Lemma 18 Let G be a connected Lie group, $H \subset G$ a connected Lie subgroup and $\mathfrak{g}$ and $\mathfrak{h}$ their Lie algebras, respectively. Then H is normal in G if and only if for all $X \in \mathfrak{g}$ and $Y \in \mathfrak{h}$ it holds:

$$(\exp X)(\exp Y)(\exp(-X)) \in H \tag{1}$$

Proof: Note that for all $X \in \mathfrak{g}$ and $Y \in \mathfrak{h}$ $\exp X, \exp(-X) = (\exp X)^{-1} \in G$ and $\exp Y \in H$. Therefore $(\exp X)(\exp Y)(\exp(-X)) \in H$ if H is normal.

Conversely, suppose that (1) holds. It is known there exist open neighborhoods $U \subset \mathfrak{g}$ of $0_{\mathfrak{g}}$ and $V \subset G$ of the identity element (in G) such that

$$\exp_{\mathfrak{g}}|_U : U \to V$$

is a diffeomorphism. We also know it holds that

$$\exp_{\mathfrak{g}}|_{\mathfrak{h}} = \exp_{\mathfrak{h}}.$$

It is possible to shrink U such that $\exp_{\mathfrak{g}}|_{U\cap\mathfrak{h}}$ is a diffeomorphism from $U \cap \mathfrak{h}$ to a heighborhood $V' \subset V$ with $0_H \in V'$. Shrink U further until it holds that $X \in U$ if and only if $-X \in U$.

For all $g \in V$ and $h \in V'$ it follows that there exist $X \in U$ and $Y \in U \cap \mathfrak{h}$ such that $\exp X = g$ and $\exp Y = h$. Since $(\exp X)(\exp Y)(\exp(-X)) \in H$ by assumption it holds that $ghg^{-1} \in H$.

By Lemma 19 it holds for all $h \in H$ that $h = \prod_{i\in I} h_i$ for $h_i \in V', i \in I \subset \mathbb{N}$. Thus we get that for alle $g \in V$ and $h \in H$ it holds that

$$ghg^{-1} = g\Big(\prod_{i\in I} h_i\Big)g^{-1} = \prod_{i\in I}(gh_ig^{-1}) \in H.$$

Similarly any $g \in G$ can be written as $g = \prod_{j\in J} g_j$ for some $g_j \in V, j \in J \subset \mathbb{N}$. It follows that $ghg^{-1} \in H$ for all $g \in G$ and $h \in H$. □

Lemma 19 Let G be a connected Lie group. For any open neighborhood $U \subset G$ of the identity element you can write any element $g \in G$ in the following way:

$$g = \prod_{j \in J} g_j$$

for some $g_j \in U$ and $J \subset \mathbb{N}$. Since G is finite dimensional J is a finite set.

Proof: Let H be the smallest Lie subgroup of G such that $U \subset H$. Define the following open subsets of H:
$U_1 := U \cup U^{-1}$, where $U^{-1} := \{g \in G : \exists u \in U \text{ with } gu = e\}$.
$U_k := U_1 U_{k-1} := \{u_1 u_2 : u_1 \in U_1, u_2 \in U_{k-1}\}$ for all $k \geq 2$.
Obviously H is the union of all U_k. Since the inversion map and the multiplication in G are diffeomorphisms U_k is open for all $k \in \mathbb{N}$. Therefore H is an open subgroup of G. But H is also closed as a Lie subgroup of G. (For this result see the report of the first talk.) Thus G is the disjoint union of the two open sets H and $G \setminus H$. Since H contains the identity and G is connected it holds that $H = G$ and by the construction of H the statement of the Lemma follows. $\square$

Definition 20 Let $\mathfrak{g}$ be a Lie algebra. A linear subspace $\mathfrak{h} \subset \mathfrak{g}$ is called an *ideal in* $\mathfrak{g}$ if $[X, Y] \in \mathfrak{h}$ for all $X \in \mathfrak{g}$ and $Y \in \mathfrak{h}$.

Theorem 21 Let G be a connected Lie group and $H \subset G$ a connected Lie subgroup. Then H is a normal subgroup if and only if $\mathfrak{h} := \mathrm{Lie}(H)$ is an ideal in $\mathfrak{g} := \mathrm{Lie}(G)$.

Proof: Consider the Lie group homomorphism $C_g(h) = ghg^{-1}$ for a $g \in G$. The following diagram commutes for any $X \in \mathfrak{g}$:

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\mathrm{Ad}_{\exp X}"] \arrow[d, "\exp"] & \mathfrak{g} \arrow[d, "\exp"] \\ G \arrow[r, "C_{\exp X}"'] & G. \end{tikzcd}$$

For $Y \in \mathfrak{g}$ it follows that

$$C_{\exp X}(\exp Y) = \exp(\mathrm{Ad}_{\exp X} Y). \tag{2}$$

Similar to (2) it holds with Theorem 17 that

$$\mathrm{Ad}_{\exp X} = \exp(\mathrm{Ad}_{*,e}(X)) = \exp(\mathrm{ad}(X))$$

if you consider the following commutating diagram:

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\mathrm{ad}"] \arrow[d, "\exp"] & \mathrm{Lie}(\mathrm{GL}(\mathfrak{g})) \arrow[d, "\exp"] \\ G \arrow[r, "\mathrm{Ad}"'] & \mathrm{GL}(\mathfrak{g}). \end{tikzcd}$$

This gives us:

$$\begin{aligned} (\exp X)(\exp Y)(\exp(-X)) &= C_{\exp X}(\exp Y) = \exp(\mathrm{Ad}_{\exp X} Y) \\ &= (\exp(\mathrm{ad}_X))Y = \sum_{k=0}^{\infty} \frac{1}{k!} (\mathrm{ad}_X)^k Y. \end{aligned}$$

Suppose that $\mathfrak{h}$ is an ideal and $Y \in \mathfrak{h}$. It holds that $\mathrm{ad}_X Y = [X, Y] \in \mathfrak{h}$ and therefore $(\mathrm{ad}_X)^k Y \in \mathfrak{h}$ for all $k \in \mathbb{N}$ by induction. All in all it follows:
$(\exp X)(\exp Y)(\exp(-X)) \in \exp \mathfrak{h} \subset H$. With Lemma 18 this gives us that H is normal.

Now suppose H is normal. For all $X \in \mathfrak{g}, Y \in \mathfrak{h}$ and $t \in \mathbb{R}$ it holds that

$$\begin{aligned}\exp t(\mathrm{Ad}_{\exp X}(Y)) &= \exp(\mathrm{Ad}_{\exp X}(tY)) = C_{\exp X}(\exp(tY)) \\ &= (\exp X)(\exp tY)(\exp(-X)) \in H.\end{aligned}$$

Therefore we know from the last talk that $\mathrm{Ad}_{\exp X}(Y) \in \mathfrak{h}$ for all $X \in \mathfrak{g}$ and $Y \in \mathfrak{h}$. Consider the smooth curve $\gamma : \mathbb{R} \to \mathfrak{g}$ with $t \mapsto \mathrm{Ad}_{\exp tX}(Y) \in \mathfrak{h}$.
Finally we want to show that $[X, Y] \in \mathfrak{h}$ for all $X \in \mathfrak{g}$ and $Y \in \mathfrak{h}$ to see that $\mathfrak{h}$ is an ideal in $\mathfrak{g}$:

$$\begin{aligned}[X, Y] = \mathrm{ad}_X Y &= \left(\frac{d}{dt}\bigg|_{t=0} (\exp t(\mathrm{ad}_X))\right) Y \\ &= \frac{d}{dt}\bigg|_{t=0} (\exp(\mathrm{ad}_{tX}))Y \\ &= \frac{d}{dt}\bigg|_{t=0} \mathrm{Ad}_{\exp tX}(Y) = \gamma'(0) \in \mathfrak{h}.\end{aligned}$$

□