Symmetric spaces

Seminar: Lie groups

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1 Definition and Properties

Definition 1.1. *A connected Riemannian manifold M is called* ***symmetric****, if for every $p \in M$ there exists an isometry $\sigma_p \colon M \to M$ such that σ_p is involutive, i.e. $\sigma_p^2 = id$, and p is an isolated fixed point of σ_p, i.e. there is a neighbourhood U of p in which p is the only fixed point of σ_p.*

As we will see soon, symmetric spaces are always homogeneous spaces. Now let us have a look at some examples.

Example 1.2. One example of a non-compact symmetric space is the euclidean space $\mathbb{R}^n$ where for any point p the isometry σ_p is just the reflection in p.
An example for a compact symmetric space is given by the n-dimensional sphere $\mathbb{S}^n$ with $\sigma_p(q) = q'$, where q and q' are equidistant points from p, lying on the geodesic through p and q. Note that if p^* is the point antipodal to p, then $\sigma_p(p^*) = p^*$. This shows that globally, p is not the only fixed point of σ_p.

Reminder. For a Riemannian manifold M, the **exponential map** $\operatorname{Exp}\colon D \subset TM \to M$ is given by

$$\operatorname{Exp}(v) = c_v(1),$$

where c_v is the unique geodesic with $c_v(0) = p$ and $\dot{c}_v(0) = v$, for $v \in T_pM$. For $p \in M$, there is a star-shaped neighbourhood $U \subset T_pM$ of the zero-vector 0_p and a neighbourhood $V \subset M$ of p, such that $\operatorname{Exp}_p \colon U \to V$ is a diffeomorphism.
If $\varphi \in \operatorname{Isom}(M, g)$, then φ maps geodesics to geodesics and hence for $v \in T_pM$

$$\varphi(\operatorname{Exp}(v)) = \varphi(c_v(1)) = \operatorname{Exp}(d\varphi_p(v)),$$

since $\varphi \circ c_v$ is the unique geodesic with $\varphi \circ c_v(0) = \varphi(p)$ and $(\varphi \dot{\circ} c_v)(0) = d\varphi_p(v)$. As a result, $\varphi \circ \operatorname{Exp} = \operatorname{Exp} \circ d\varphi$.

Lemma 1.3. *Let M be a Riemannian manifold, $p \in M$ and σ_p an involutive isometry with p as isolated fixed point. Then $d\sigma_p|_p(X_p) = -X_p$ and $\sigma_p(\operatorname{Exp}(X_p)) = \operatorname{Exp}(-X_p)$ for all $X_p \in T_pM$.*

Proof. In the following, we write $d\sigma_p$ instead of $d\sigma_p|_p$, always referring to the homomorphism $d\sigma_p|_p\colon T_pM \to T_pM$. Since $\sigma_p^2 = id_M$, it follows that $d\sigma_p$ has the property $d\sigma_p^2 = id_{T_pM}$, and hence the eigenvalues of $d\sigma_p$ have to be ± 1. Assume that $+1$ is an eigenvalue. Then there is $X_p \in T_pM, X_p \neq 0$, satisfying $d\sigma_p(X_p) = X_p$. Since σ_p is an isometry, we have

$$\sigma_p(\mathrm{Exp}(tX_p)) = \mathrm{Exp}(d\sigma_p(tX_p)) = \mathrm{Exp}(tX_p).$$

As a result, the geodesic through p with initial direction X_p (given by $\mathrm{Exp}(tX_p)$) is fixed pointwise by σ_p, so p is not an isolated fixed point of σ_p - a contradiction.
Now the only eigenvalue of $d\sigma_p$ is -1, i.e. for all $X_p \in T_pM$, $d\sigma_p(X_p) = -X_p$, and $\sigma_p(\mathrm{Exp}(X_p)) = \mathrm{Exp}(d\sigma_p(X_p)) = \mathrm{Exp}(-X_p)$. In summary, σ_p takes each geodesic through p onto itself with directions reversed. □

To prove the following corollary, we need a more general statement on Riemannian manifolds.

Lemma 1.4. *Let M be a complete connected Riemannian manifold, $F_1, F_2\colon M \to M$ isometries satisfying $F_1(p) = F_2(p)$ and $dF_1 = dF_2$ on T_pM for some $p \in M$. Then $F_1 = F_2$.*

Proof. See [Boo86, p.350] □

Corollary 1.5. *Let M be a complete connected Riemannian manifold, $p \in M$. Then there is at most one involutive isometry σ_p with p as isolated fixed point.*

Proof. Let σ_p, σ'_p be two involutive isometries with p as isolated fixed point. Then by Lemma 1.3 $d\sigma_p = d\sigma'_p$ on T_pM and hence $\sigma_p = \sigma'_p$ by Lemma 1.4. □

Theorem 1.6. *A symmetric Riemannian manifold M is necessarily complete. Further, for $q_1, q_2 \in M$, there is $r \in M$ such that $\sigma_r(q_1) = q_2$.*

Proof. We have to show that every geodesic segment can be extended to infitite length. So let $c(s)$ be a geodesic ray defined on the open intervall $(0, b)$ and let $s_0 := \frac{3}{4}b$. The isometry $\sigma_{c(s_0)}$ in $c(s_0)$ takes any geodesic through $c(s_0)$ to itself with initial direction reversed; in particular, it takes c to the geodesic through $c(s_0)$ with tangent vector $-\left(\frac{\mathrm{d}c}{\mathrm{d}s}\right)_{s_0}$ at $c(s_0)$. This geodesic coincides with c on $\frac{1}{2}b < s < b$ and extends it to length $< \frac{3}{2}b$. This proves the first statement. If we take r to be the midpoint of the geodesic from q_1 to q_2, it follows that $\sigma_r(q_1) = q_2$, since σ_r carries the geodesic onto itself, preserving distances between points on it. □

The isometries of a Riemannian manifold form a group $\mathrm{Isom}(M)$, that is a subgroup of the group of all diffeomorphisms on M. It is a Lie group due to a result from Myers and Steenrod (see [Hel78, Ch.IV.3]), and according to Th. 1.6, its action is transitive, if M is a symmetric space. Hence, any symmetric space is also homogeneous.

2 Lie Groups as Symmetric Spaces

Reminder. A Riemannian metric ϕ on a Lie group G is called **bi-invariant**, if it is invariant under both left- and right-translation, i.e. $L_g^*\phi_{gp} = \phi_p = R_g^*\phi_{pg}$ for all $g, p \in G$.

Theorem 2.1. *Let G be a compact connected Lie Group with a bi-invariant metric. Then G is a symmetric space with respect to this metric.*

Proof. First, we remark that the assumption that G is compact and connected is necessary to guarantee the existence of a bi-invariant metric on G. Let $\psi\colon G \to G$ be the inversion, i.e. $\psi(x) = x^{-1}$ for all $x \in G$. Clearly, ψ is involutive. We claim that it is an isometry of G with the identity element e of G as isolated fixed point. To prove this, recall that for any $X_e \in T_eG$, there is a unique one-parameter subgroup, i.e. a group homomorphism $g\colon (\mathbb{R}, +) \to G,\ t \mapsto g(t)$ satisfying $\dot{g}(0) = X_e$. Applying ψ, this gives us

$$\psi(g(t)) = (g(t))^{-1} = g(t^{-1}) = g(-t),$$

and for the differential we obtain (using the chain rule)

$$d\psi_e(X_e) = d\psi_e(\dot{g}(0)) = d\psi_e(dg_0) = d\,(\psi \circ g)_0 = \frac{\partial}{\partial t}\,(\psi(g(t))) = -\dot{g}(0) = -X_e.$$

Hence $d\psi_e = -I$, so it preserves any inner product on T_eG.
Let now $g \in G$ be arbitrary, and L_g, R_g denote the left and right translations by g. Then for all $x \in G$

$$\psi(x) = x^{-1} = (g^{-1}x)^{-1}g^{-1} = \left(R_{g^{-1}} \circ \psi \circ L_{g^{-1}}\right)(x),$$

which gives us for the differential $d\psi_g\colon T_g(G) \to T_{g^{-1}}G$

$$d\psi_g = d(R_{g^{-1}})_e \circ d\psi_e \circ d(L_{g^{-1}})_g.$$

Since we consider a bi-invariant metric on G, $L_{g^{-1}}$ and $R_{g^{-1}}$ induce isometries between the tangent spaces. As $d\psi_e$ is an isometry on T_eG, also $d\psi_g$ is an isometry for any g in G as composition of isometries. So indeed, the inversion ψ is an isometry of the Lie group G. To see that is has the identity element e as an isolated fixed point, we consider a neighbourhood N of e, that is the diffeomorphic image under Exp_p of some star-shaped neighbourhood of $0_p \in T_pM$. Using the fact that ψ is an isometry and that $d\psi_e(X_e) = -X_e$, we obtain $\psi(\mathrm{Exp}(X_e)) = \mathrm{Exp}(d\psi_e(X_e)) = \mathrm{Exp}(-X_e)$. In the neighbourhood N, we can identify $\mathrm{Exp}(X_e)$ uniquely with $X_e \in T_eG$, so in local coordinates, ψ is just a reflection in the origin and therefore has e as an isolated fixed point.
Let now $g \in G$ be arbitrary and define $\sigma_g\colon G \to G$ by $\sigma_g = L_g \circ R_g \circ \psi$, so $\sigma_g(x) = gx^{-1}g$ for all $x \in G$. As a composition of isometries, σ_g itself is an isometry. Further, it is involutive and has g as isolated fixed point. This completes the proof that G is a symmetric space. □

Now that we familiarized ourselves with the notion of a symmetric space, let us consider a concrete example. For that, we need a result guaranteeing the existence of a bi-invariant Riemannian metric on a Lie group G:

Lemma 2.2. *Let ψ_e be an inner product on T_eG. Then ψ_e determines a bi-invariant Riemannian metric on G if and only if $d(Ad(g))\psi_e = \psi_e$ for all $g \in G$, where $Ad(g)\colon T_eG \to T_eG$ is the homomorphism induced by the conjugation map $h \mapsto ghg^{-1}$ (cf. Talk 4 on the Lie Correspondence).*

For the proof, see [Boo86, p.246].

Example 2.3. Let $G = SO(n,\mathbb{R})$. Our aim is to show that G indeed is a symmetric space. We start with constructing a bi-invariant Riemannian metric on G and want to use the lemma stated above. Therefore, we have to find an inner product on T_eG that is invariant under $Ad(B)$ for all $B \in G$. We know that T_eG can be identified with the group of all skew-symmetric matrices $\{A \in GL(n,\mathbb{R})|\ A = -A^T)\}$ in the sense that $X_e = \sum_{i,j} a_{ij}\frac{\partial}{\partial x_{ij}}$ with $A = (a_{ij})$ skew-symmetric, is tangent to the identity element $e = I$ of $SO(n,\mathbb{R})$ considered as submanifold of $GL(n,\mathbb{R})$. The unique one-parameter subgroup corresponding to the left-invariant vectorfield with value X_e in e is of the form $Z(t) = e^{tA}$ (it is easy to see that Z is a group homomorphism and that $\dot{Z}(0) = A$). For $B \in G = SO(n,\mathbb{R})$, $Ad(B)\colon T_eG \to T_eG$ is induced by the map $Z \mapsto BZB^{-1}$. Since X_e corresponds to the one-parameter subgroup e^{tA}, $Ad(B)(X_e)$ corresponds to $Be^{tA}B^{-1} = e^{tBAB^{-1}}$, where the last equality can easily be verified by the definition of e^{tA}. Hence, $Ad(B)(X_e)$ has component matrix BAB^{-1}.
We now define an inner product on T_eG. For $X_e = \sum_{i,j} a_{ij}\frac{\partial}{\partial x_{ij}}$ and $Y_e = \sum_{i,j} c_{ij}\frac{\partial}{\partial x_{ij}}$, we set

$$\langle X_e, Y_e\rangle := tr(A^TC) = \sum_{i,j} a_{ij}c_{ij}$$

which is clearly bilinear, symmetric and positive definit since $\langle X_e, X_e\rangle = \sum_{i,j} a_{ij}^2 \geq 0$. Now for $B \in SO(n,\mathbb{R})$, we have

$$\langle Ad(B)(X_e), Ad(B)(Y_e)\rangle = tr((BAB^{-1})^T(BCB^{-1})) = tr(BA^TCB^{-1}) = tr(A^TC) = \langle X_e, Y_e\rangle.$$

So the inner product $\langle\cdot,\cdot\rangle$ is invariant under $Ad(B)$. By Lemma 2.2, it determines a bi-invariant Riemannian metric on G, which by Theorem 2.1 makes G a symmetric space.

Our next goal is to show that the geodesics through the identity element e with initial direction $X_e \in T_eG$ are exactly the one-parameter subgroups determined by X_e. Let M be a symmetric Riemannian manifold and $p(t)$, $-\infty < t < \infty$ a geodesic on M. As we have seen, for any t, $\sigma_{p(t)}$ maps p onto itself and reverses its sense. Hence for $c \in \mathbb{R}$ fixed

$$\tau_c := \sigma_{p(c)} \circ \sigma_{p(\frac{c}{2})}$$

is an isometry that maps p onto itself and preserves its sense, so $\tau_c(p(t)) = p(t + const)$. As $\tau_c(p(0)) = \sigma_{p(c)}(p(c)) = p(c)$, we obtain $\tau_c(p(t)) = p(t + c)$ for all t. This gives us the first part of the follwing theorem:

Theorem 2.4. *Let M, p and τ_c be as above. Then $\tau_c(p(t)) = p(t+c)$ and for $X_{p(0)} \in T_{p(0)}M$, the vectorfield*

$$X_{p(t)} := d\tau_t X_{p(0)}$$

is the associated parallel vectorfield along $p(t)$, i. e. $d\tau_t \colon T_{p(0)}M \to T_{p(t)}M$ is the parallel translation along the geodesic p.

Proof. The first statement was already shown above. For the second, define $X_{p(t)}$ as in the statement and let $\tilde{X}_{p(t)}$ denote the unique parallel vectorfield along p with $\tilde{X}_{p(0)} = X_{p(0)}$. We claim that the vectorfields X and $\tilde{X}$ coincide. For any $t_0 \in \mathbb{R}$, $d\sigma_{p(t_0)}\tilde{X}_{p(t)}$ is parallel along p, since $\sigma_{p(t_0)}$ is an isometry. By Lemma 1.3, $d\sigma_{p(t_0)}\tilde{X}_{p(t_0)} = -\tilde{X}_{p(t_0)}$. Hence both $-\tilde{X}_{p(t_0)}$ and $d\sigma_{p(t_0)}\tilde{X}_{p(t_0)}$ are parallel vectorfields along p and agree at the point $p(t_0)$. By uniqueness, this vectorfields are equal. This gives us

$$\begin{aligned} d\tau_c(\tilde{X}_{p(t)}) &= d(\sigma_{p(c)} \circ \sigma_{p(\frac{c}{2})})(\tilde{X}_{p(t)}) \\ &= d\sigma_{p(c)}\big|_{\sigma_p(\frac{c}{2})(p(t))} \left(d\sigma_{p(\frac{c}{2})}\tilde{X}_{p(t)}\right) \\ &= d\sigma_{p(c)}\big|_{\sigma_p(\frac{c}{2})(p(t))} \left(-\tilde{X}_{\sigma_p(\frac{c}{2})(p(t))}\right) \\ &= \tilde{X}_{\tau_c(p(t))} = \tilde{X}_{p(t+c)}. \end{aligned}$$

In particular, for $t = 0$ and $c = t$, this leads to

$$X_{p(t)} = d\tau_t X_{p(0)} = d\tau_t \tilde{X}_{p(0)} = \tilde{X}_{p(t)},$$

so the two vectorfields coincide everywhere, i.e. $d\tau_t$ is the parallel translation along p. ☐

Remark. If $p_1 = p(c_1)$ and $p_2 = c(p_2)$ are two points on the geodesic, than just as above, $\sigma_{p_2} \circ \sigma_{p_1}(p(t)) = p(t + 2(c_2 - c_1))$ and $d(\sigma_{p_2} \circ \sigma_{p_2})$ maps any parallel vectorfield along p to another parallel vectorfield.

Now we are ready to prove the result mentioned above on geodesics and one-parameter subgroups.

Theorem 2.5. *Let $M = G$ be a compact connected Lie group with bi-invariant metric and $X_e \in T_eG$. Then the unique geodesic $p(t)$ with $p(0) = e$ and $\dot{p}(0) = X_e$ is precisely the one-parameter subgroup determined by X_e. All other geodesics are left (or right) cosets of these one-parameter subgroups.*

Proof. Let p be as above. We have to show that p is a group homomorphism from $(\mathbb{R}, +)$ to G. As we have just seen, the isometry $\sigma_{p(s)} \circ \sigma_{p(0)}$ maps p onto itself with $p(t) \mapsto p(t + 2s)$. Remember that $\sigma_p(x) = px^{-1}p$ (see Th. 2.1) and $p(0) = e$. It follows that

$$p(t + 2s) = \sigma_{p(s)} \circ \sigma_{p(0)}(p(t)) = \sigma_{p(s)}(p(0)p(t)^{-1}p(0)) = p(s)p(t)p(s)$$

and inductively, using various t, we find $p(s)^n = p(ns)$. In particular, for $a, b, c, c \in \mathbb{Z}$ with $bd \neq 0$ we have

$$p\left(\frac{a}{b} + \frac{c}{d}\right) = p\left(\frac{ad+bc}{bd}\right) = p\left(\frac{1}{bd}\right)^{ad+bc} = p\left(\frac{1}{bd}\right)^{ad} p\left(\frac{1}{bd}\right)^{bc} = p\left(\frac{a}{b}\right) p\left(\frac{c}{d}\right).$$

By continuity, $p(t+s) = p(t)p(s)$ for all $s, t \in \mathbb{R}$, and hence any geodesic with $p(0) = e$ is a one-parameter subgroup. Since there exists a unique geodesic and a unique one-parameter subgroup with $\dot{p}(0) = X_e$, we have proven the first claim. The second claim follows, sincet left and right translations are isometries with respect to the bi-invariant metric and therefor preserve geodesics. A geodesic through $g \in G$ is uniquely determined by its tangent vector at g, including parametrization. We can find a translation mapping this geodesic to one through e, corresponding to a one-parameter subgroup. Hence the original geodesic is the image of this under a translation and therefor a left (or right) coset of the one-parameter subgroup. □

Corollary 2.6. *Let G be a compact connected Lie group. Then any $g \in G$ lies on a one-parameter subgroup.*

Proof. According to Th.2.1, G is a Riemannian manifold with respect to the bi-invariant metric, and by Th.1.6 it is complete. Hence we can join any $g \in G$ to e by a geodesic segment, that is a one-parameter subgroup by Th.2.5. □

Let us briefly get back to $SO(n, \mathbb{R})$ considered in Example 2.3.

Example 2.7. In $G = SO(n, \mathbb{R})$ the geodesics relative to the bi-invariant metric constructed in Example 2.3, are exactly the one-parameter subgroups, so the curves $p(t) = e^{tA}$ with A being any skew-symmetric matrix, and their cosets.

3 From homogeneous to symmetric spaces

In the first part of the talk, we found out that any symmetric space is a homogeneous space for its group of isometries. We now want to examine under what circumstances we can be sure that a given homogenous space is symmetric.

Theorem 3.1. *Let G be a Lie group acting transitively on a manifold M. If the stabilizer H of a point $p \in M$ is a connected compact Lie subgroup of G, then M admits a Riemannian metric such that the transformation determined by each element of G is an isometry.*

Proof. Let $p \in M$ fix and H be as above. We denote the action of G on M by $\theta\colon G \times M \to M$, and by $\theta_g\colon M \to M$ the diffeomorphism given by $\theta_g(q) = \theta(g, q)$ for $q \in M$. For $g \in H$, θ_g induces a linear mapping $d\theta_g\colon T_pM \to T_pM$. Since $\theta_{g_1} \circ \theta_{g_2} = \theta_{g_1 g_2}$, we have $d\theta_{g_1} \circ d\theta_{g_2} = d\theta_{g_1 g_2}$ and hence the map $\psi\colon g \mapsto d\theta_g$ from H into the group of linear transformations $Aut(T_pM)$ of T_pM is a homomorphism. Since θ is C^∞, so is ψ (this can be seen using a basis of T_pM). Hence, ψ is a smooth group homomorphism from H to $Aut(T_pM)$, i.e. a representation of H on T_pM.

By Theorem VI 3.9 [Boo86], this guarantees the existence of an inner product on T_pM, denoted by ϕ_p, that is H-invariant. Note that for applying this theorem, we need that H is connected and compact.
Since G acts transitive by assumption, for any $q \in M$ there is $g \in G$ with $\theta_g(q) = p$. Define

$$\phi_q(X_q, Y_q) := \theta_g^* \phi_p(X_q, Y_q) = \phi_p(d\theta_g X_q, d\theta_g Y_q) \quad \forall X_q, Y_q \in T_qM.$$

We claim that this gives us a Riemannian metric ϕ on M with respect to which each θ_g is an isometry of M. First, ϕ_q is well-defined: Let $g_1, g_2 \in G$ with $\theta_{g_1}(q) = p = \theta_{g_2}(q)$. Then $g_2^{-1}g_1 \in H$ and hence $\theta_{g_2^{-1}g_1}\phi_p = \phi_p$, so

$$\theta_{g_2}^*\phi_p = \theta_{g_2}^*\theta_{g_2^{-1}g_1}^*\phi_p = \theta_{g_2}^*\theta_{g_2^{-1}}^*\theta_{g_1}^*\phi_p = \theta_{g_1}^*\phi_p.$$

Second, for any $q \in M$, ϕ_q is positive definit, since ϕ_p is positive definite and θ_g is a diffeomorphism. Further, one checks that ϕ is G-invariant on M. Let $g \in G$, $q_1, q_2 \in M$ with $\theta_g(q_1) = q_2$ and $g_1, g_2 \in G$ with $\theta_{g_i}(q_i) = p$ for $i = 1, 2$. It holds that $g_2 g g_1^{-1}$ fixes p, so this in an element h in H. Since ϕ_p is H-invariant, $\phi_p(d\theta_h(X_p), d\theta_h(Y_p)) = \phi_p(X_p, Y_p)$ for all X_p, Y_p in T_pM, and in particular $\phi_p(d\theta_{g_2 g}X_{q_1}, d\theta_{g_2 g}Y_{q_1}) = \phi_p(d\theta_{g_1}(X_{q_1}), d\theta_{g_1}(Y_{q_1}))$ for $X_{q_1}, Y_{q_1} \in T_{q_1}M$. As a result, we find

$$\begin{aligned}
\theta_g^*\phi_{q_2}(X_{q_1}, Y_{q_1}) = & \quad \theta_g^*\theta_{g_2}^*\phi_p(X_{q_1}, Y_{q_1}) = \theta_{g_2 g}^*\phi_p(X_{q_1}, Y_{q_1}) = \phi_p(d\theta_{g_2 g}X_{q_1}, d\theta_{g_2 g}Y_{q_1}) \\
= & \quad \phi_p(d\theta_{g_1}(X_{q_1}), d\theta_{g_1}(Y_{q_1})) = \theta_{g_1}^*\phi_{q_2}(X_{q_1}, Y_{q_1}) = \phi_{q_1}(X_{q_1}, Y_{q_1}),
\end{aligned}$$

so ϕ is G-invariant on M. Lastly, we need to show that for vector fields X, Y on M the map

$$M \to \mathbb{R}, \quad q \mapsto \phi_q(X_q, Y_q) = \phi_p(d\theta_g X_q, d\theta_g Y_q) \quad \text{with } g \in G \text{ s.t. } \theta(g, q) = g$$

is smooth. For that, we have to show that the assignment $q \mapsto d\theta_g X_q$ is smooth, which is not clear at first sight, since there are many different choices for g. To fix this, we consider the natural identification of M with the quotient space:

$$F\colon\ G/H \to M, \quad gH \mapsto \theta(g, p),$$

which is a diffeomorphism and commutes with left-translation on G/H (see Talk on Homogeneous Spaces, Theorem 5). For $gH \in G/H$, there is a C^∞-section $S\colon V \to G$ defined on a neighbourhood V of gH such that $\pi \circ S = id_V$, where π denotes the projection of G onto G/H. Since F is a diffeomorphism, $\tilde{S} := S \circ F^{-1}\colon M \to G$ is a C^∞-section into M. The following diagramm illustrates the situation.

$$\begin{tikzcd}
G \arrow[d, shift left, "\pi"] & \\
G/H \arrow[u, shift left, "S"] \arrow[r, "F"] & M \arrow[ul, "\tilde{S}=S\circ F^{-1}"']
\end{tikzcd}$$

Note that the sections S and $\tilde{S}$ are not defined on the whole of ${}^{G}/_{H}$ and M respectively, but only on some neighbourhoods. For $q \in M$, let $q_1 \in {}^{G}/_{H}$ be the preimage of q under F, i.e. $F(q_1) = q$. Then the map $\tilde{S}$ satisfies

$$\theta(\tilde{S}(q), p) = \theta(S \circ F^{-1}(F(q_1)), p) = \theta(S(q_1), p) = F(q_1) = q,$$

by the definition of F. Now $\tilde{S}$ is smooth, so also $d\theta_{\tilde{S}(q)}$ depends smoothly on q. Hence by well-definedness of ϕ_q, we can write $\phi_q(X_q, Y_q) = \phi_p(d\theta_{\tilde{S}(q)} X_q, d\theta_{\tilde{S}(q)} Y_q)$, which depends smoothly on q. This concludes the proof that ϕ is a Riemannian metric with respect to which each θ_g is an isometry of M. □

In the following, we always suppose that H is compact and connected and that the action of G on M is faithful, i.e. if $\theta(g, m) = m$ for all $m \in M$, then $g = e$.

Theorem 3.2. *Let G, H, p and M be as above and let $\alpha\colon G \to G$ be an involutive automorphism of G with fixed set H. Then*

$$\tilde{\alpha}\left(\theta(g, p)\right) := \theta(\alpha(g), p)$$

defines an involutive isometry of M onto M with p as isolated fixed point.

Proof. We start with showing that $\tilde{\alpha}$ defines a mapping of M onto itselt. For $q \in M$ arbitrary, there is $g \in G$ with $\theta(g, p) = q$ by transitivity. If g' is another element of G satisfying $\theta(g', p) = q$, then the element $h := g^{-1}g'$ lies in the stabilizer H of p, which by assumption is also the fixed set of α. This gives us $g' = gh$ and $\alpha(g') = \alpha(g)\alpha(h) = \alpha(g)h$, so we find

$$\tilde{\alpha}(\theta(g', p)) = \theta(\alpha(g)h, p) = \theta(\alpha(g), \theta(h, p)) = \theta(\alpha(g), p).$$

As a result, $\tilde{\alpha}$ is well-defined. Further, since

$$\tilde{\alpha}^2(\theta(g, p)) = \tilde{\alpha}(\theta(\alpha(g), p)) = \theta(\alpha^2(g), p) = \theta(g, p)$$

for all $g \in G$, $\tilde{\alpha}^2 = id_M$, so $\tilde{\alpha}$ is involutive. It follows that $\tilde{\alpha}$ is onto.
Now let us assume we already showed that $\tilde{\alpha}$ is C^∞, has p as isolated fixed point and that $d\tilde{\alpha}\colon T_pM \to T_pM$ equals $-I$, i.e. $d\tilde{\alpha}(X_p) = -X_p$ for all $X_p \in T_pM$. Then $d\tilde{\alpha}$ preserves the inner product ϕ_p on T_pM:

$$\phi_p(d\tilde{\alpha}(X_p), d\tilde{\alpha}(Y_p)) = \phi_p(-X_p, -Y_p) = \phi_p(X_p, Y_p) \quad \forall X_p, Y_p \in T_pM.$$

Further, for $q \in M$, $q \neq p$, we choose $g \in G$ with $\theta(g, p) = q$ (which is possible since the action is transitive). Then

$$\tilde{\alpha}(q) = \tilde{\alpha}(\theta(g, p)) = \theta(\alpha(g), p) = \theta_{\alpha(g)}\left(\theta_{g^{-1}}(q)\right),$$

so $\tilde{\alpha} = \theta_{\alpha(g)} \circ \theta_{g^{-1}}$. Hence $d\tilde{\alpha}_q = d\theta_{\alpha(g)} \circ d\theta_{g^{-1}}$, which are both isometries of tangent spaces. It follows that $d\tilde{\alpha}_q$ is an isometry. So once we checked the properties assumed, we know that $\tilde{\alpha}$

is an isometry of M. The smoothness of $\tilde{\alpha}$ can be proven using the natural identification of M with ${}^{G}\!/_{H}$, quite similar to the method we used in the proof of the previous theorem. To show that the p is an isolated fixed point, we make use of the fact that the exponential map is a diffeomorphism on some neighbourhood of p. For the details, see [Boo86, p.359f]. $\square$

Corollary 3.3. *Under the assumptions of Theorem 3.2, M is a symmetric space with involutive isometries $\sigma_p = \tilde{\alpha}$ and $\sigma_q = \theta_g \circ \tilde{\alpha} \circ \theta_{g^{-1}}$, where $q = \theta(g, p)$.*

The proof of this Corollary is immediate, since θ_g is an isometry.

Now we can consider more complicated examples of symmetric manifolds.

Example 3.4. Let $M \subset \mathrm{Mat}(n \times n, \mathbb{R})$ be the collection of all symmetric, positive definit real matrices of determinant 1, and $G = SL(n, \mathbb{R})$ the $n \times n$-matrices of determinant 1. Then G acts on M by $\theta(g, s) = gsg^T$. As base point p, we fix the $n \times n$-identity matrix $I \in M$. The stabilizer H of I is given by

$$H = \{g \in SL(n, \mathbb{R}) \mid gg^T = I\} = SO(n, \mathbb{R}),$$

i.e. the orthogonal $n \times n$-matrices. Note that this is a compact connected subgroup of G. We want to apply Theorem 3.2. An involutive automorphism of G is given by $\alpha(g) = (g^{-1})^T$. We have

$$\alpha(g) = g \iff g^{-1}(g^{-1})^T = I \iff g \in SO(n, \mathbb{R}),$$

so $SO(n, \mathbb{R})$ is also the fixed point set of α. Since the action of G on M is faithful, it remains to show transitivity. This immediately follows from the fact that any positive definite symmetric matrix $q \in M$ may be written as $q = gg^{-1} = gIg^{-1}$ for some $g \in SL(n, \mathbb{R})$. By Theorem 3.2 (and its corollary), M is a symmetric space with respect to an $SL(n, \mathbb{R})$-invariant metric. By what we know about homogeneous spaces, we can identify

$$M \cong {}^{SL(n, \mathbb{R})}\!\big/_{SO(n, \mathbb{R})}.$$

Let us have a closer look at the isometries of M, in particular at the isometry at $p = I$, which gives rise to all the other isometries. With notations as before, $\tilde{\alpha} \colon M \to M$ is given by $\tilde{\alpha}(\theta(g, p)) = (g^{-1})^T p g^{-1}$. Now $q \in M$ can be written as gIg^T with $g \in SL(n, \mathbb{R})$, so

$$\tilde{\alpha}(q) = \tilde{\alpha}(\theta(g, I)) = (g^{-1})^T g^{-1} = (gg^T)^{-1} = q^{-1},$$

so $\tilde{\alpha}$ takes any element in M to its inverse. Hence, the only fixed point of $\tilde{\alpha}$ indeed is the identity I.

A special case of this example is the following.

Example 3.5. Let $\mathbb{H} = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$ be the upper half plane in $\mathbb{R}^2$. If we identify $\mathbb{R}^2$ with $\mathbb{C}$, then $SL(2, \mathbb{R})$ acts on $\mathbb{H}$ by Moebiustransformations:

$$\theta \colon SL(2, \mathbb{R}) \times \mathbb{H} \to \mathbb{H}, \quad \theta(M, z) = \frac{az + b}{cz + d} \quad \text{with } M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(n, \mathbb{R}).$$

It is easy to verify that this action is well-defined, transitive and faithful. A Riemannian metric on $\mathbb{H}$ is given by

$$g(x,y) = \begin{pmatrix} \frac{1}{y^2} & 0 \\ 0 & \frac{1}{y^2} \end{pmatrix}.$$

One can show that this is invariant under the action of $SL(2,\mathbb{R})$. Hence, $SL(2,\mathbb{R})$ acts on M as group of isometries. Further, the stabilizer of the point $p = i$ is given by $SO(2,\mathbb{R})$, so we have

$$\mathbb{H} \cong SL(2,\mathbb{R}) \big/ SO(2,\mathbb{R}) .$$

A further example is the Grassman manifold $G(k,n)$ consisting of k-planes through the origin in $\mathbb{R}^n$. Details of this can be found in [Boo86, p.362f].

References

[Boo86] William M. Boothby. *An introduction to differentiable manifolds and Riemannian geometry*. Academic Press, 1986.

[Hel78] Sigurdur Helgason. *Differential Geometry, Lie Groups and Symmetric Spaces*. Academic Press, 1978.