The Farey Tessellation

Seminar: Geometric Structures on manifolds

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Introduction

In this paper, we are going to introduce the Farey tessellation. Since it is closely related to the once-punctured torus, we will start with the construction of a hyperbolic once-punctured torus. Then, we will get to know tessellations. Intuitively, a tessellation is some special cover, that is similar to the tiling of a kitchen floor. The Tessellation theorem gives a connection between the once-punctured torus and a special tessellation of the hyperbolic plane $\mathbb{H}^2$. Subsequently, we get to know the Farey circle packing, a collection of circles in $\mathbb{H}^2$. From this circle packing, we construct the so-called Farey Tessellation and show that it is indeed a tessellation of $\mathbb{H}^2$. Finally, we investigate how we can vary certain parameters in the construction of this tessellation, such that the result again is a tessellation.

1. The once-punctured torus

The euclidean torus is obtained by gluing the sides of a square, usually the unit square. If we now consider the one-punctured torus, we just remove one point. Without loss of generality, this point is the point corresponding to the four vertices glued together. However, this surface, equipped with the euclidean metric, is not complete, since we can find a Cauchy-sequence that does not converge.

Fortunately, this is not the only way to construct the once-punctured torus. In this section, we will glue the sides of a hyperbolic square to obtain a surface equipped with a hyperbolic metric that is (homeomorphic to) the once-punctured torus. We will later show that it is complete.

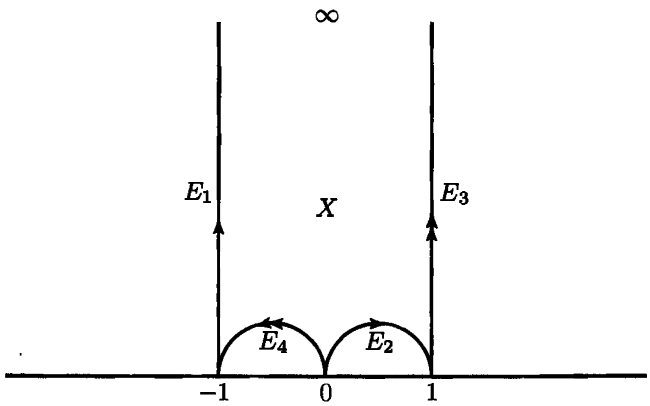

Figure 1: The sides of the ideal rectangle X with vertices $-1, 0, 1$ and ∞ can be glued to obtain a once-punctured torus.

Definition 1.1. *A vertex of a hyperbolic polygon is an* ***ideal vertex****, if it is at infinity of $\mathbb{H}^2$, i.e. if it lies in $\mathbb{R} \cup \{\infty\}$. A hyperbolic polygon is an* ***ideal polygon****, if all its vertices are ideal.*

We consider the ideal polygon X in $\mathbb{H}^2$ with vertices at $-1, 0, 1$ and ∞. We label the edges with $E_1, \ldots, E_4$ and orient them as indicated in Figure 1. We now want to glue opposite edges of X using hyperbolic isometries. There are various ways to do so, we decide to choose the following gluing isometries:

$$\varphi_1 \colon E_1 \to E_2,\ \varphi(z) = \frac{z+1}{z+2}; \qquad \varphi_3 \colon E_3 \to E_4,\ \varphi(z) = \frac{z-1}{-z+2}. \tag{1.1}$$

Then φ_1 sends -1 to 0 and ∞ to 1, φ_3 sends 1 to 0 and ∞ to -1. We further set $\varphi_2 := \varphi_1^{-1}$ and $\varphi_4 := \varphi_3^{-1}$.
We denote by d_X the hyperbolic metric on the polygon X:

$$d_X(p, q) := \inf\{l_{\text{hyp}}(c) \mid c \text{ curve from } p \text{ to } q \text{ in } X\}.$$

Since X is convex, this is just the restriction of the hyperbolic metric d_{hyp}. Now the quotient metric space $(\bar{X}, \bar{d}_X)$ obtained from the metric space (X, d_X) by performing the edge gluings is a hyperbolic surface, and it is homeomorphic to the once-punctured torus.

Remark. The metric $\bar{d}_X$ on the quotient space is defined as follows: Let $\bar{P}, \bar{Q} \in \bar{X}$. A discrete walk from $\bar{P}$ to $\bar{Q}$ is a sequence of points $\omega = P_1, Q_1, P_2, Q_2, \ldots, P_n, Q_n$ in X such that Q_i and P_i are glued together and P_1 corresponds to $\bar{P}$, Q_n corresponds to $\bar{Q}$. The length of the discrete walk ω is defined as

$$l_{\bar{X}}(\omega) = \sum_{i=1}^{n} d_X(P_i, Q_i).$$

Then one can prove that

$$\bar{d}_X(\bar{P}, \bar{Q}) := \inf\{l_{\bar{X}}(\omega) \mid \omega \text{ discrete walk from } \bar{P} \text{ to } \bar{Q}\}$$

defines a semi-metric on $\bar{X}$. In all cases we consider, it is a metric.

2. Tessellations

In this section, we will first introduce the notion of a tessellation of the hyperbolic plane. The so-called Tessellation theorem gives a connection between tessellations and surfaces obtained by gluing polygon edges. Poincaré's polygon theorem will give us the means to decide whether such a surface is complete. To explore the meaning of these theorems in practice, we will apply them to the once-punctured torus considered in Section 1.

Definition 2.1. *A **tessellation** of the hyperbolic plane is a family of **tiles** $(X_i)_{i \in I}$, with some index set I, such that*

(i) for all $i \in I$, X_i is a connected polygon in the hyperbolic plane $\mathbb{H}^2$;
(ii) any two X_i, X_j are isometric;
(iii) the union of all X_i covers the whole of $\mathbb{H}^2$;
(iv) the intersection of X_i and X_j for $i \neq j$ consists only of edges and vertices of X_i which are also vertices or edges of X_j, i.e. the interiors of X_i, X_j are disjoint;
(v) (Local Finiteness) for all points $P \in \mathbb{H}^2$ there is $\varepsilon > 0$ such that the hyperbolic ball $B_{d_{\mathrm{hyp}}}(P, \varepsilon)$ meets only finitely many tiles X_i.

Analogously, one can define tessellations of the euclidean space $\mathbb{R}^2$ or the sphere $\mathbb{S}^3$.

Let now X be a polygon in $\mathbb{H}^2$ with edges $E_1, \ldots, E_{2p}$ grouped in pairs $\{E_{2k-1}, E_{2k}\}$ together with gluing isometries $\varphi_{2k-1} \colon E_{2k-1} \to E_{2k}$ and $\varphi_{2k} := \varphi_{2k-1}^{-1}$. We can extend each φ_i to an isometry of $\mathbb{H}^2$ such that $\varphi_i(X)$ is on the side of $\varphi_i(E_i)$ that is opposite of X. In other words, φ maps a point in the interior of X to a point outside of X. For instance, X could be the ideal rectangle considered in Section 1, with gluing isometries as in (1.1).

Definition 2.2. *The **tiling group** associated to X and gluing isometries φ_i is the group generated by the φ_i, i.e.*

$$\Gamma := \{\varphi \in \mathrm{Isom}(\mathbb{H}^2) \mid \varphi = \varphi_{i_l} \circ \cdots \circ \varphi_{i_1} \text{ with } i_j \in \{1, \ldots, 2p\} \ \forall j \leq l\}.$$

Note that the identity is contained in Γ, as it is the composition of zero gluing maps.

Let d_X be the metric on X as before, and let $(\bar{X}, \bar{d}_X)$ be the quotient metric space obtained from X by performing the edge gluings.

Theorem 2.3 (**Tessellation theorem**). *Let X be a hyperbolic connected polygon with gluing data as above. Suppose that for each vertex of X the angles of X at all vertices that are glued to a point $P \in \bar{X}$ sum up to $\frac{2\pi}{n}$ for some $n > 0$ depending on the point P. In particular, $(\bar{X}, \bar{d}_X)$ then is a hyperbolic surface with cone singularities. Suppose further that $(\bar{X}, \bar{d}_X)$ is complete. Then the family of polygons $\{\varphi(X)\}_{\varphi\in\Gamma}$ forms a tessellation of the hyperbolic plane.*

Clearly, all $\varphi(X)$ for $\varphi \in \Gamma$ are connected hyperbolic polygons and since all elements of Γ are hyperbolic isometries, any two of them are isometric. To prove that their union indeed covers the whole hyperbolic plane, their interiors are mutually disjoint and that they fulfill the Local Finiteness property requires some more thought. The idea behind the proof is to start with one tile, namely X, and progressively set one tile after the other - just as one would do tiling a kitchen floor. Potential problems that have to be ruled out are that the tiles do not cover all of $\mathbb{H}^2$ or that at some point, they overlap.

Definition 2.4. *A tile $\varphi(X)$ is called **adjacent** to X at a point P, if there exists a sequence $\varphi_{i_1}, \dots, \varphi_{i_l}$ of gluing maps such that*

$$\varphi_{i_{j-1}} \circ \cdots \circ \varphi_{i_1}(P) \in E_{i_j} \ \forall\ j \leq l \quad \text{and} \quad \varphi = \varphi_{i_1}^{-1} \circ \cdots \circ \varphi_{i_l}^{-1}.$$

*More generally, two tiles $\varphi(X)$ and $\psi(X)$ are **adjacent** at $P \in \varphi(X) \cap \psi(X)$, if $\psi^{-1} \circ \varphi(X)$ is adjacent to X at $\psi^{-1}(P)$.*

Intuitively, the tiles adjacent to X at a point P are just the tiles that are 'neighbours' of X and contain P. Let for instance X be the ideal rectangle from Section 1 and let $P \in E_1$ be no vertex. Then the only tiles adjacent to X at P are X itself (with gluing isometry $\varphi = id$) and $\varphi_2(X) = \varphi_1^{-1}(X)$. Note that here, $l = 1$ and $\varphi_{i_1} = \varphi_1$. The first condition in the definition of adjacency is satisfied since $\varphi_1(P) \in E_2$. For $P \in \text{int}(X)$, X is the only tile adjacent to X at P.

Lemma 2.5. *There are only finitely many tiles that are adjacent to X at a point $P \in X$.*

Proof. See [Bon00, p.137ff]. □

Corollary 2.6. *Under the hypothesis of the Tessellation theorem 2.3, for every $P \in X$, there is $\varepsilon > 0$ such that the tiles $\varphi(X)$ adjacent to X at P decompose the disk $B_d(P, \varepsilon) \subset \mathbb{H}^2$ into finitely many hyperbolic disc sectors with disjoint interiors.*

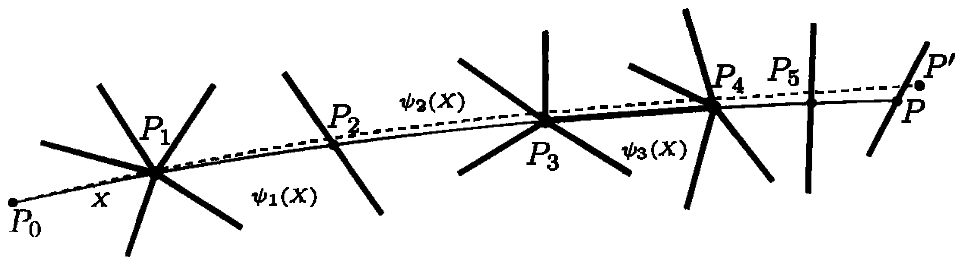

Figure 2: To show that a point P is covered by some tile, we progressively set tiles along the geodesic g connecting P to a base point P_0.

Lemma 2.7. *Every $P \in \mathbb{H}^2$ is covered by some $\varphi(X)$ with $\varphi \in \Gamma$.*

Proof. We pick a base point $P_0 \in \operatorname{int}(X)$. Let $P \in \mathbb{H}^2$ be an arbitrary point and let g be the unique hyperbolic geodesic joining P_0 to P (Figure 2). Denote by P_1 the point where g leaves X. At P_1, g enters one of the finitely many tiles that are adjacent to X at P_1. We denote it by $\psi_1(X)$. Repeating this process yields a sequence $(P_n)_{n\in\mathbb{N}}$ of points on g and of tiles $(\psi_n(X))_{n\in\mathbb{N}}$ such that g leaves $\psi_{n-1}(X)$ and enters $\psi_n(X)$ at P_n. Note that ψ_n is not always uniquely determined, i.e. when g follows an edge seperating two tiles adjacent to $\psi_{n-1}(X)$ at P_n. When g enters a tile $\psi_n(X)$ and never leaves it, this process terminates. This happens exactly when $P \in \psi_n(X)$. The proof that this indeed is the case after finitely many steps makes use of the fact that the interior angles glued together sum up to $\frac{2\pi}{n}$, such that $(\bar{X}, \bar{d}_X)$ is a hyperbolic surface, and in particular (and this is the crucial part) uses the compactness of $(\bar{X}, \bar{d}_X)$. For details, see [Bon00, p.141ff]. $\square$

Definition 2.8. *$\varphi(X)$ is called* ***canonical tile*** *for P with respect to the base point P_0, if $\varphi(X)$ is adjacent to $\psi_n(X)$ at P and $\psi_n(X)$ is the last tile needed to cover g.*

In particular, if P is an interior point of $\psi_n(X)$, then $\psi_n(X)$ is the only canonical tile for P.

Lemma 2.9. *For every $P \in \mathbb{H}^2$, there is $\varepsilon > 0$ such that for all $P' \in B_{d_{\mathrm{hyp}}}(P, \varepsilon)$ the canonical tiles for P' are exactly the canonical tiles for P containing P'.*

Sketch of proof. With the notation as before, let P be contained in the tile $\psi_n(X)$ and let T be the collection of all the tiles $\psi_i(X)$ for $i = 0, \ldots, n$ and all tiles adjacent to them. Then for any point $Q \in g$, there is an $\varepsilon > 0$ such that the ball $B_{d_{\mathrm{hyp}}}(Q, \varepsilon)$ is contained in the union $\bigcup_{Y \in T} Y$. If we move P to a point P' that is close to P (for instance at distance $< \varepsilon$), then g moves to a geodesic g' (close to g) joining P_0 to P' (Figure 2). The tiling process of g' only involves tiles of T, so in particular, the final tile $\psi'_n(X)$ is adjacent to $\psi_n(X)$. If $P \in \operatorname{int}(X)$, then (choosing ε sufficiently small) also $P' \in \operatorname{int}(X)$ and the claim follows. If P lies on an edge of $\psi_n(X)$, but is not a vertex, there are exactly two tiles adjacent to $\psi_n(X)$ at P. Since P' is close to P, it lies in one (or both) of these tiles, and the claim follows. If now P is a vertex of $\psi_n(X)$, P' is not a vertex of $\psi'_n(X)$ (for ε sufficiently small), but is contained in one or two tiles adjacent to $\psi_n(X)$ at P, regarding to whether it is interior point of $\psi'_n(X)$ or lies on an edge. In any case, the canonical tiles for P' are exactly the tiles adjacent to $\psi_n(X)$ at P containing P'. □

Lemma 2.10. *Let $\varphi \in \Gamma$ and $P, Q \in \mathrm{int}(\varphi(X))$. If $\varphi(X)$ is canonical for P it is also canonical for Q. Further, $\varphi(X)$ is the only canonical tile for P and Q.*

Proof. See [Bon00, p.144]. □

Lemma 2.11. *Every tile $\varphi(X)$ is canonical for some P in its interior.*

Proof. Assume first that the tile $\varphi(X)$ is canonical for some P in its interior, and let φ_i be a gluing map. We claim that $\varphi \circ \varphi_i(X)$ is canonical for some P' in its interior. The tiles $\varphi(X)$ and $\varphi \circ \varphi_i(X)$ meet at an edge $\varphi \circ \varphi_i(E_i)$. Let Q be a point on this edge that is not a vertex. Let $P' \in \mathrm{int}(\varphi(X))$ be sufficiently close to Q. By Lemma 2.10, $\varphi(X)$ is the only canonical tile for P'', and by Lemma 2.9, $\varphi(X)$ is also canonical for Q. Any canonical tile for Q now has to be adjacent to $\varphi(X)$. Since Q is not a vertex, the only canonical tiles for Q are $\varphi(X)$ and $\varphi \circ \varphi_i(X)$. Let P' be a point in $\varphi \circ \varphi_i(X)$. If P' is sufficiently close to Q, again Lemma 2.9 gives that the canonical tile for P' has to be $\varphi(X)$ or $\varphi \circ \varphi_i(X)$. Since $P' \notin \varphi(X)$, $\varphi \circ \varphi_i(X)$ is canonical for P', as we claimed. Since the tile X is canonical for the point P_0, it now follows inductively that any tile $\varphi(X)$ is canonical for some point P in its interior. □

Thanks to all the auxiliary lemmata we have just proven, we are now ready to prove our main theorem, the Tessellation theorem.

Proof of Theorem 2.3. We already know that the family $\{\varphi(X)\}_{\varphi \in \Gamma}$ satisfies (i) and (ii) from Definition 2.1. By Lemma 2.7 the union of these tiles covers all of $\mathbb{H}^2$, so (iii) is satisfied as well. Note that for applying Lemma 2.7, we make use of the assumptions that the interior angles sum up to $\frac{2\pi}{n}$ and that the resulting surface is compact. Now suppose that $P \in \mathrm{int}(\varphi(X)) \cap \mathrm{int}(\varphi'(X))$. Then by Lemma 2.10, $\varphi(X) = \varphi'(X)$ since there is a unique canonical tile for P. In particular, the interiors of $\varphi(X)$ and $\varphi'(X)$ are disjoint, hence (iv) is satisfied. To prove local finiteness, one consideres one point $P \in \mathbb{H}^2$. P has only finitely many canonical tiles, so there exists some ball $B_{d_{\mathrm{hyp}}}(P, \varepsilon)$ contained in the union of these tiles. No other tile $\varphi(X)$ can meet this ball, since otherwise, its interior would meet the interior of one of the canonical tiles for P. By (iv), this cannot occur. In total, the family $\{\varphi(X)\}_{\varphi\in\Gamma}$ satisfies (i)-(v) from Definition 2.1, so it forms a tessellation of $\mathbb{H}^2$. $\square$

From Theorem 2.3 we know that, whenever a quotient space $(\bar{X}, \bar{d}_X)$ as above obtained from gluing a polygon is complete (and satisfies the angle-sum condition), it gives rise to a tessellation of $\mathbb{H}^2$. Hence, it would be useful to have some easy criterion to check whether or not such a quotient space is complete. If the polygon X is bounded, then the quotient space $\bar{X}$ is compact and hence complete. That is why in the following, we only consider unbounded polygons with one or several ideal vertices. A criterion to check whether a quotient space obtained in this way is complete is given by Poincaré's polygon theorem. To state it, we first need some notation.

Definition 2.12. *A **horocircle** centered at $\zeta \in \mathbb{R} \cup \{\infty\}$ is a curve $C - \{\zeta\}$, where C is an euclidean circle in $\mathbb{H}^2$ tangent to the real line at ζ. A horocircle centered at ∞ is just a horizontal line. An isometry φ of $\mathbb{H}^2$ is called **horocyclic at** ζ if it respects some horocircle centered at ζ. For $\zeta = \infty$, φ then is a horizontal translation $z \mapsto z + b$ or a reflection across at a vertical line $z \mapsto \bar{z} + b$.*

Remark. Since an isometry φ of $\mathbb{H}^2$ sends generalized (euclidean) circles to generalized (euclidean) circles, φ sends a horocircle centered at ζ to some hororcircle centered at $\varphi(\zeta)$. If φ is horocyclic at some $\zeta \in \mathbb{R} \cup \infty$, then it respects every horocircle centered at ζ. This can easily be seen in the case $\zeta = \infty$. For an arbitrary ζ the claim follows from this special case by applying an isometry of $\mathbb{H}^2$ sending ζ to ∞.

Let ζ be an ideal vertex of X, endpoint of the edge E_i. The gluing isometry φ_i sends ζ to another ideal vertex $\varphi_i(\zeta)$. We denote the element in the quotient space $\bar{X}$ corresponding to ζ by $\bar{\zeta}$ and write $\bar{\zeta} = \{\zeta_1, \ldots, \zeta_k\}$, meaning that the ideal vertices ζ_i are glued to $\bar{\zeta}$.

Lemma 2.13. *The indexing of the ideal vertices in $\bar{\zeta} = \{\zeta_1, \ldots, \zeta_k\}$ can be chosen such that there exist gluing maps $\varphi_{i_j} \colon E_{i_j} \to E_{i'_{j+1}}$ with $E_{i'_{j+1}} := E_{i_{j\pm 1}}$ depending on φ_{i_j} for $j = 1, \ldots, k$ satisfying*

(i) ζ_j is endpoint of E_{i_j}, ζ_{j+1} is an endpoint of $E_{i'_{j+1}}$ and $\varphi_{i_j}(\zeta_j) = \zeta_{j+1}$;

(ii) the edges E_{i_j} and $E_{i'_j}$ adjacent to ζ_j are disjoint, i.e. $\varphi_{i_j} \neq \varphi^{-1}_{i_{j-1}}$ for all $1 < j < k-1$;

(iii) exactly one of the following holds:

- *(a) there is a map $\varphi_{i_k} \colon E_{i_k} \to E_{i'_1}$ such that φ_{i_k} sends ζ_k to ζ_1, ζ_k is an endpoint of E_{i_k} and E_{i_k} is not the image $E_{i'_k}$ of $\varphi_{i_{k-1}}$ and such that ζ_1 is an endpoint of $E_{i'_1}$ and $E_{i'_1}$ is not the domain E_{i_1} of φ_{i_1}* or
- *(b) each of ζ_1, ζ_k is adjacent to a unique edge of X, namely E_{i_1} and E_{i_k} respectively.*

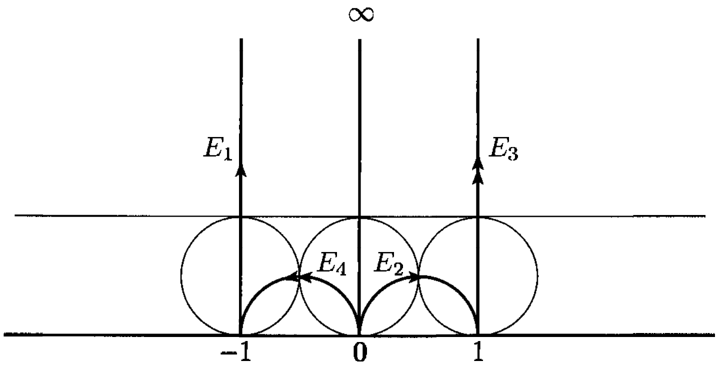

Figure 3: These horocircles at the vertices of X fulfill the horocircle-condition. The figure contains an additional edge, the diagonal from 0 to ∞ which we will consider later on.

Sketch of proof. In the case that any vertex of ζ_i is adjacent to exactly two edges, the proof is straight-forward. We start with an arbitrary $\zeta := \zeta_1 \in \bar{\zeta}$ adjacent to some edge E_{i_1} and consider the gluing isometry $\varphi_{i_1} : E_{i_1} \to E_{i'_1}$. We set $\zeta_2 := \varphi_{i_1}(\zeta_1)$ and go on like this. As there are only finitely many edges glued to $\bar{\zeta}$ we eventually reach some k with $\zeta_{k+1} = \zeta_j$ for some $j \leq k$. If k is the smallest such index, assume that $j > 1$. It then follows that ζ_1 is adjacent to only one edge, contradicting our assumption. Hence $j = 1$ and we are in case *(iii)(a)*. The other case, leading to *(iii)(b)* requires more thought. However, we are only going to use the first case. For details, see [Bon00, p.170f]. □

Lemma 2.14. *The following properties are equivalent:*

(i) Horocircle condition: *At each ideal vertex ζ of X, one can choose a horocircle C_ζ centered at ζ such that whenever $\varphi_i \colon E_i \to E_{i\pm1}$ sends ζ to another ideal vertex ζ', then $\varphi_i(C_\zeta) = C_{\zeta'}$.*

(ii) Edge cycle condition: *For every edge cycle around an ideal vertex $\bar{\zeta} = \{\zeta_1, \dots, \zeta_k\}$ with gluing maps $\varphi_{i_j} \colon E_{i_j} \to E_{i_{j+1}}$ sending ζ_j to ζ_{j+1} for $j = 1, \dots, k$, the corresponding composition $\varphi_{i_k} \circ \cdots \circ \varphi_{i_1}$ is horocyclic at ζ_1.*

Proof. If the horocircle condition is valid, then for any edge cycle around $\bar{\zeta}$ as above, the composition $\varphi := \varphi_{i_k} \circ \cdots \circ \varphi_{i_1}$ sends ζ_1 to itself, in particular it sends C_ζ to C_ζ, so it is horocyclic at ζ_1.

If on the other hand, the edge cycle condition holds and the gluing data around an ideal vertex $\bar{\zeta}$ is arranged as above, pick an arbitrary horocircle centered at ζ_1. Then $C_{\zeta_j} := \varphi_{i_{j-1}} \circ \dots \varphi_{i_1}(C_{\zeta_1})$ for $j \leq k$ is a horocircle centered at ζ_j by construction. One easily checks that these horocircles fulfill the horocircle condition. Performing this construction for every ideal vertex proves the claim. □

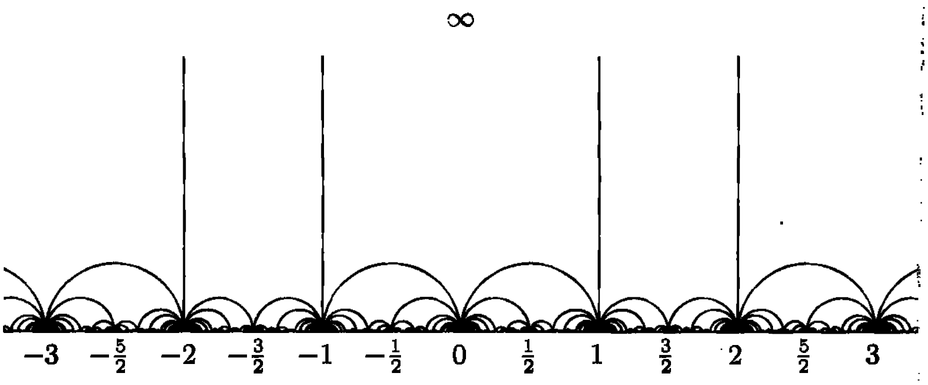

Figure 4: The images of X under the group Γ form a tessellation of $\mathbb{H}^2$.

We are now ready to state the polygon theorem.

Theorem 2.15 (Poincaré's polygon theorem)**.** *Let $(\bar{X}, \bar{d}_X)$ be the quotient space obtained by gluing the edges of a polygon (X, d_X) in $\mathbb{H}^2$ using gluing maps $\varphi_i \colon E_i \to E_{i\pm 1}$. Then $(\bar{X}, \bar{d}_X)$ is complete if and only if one of the equivalent conditions of Lemma 2.14 is satisfied.*

For the proof, we refer to [Bon00, Chapter 6.8].

To conclude the section on tessellations, we apply our two main theorems on the once-punctured torus. Let X be the hyperbolic polygon with vertices $-1, 0, 1$ and ∞ considered in Section 1 with gluing isometries as in (1.1). We show that the horocircle condition from Lemma 2.14 is satisfied in this case.

Let C_∞ be the horizontal line given by $\mathrm{Im}(z) = 1$, and let C_ζ be the horocircle centered at ζ with radius $\frac{1}{2}$ for $\zeta = -1, 0, 1$ (Figure 3). Then φ_1 sends ∞ to 1 and the point $-2 + i \in C_\infty$ to $1 + i$. Hence, $\varphi_1(C_\infty) = C_1$. A similar computation gives the same result for the other gluing isometries. So X satisfies the horocircle condition and therefore, by Poincaré's polygon theorem, the once-punctured torus $(\bar{X}, \bar{d}_X)$ is complete. Since the inner angle at all ideal vertices of X is 0, we can apply the Tessellation theorem and obtain: The family $\{\varphi(X)\}_{\varphi \in \Gamma}$, where Γ is generated by the gluing isometries φ_i, forms a tessellation of the hyperbolic plane (Figure 4).

3. The Farey circle packing and tessellation

In this section, we get to know the Farey circle packing and the corresponding Farey tessellation and discover that it is closely related to the once-punctured torus. In the following, X will denote the ideal hyperbolic polygon considered in Section 1.

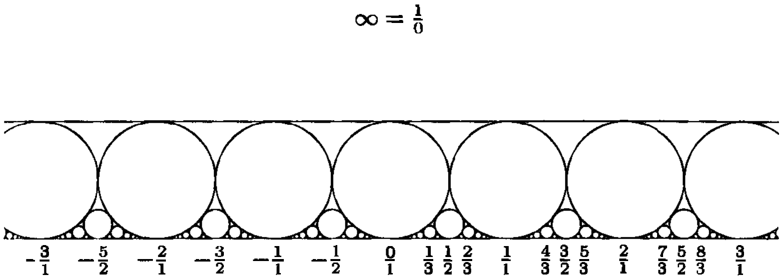

Figure 5: The Farey circle packing.

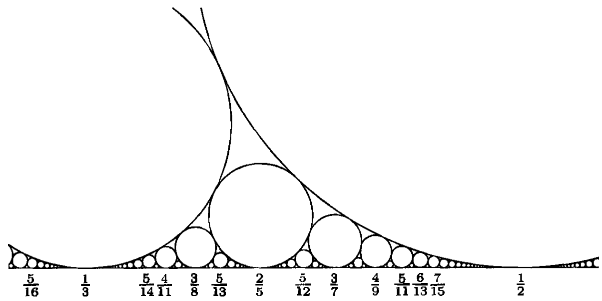

Figure 6: Zooming in on the Farey Circle packing.

For any $\frac{p}{q} \in \mathbb{Q}$ considered, let p, q be coprime and $q > 0$. For any such $\frac{p}{q}$ draw in $\mathbb{R}^2$ the circle $C_{\frac{p}{q}}$ of diameter $\frac{1}{q^2}$ that is tangent to the x-axis at $(\frac{p}{q}, 0)$ and lies in the upper half-plane (Figure 5). We make the following observations:

(i) The circles $C_{\frac{p}{q}}$ have disjoint interiors.

(ii) $C_{\frac{p}{q}}$ and $C_{\frac{p'}{q'}}$ are tangent if and only if $pq' - p'q = \pm 1$. We say that $\frac{p}{q}$ and $\frac{p'}{q'}$ form a **Farey pair**.

(iii) $C_{\frac{p}{q}}$, $C_{\frac{p'}{q'}}$ and $C_{\frac{p''}{q''}}$ with $\frac{p}{q} < \frac{p''}{q''} < \frac{p'}{q'}$ are pairwise tangent to each other if and only if $\frac{p''}{q''} = \frac{p}{q} \oplus \frac{p'}{q'}$, where $\frac{p}{q} \oplus \frac{p'}{q'} := \frac{p+p'}{q+q'}$ is called the **Farey sum** of $\frac{p}{q}$ and $\frac{p'}{q'}$.

The same holds if we consider $\infty = \frac{1}{0} = \frac{-1}{0}$ and set $C_\infty := \{(x, y) \in \mathbb{R}^2 \mid y = 1\}$ with interior points (x, y) with $y > 1$.

If $\frac{p}{q}$ and $\frac{p'}{q'}$ form a Farey pair, i.e. if $C_{\frac{p}{q}}$ and $C_{\frac{p'}{q'}}$ are tangent, we now connect $(\frac{p}{q}, 0)$ and $(\frac{p'}{q'}, 0)$ by a semi-circle. Erasing the circles $C_{\frac{p}{q}}$, we are left with a collection of hyperbolic geodesics that looks similar to the tessellation belonging to the once-puntured torus. We call this the **Farey tessellation of** $\mathbb{H}^2$ of the hyperbolic plane - even if we do not know yet if it actually is a tessellation (Figure 7).

Let us get back to the ideal polygon X. We split X along the diagonal from 0 to ∞ into two triangles T^+ and T^-. By the isometry $z \mapsto -\bar{z}$, both triangles are isometric. Hence,

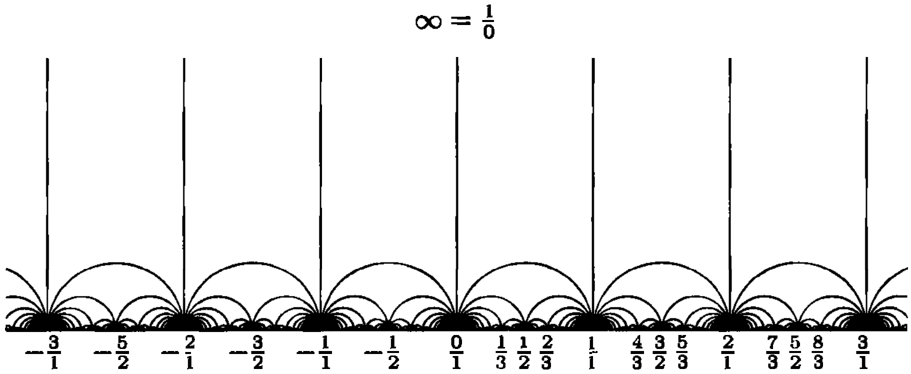

Figure 7: Connection the end-points of circles that are tangent to each other, we obtain a family of geodesics in $\mathbb{H}^2$, the so-called Farey-tessellation.

since the family $\varphi(X)_{\varphi\in\Gamma}$ forms a tessellation of $\mathbb{H}^2$, also

$$\mathcal{T} := \{\varphi(T^+), \varphi(T^-)\}_{\varphi\in\Gamma}$$

forms a tessellation of $\mathbb{H}^2$.

Theorem 3.1. *$\mathcal{T}$ is equal to the Farey tessellation, i.e. its edges are hyperbolic geodesics joining $\frac{p}{q}$ to $\frac{p'}{q'}$ whenever $pq' - p'q = \pm 1$.*

For the proof of this theorem, we make use of two lemmata. Remember that $\varphi \in \mathrm{PSL}_2(\mathbb{Z})$ is the group of linear fractional maps of the form

$$\varphi(z) = \frac{az+b}{cz+d} \quad \text{with } a, b, c, d \in \mathbb{Z},\ ad - bc = 1.$$

Lemma 3.2. *Let $\frac{p}{q}, \frac{p'}{q'} \in \mathbb{Q} \cup \{\infty\}$ form a Farey pair. Then for any $\varphi \in \mathrm{PSL}_2(\mathbb{Z})$, the points $\varphi(\frac{p}{q})$ and $\varphi(\frac{p'}{q'})$ form a Farey pair as well.*

Proof. Simple computation. $\square$

The pairs $\{0,\infty\}, \{0,1\}$ and $\{1,\infty\}$ form Farey pairs, and similar the vertices of T^-. It follows that the endpoints of each edge of $\mathcal{T}$ form a Farey pair, since they are all images of edges of T^+ or T^- under elements of $\Gamma \subseteq \mathrm{PSL}_2(\mathbb{Z})$.

Lemma 3.3. *Let g_1, g_2 be distinct geodesics in $\mathbb{H}^2$ whose endpoints form Farey pairs. Then g_1 and g_2 are disjoint.*

Proof. Let the endpoints of g_1 be $\frac{p_1}{q_1}$ and $\frac{p_1'}{q_1'}$ with indexing chosen such that $p_1'q_1 - p_1q_1' = 1$. Then

$$\varphi(z) := \frac{q_1 z - p_1}{-q_1' z + p_1}$$

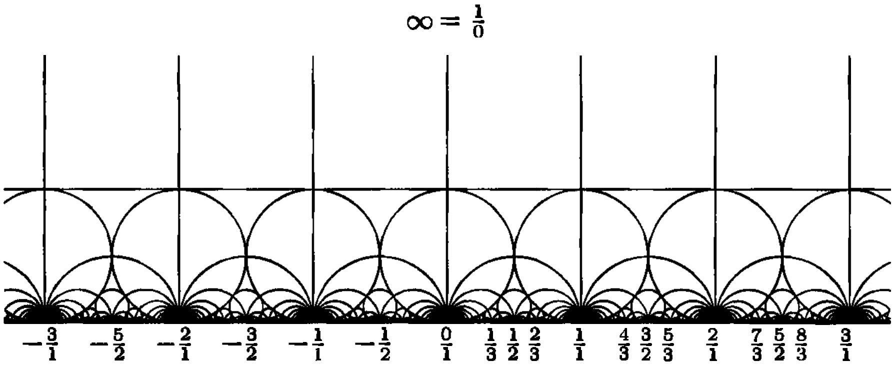

Figure 8: The Farey tessellation together with the corresponding horocircles, which coincide with the circles forming the Farey circle packing.

sends g_1 to the geodesic with endpoints 0 and ∞. Let $\varphi(g_2)$ have endpoints $\frac{p_2}{q_2}$ and $\frac{p_2'}{q_2'}$. Suppose that g_1 and g_2 meet. Then $\varphi(g_1)$ and $\varphi(g_2)$ meet as well, in particular $\varphi(g_2)$ crosses the line from 0 to ∞, so p_2 and p_2' have different signs, as $q_2, q_2' > 0$ by assumption. From Lemma 3.2 we know that $\frac{p_2}{q_2}$ and $\frac{p_2'}{q_2'}$ satisfy the relation $p_2q_2' - p_2'q_2 = \pm 1$, what is not possible if p_2, p_2' have different signs. Hence, g_1 and g_2 have to be disjoint. $\square$

Lemma 3.3 shows that any geodesic whose endpoints form a Farey pair must be an edge of the Farey tessellation: Since the tiles of $\mathcal{T}$ are ideal triangles, their interiors cannot contain any complete geodesic. If now g is a hyperbolic geodesic whose endpoints form a Farey pair and g meets an edge g' of $\mathcal{T}$, then $g = g'$ by Lemma 3.3. As a result, the edges of the tessellation $\mathcal{T}$ are exactly the complete geodesics whose endpoints form a Farey pair, so $\mathcal{T}$ coincides with the Farey tessellation. This proves Theorem 3.1. The denomination *Farey tessellation* is now justified, since it is indeed a tessellation of $\mathbb{H}^2$.

Remark. We now may ask if, given the tessellation of $\mathbb{H}^2$ corresponding to the once-punctured torus, we can reconstruct the Farey circle packing. This indeed is possible: Recall the horocircles C_∞, C_{-1}, C_0 and C_1 from the end of Section 2. Two of these horocircles meet if and only if they are tangent and this is exactly when their centers are the ends of an edge of the Farey tessellation. In the tessellation of the hyperbolic plane by the tiles $\varphi(X)$, $\varphi \in \Gamma$, the images of these horocircles C_ζ for $\zeta = -1, 0, 1, \infty$ form a family of horocircles, all centered at the images of $-1, 0, 1$ and ∞ under φ (Figure 8). These are points $\frac{p}{q} \in \mathbb{Q}$, and, just as their pre-images, two of these hororcircles meet only when they are tangent, i.e. when their centers are the ends of an edge of the Farey tessellation. Computing the image of C_∞ under an isometry $\varphi(z) = \frac{az+b}{cz+d} \in \mathrm{PSL}_2(\mathbb{Z})$ exactly, one finds that they are horocircles centered of $\frac{a}{c}$ with diameter $\frac{1}{c^2}$. Hence, this family of horocircles coincides with the Farey circle packing.

4. Shearing the Farey tessellation

In Section 1, we glued the ideal polygon X using gluing isometries φ_i as in (1.1). We are now going to modify these gluing isometries slightly and check if this construction again gives rise to a tessellation of $\mathbb{H}^2$. We start with a basic result from hyperbolic geometry.

Lemma 4.1. *Given a triple $\zeta_1, \zeta_2, \zeta_3 \in \mathbb{R} \cup \{\infty\}$ of distinct points and another such triple $\zeta_1', \zeta_2', \zeta_3' \in \mathbb{R} \cup \{\infty\}$, there is a unique isometry of $\mathbb{H}^2$ sending each ζ_i to the corresponding ζ_i'. Also, at each ζ_i there is a unique horocircle C_i centered at ζ_i such that any two C_i, C_j are tangent to each other and meet at a point of the complete hyperbolic geodesic going from ζ_i to ζ_j.*

Proof. For the first part, without loss of generality, we can assume that $\zeta_1' = 0, \zeta_2' = 1$ and $\zeta_3' = \infty$. The general case then is given by composition of isometries obtained in this special case. If $\infty \notin \{\zeta_1, \zeta_2, \zeta_3\}$, an isometry φ satisfying $\varphi(\zeta_i) = \zeta_i'$ is

$$\varphi(z) = \frac{z - \zeta_1}{z - \zeta_3} \frac{\zeta_2 - \zeta_3}{\zeta_2 - \zeta_1}.$$

If one of the ζ_i is equal to ∞, for instance $\zeta_1 = \infty$, φ can be defined by

$$\varphi(z) = \frac{\zeta_2 - \zeta_3}{z - \zeta_3}.$$

The cases that ζ_2 or ζ_3 are equal to ∞ are similar. For the second claim, it suffices to check for the points $0, 1$ and ∞, since by the first part, we can always get back to this case using a unique hyperbolic isometry. Here, we first consider the horocircle C_∞ - it is a straight line given by the equation $\mathrm{Im}(z) = k$ for some $k > 0$. Since the horocircles C_0 and C_1 are supposed to be tangent to C_∞, we conclude that both have radius $\frac{k}{2}$. The only k for which C_0 and C_1 are tangent then is $k = 1$, hence the C_i are uniquely determined. The point where C_∞ and C_0 meet clearly lies on the geodesic from 0 to ∞, the same holds for C_∞ and C_1. C_0 and C_1 meet at the point $\frac{1}{2} + \frac{1}{2}i$, which is on the geodesic from 0 to 1 as well. □

For any edge of an ideal triangle, we can now by Lemma 4.1 fix a base point, namely the point where the unique horocircles centered at its endpoints meet.

Recall the hyperbolic polygon X from Section 1 with gluing isometries φ_i as in (1.1). We can modify the gluing isometries as follows:

$$\varphi_1 \colon E_1 \to E_2,\ \varphi(z) = \frac{z+1}{z+a}; \qquad \varphi_3 \colon E_3 \to E_4,\ \varphi(z) = \frac{z-1}{-z+b}. \tag{4.1}$$

Until now, we considered the special $a = b = 2$ and obtained a complete hyperbolic surface. However, for any $a, b \in \mathbb{Z}$, φ_1 sends the edge E_1 going from -1 to ∞ to the edge E_2 going from 0 to 1, similarly φ_3 sends E_3 to E_4. We can therefor work with these more general gluing isometries; in this case, we do not know yet if the resulting surface is complete. We split X again as in Section 3 and obtain triangles T^+ and T^-. Using Lemma 4.1, we find base points on each of the edges of X. The base points P_i on the edges E_i are given as

$$P_1 = -1 + i, \quad P_2 = \frac{1}{2} + \frac{1}{2}\, i, \quad P_3 = 1 + i \quad \text{and} \quad P_4 = -\frac{1}{2} + \frac{1}{2}\, i.$$

One easily checks that for the case $a = b = 2$, the gluing isometries send base points to base point. This is why the horocircles corresponding to the tessellation fit nicely side-by-side (Figure 8). For arbitrary a, b, this property does not hold. To see this, fix a and b. Recall that $\varphi_2(z) := \varphi_1^{-1}(z) = \frac{az-1}{-z+1}$. Then the basepoint of $E_1 = \varphi_2(E_2)$ corresponding to horocircles of $\varphi_2(X)$ is $\varphi_2(P_2) = -1 + i(a-1)$. Note that $\varphi_2(P_2) \neq P_1$ for $a \neq 2$. Since E_1 is a vertical line, by the formula for hyperbolic distance, we find

$$d_{\mathrm{hyp}}(\varphi_2(P_2), P_1) = |\log(a-1)|.$$

Seen from the interior of X, $\varphi_2(P_2)$ is at signed distance $s_1 := -\log(a-1)$ to the left of P_1 (compare Figure 9). Since φ_1 is a hyperbolic isometry, also $P_2 = \varphi_1(\varphi_2(P_2))$ is at distance $|s_1|$ from $\varphi(P_1)$. If we consider signed distances, we have to keep in mind that φ_1 sends the interior of X to the side of E_2 that is opposite to X. Hence, as isometries are orientation-preserving, P_2 is at signed distance s_1 to the left of $\varphi_1(P_1)$ seen from *outside* of X. Seen from the interior, $\varphi_1(P_1)$ is at signed distance s_1 to the left of P_2. Similarly, the basepoint $\varphi_4(P_4)$ on E_3 determined by $\varphi_4(X)$ is at signed distance $s_3 := log(b-1)$ to the left of P_3 and on E_4, the base point $\varphi_3(P_3)$ is at signed distance s_3 to the left of P_4, both seen from the interior of X. We now consider any edge E of the partial tessellation of $\mathbb{H}^2$ associated to X and gluing isometries φ_1, φ_3. Then $E = \varphi(E_1)$ or $E = \varphi(E_3)$ for some element $\varphi \in \Gamma$, and E seperates the tile $\varphi(X)$ from a tile $\psi(X)$ for some $\psi \in \Gamma$. If we transport our considerations from above to the tile $\varphi(X)$, we see that, seen from the interior of $\varphi(X)$, the base point determined by $\psi(X)$ is at signed distance s_1 to the left of the base point determined by $\varphi(X)$ if $E = \varphi(E_1)$ and at signed distance s_3 to the left of the base point determined by $\varphi(X)$ if $E = \varphi(E_3)$.

Figure 9 illustrates the case $s_1 = 0.25$ and $s_3 = -1$. We denote the partial tessellation of $\mathbb{H}^2$ obtained in this way by $\mathcal{T}_{s_1,s_3}$. We do not know yet if $\mathcal{T}_{s_1,s_3}$ is a tessellation, as it does not necessarily need to cover the whole of $\mathbb{H}^2$. Every tile of $\mathcal{T}_{s_1,s_3}$ corresponds to a tile of the tessellation $\mathcal{T} = \mathcal{T}_{0,0}$. To obtain $\mathcal{T}_{s_1,s_3}$ from $\mathcal{T}$, we progressively slide all tiles to the

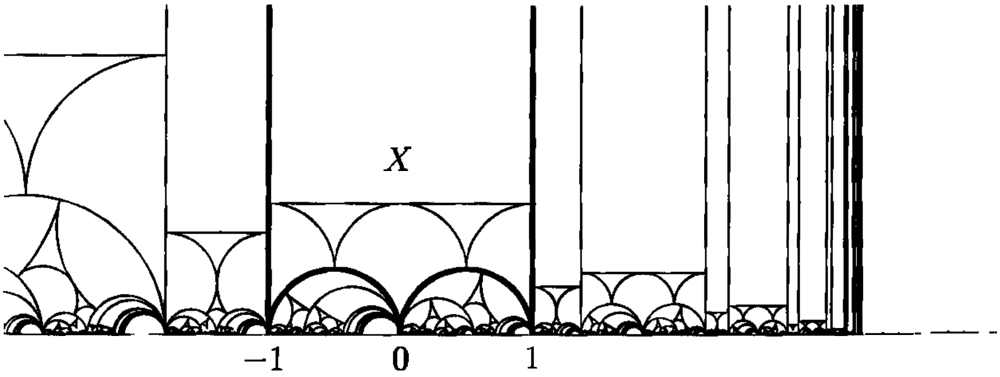

Figure 9: If we shear the Farey tessellation according to shear parameters $s_1 = 0.25$ and $s_3 = -1$, the result is a partial tessellation of $\mathbb{H}^2$.

left along te edges by signed distance s_1 or s_3, according to whether the edge considered is an image of E_1 or E_3.

Definition 4.2. *The partial tessellation $\mathcal{T}_{s_1,s_3}$ is obtained by* ***shearing*** *$\mathcal{T}$ according to the* ***shear parameters*** *s_1 and s_3.*

We can generalize this construction once more by introducing an additional edge E_5, the diagonal from 0 to ∞. As before, we obtain two triangles T^+ and T^-. We replace T^- by its image under the isometry φ_5 defined by $z \mapsto \exp^{-s_5}$ for some shear parameter s_5. Seen from the interior of T^+, we slide T^- to the left along E_5 by distance s_5. We obtain a new ideal polygon $\tilde{X} := \varphi_5(T^-) \cup T^+$ (Figure 10).

Starting with this sheared polygon, we construct a partial tessellation of $\mathbb{H}^2$ as before, using the shear parameters s_1 and s_3. Remember that $s_1 = -\log(a-1)$ and $s_3 = \log(b-1)$, hence $a = e^{-s_1} + 1$ and $b = e^{s_3} + 1$. For gluing the sides of $\tilde{X}$, we use the isometries

$$\tilde{\varphi}_1(z) := \varphi_1 \circ \varphi_5^{-1}(z) = \frac{e^{s_5} z + 1}{e^{s_5} z + e^{-s_1} + 1},$$

$$\tilde{\varphi}_3(z) := \varphi_5 \circ \varphi_3(z) = e^{-s_5} \frac{z - 1}{-z + e^{s_3} + 1}.$$

Lemma 4.3. *The images of $\tilde{X}$ under the tiling group $\tilde{\Gamma}$ generated by $\tilde{\varphi}_1$ and $\tilde{\varphi}_3$ cover the whole hyperbolic plane if and only if $s_1 + s_3 + s_5 = 0$.*

Proof. By Poincaré's polygon theorem 2.15, the quotient space $(\bar{\tilde{X}}, \bar{d}_{\tilde{X}})$ is complete if and only if the horocircle condition from Lemma 2.14 holds. The only edge cycle around an ideal vertex $\bar{\zeta}$ in $\bar{\tilde{X}}$ consists of the vertex $\bar{\infty} = \{\infty, 1, 0, -e^{-s_5}\}$. The composition

$\tilde{\varphi}_4 \circ \tilde{\varphi}_2 \circ \tilde{\varphi}_3 \circ \tilde{\varphi}_1$ sends ∞ to ∞:

$$\begin{aligned}\tilde{\varphi}_4 \circ \tilde{\varphi}_2 \circ \tilde{\varphi}_3 \circ \tilde{\varphi}_1(\infty) &= \tilde{\varphi}_4 \circ \tilde{\varphi}_2 \circ \tilde{\varphi}_3(1) \\ &= \tilde{\varphi}_4 \circ \tilde{\varphi}_2(0) \\ &= \tilde{\varphi}_4(-e^{-s_5}) \\ &= \infty,\end{aligned}$$

so the gluing maps corresponding to this edge cycle are $\tilde{\varphi}_1\colon E_1 \to E_2$, $\tilde{\varphi}_3\colon E_3 \to E_4$, $\tilde{\varphi}_2 = \tilde{\varphi}_1^{-1}\colon E_2 \to E_1$ and $\tilde{\varphi}_4 = \tilde{\varphi}_3^{-1}\colon E_4 \to E_3$. By Poincaré's polygon theorem 2.15, the quotient space $\bar{\tilde{X}}, \bar{d}_{\tilde{X}}$ is complete if and only if the composition $\tilde{\varphi}_4 \circ \tilde{\varphi}_2 \circ \tilde{\varphi}_3 \circ \tilde{\varphi}_1$ is horocylic at ∞. Computing the composition explicitly, we find

$$\tilde{\varphi}_4 \circ \tilde{\varphi}_2 \circ \tilde{\varphi}_3 \circ \tilde{\varphi}_1(z) = e^{2(s_1+s_3+s_5)}z + (1 + e^{s_3} + e^{s_3+s_1} + e^{s_3+s_1+s_5} + e^{2s_3+s_1-s_5} + e^{2s_3+2s_1+s_5}).$$

Remember that an isometry φ is horocyclic at ∞ if is a translation $z \mapsto z+b$ or a reflection $z \mapsto -\bar{z} + b$. The second case cannot occur here, hence the composition map is horocyclic at ∞ if and only if $s_1 + s_3 + s_5 = 0$.

Let now $s_1 + s_3 + s_5 = 0$. Then $(\bar{\tilde{X}}, \bar{d}_{\tilde{X}})$ is complete by Poincaré's polygon theorem, and since $\tilde{X}$ is an ideal quadrangle, it follows by the Tessellation theorem 2.3 that the family $\{\varphi(\tilde{X})\}_{\varphi\in\tilde{\Gamma}}$ forms a tessellation of $\mathbb{H}^2$. Conversely, let the images of $\tilde{X}$ under $\tilde{\Gamma}$ tessellate $\mathbb{H}^2$. By Theorem A.1, the space $(\bar{\tilde{X}}, \bar{d}_{\tilde{X}})$ is isometric to the quotient $(\mathbb{H}^2/\tilde{\Gamma}, \bar{d}_{\tilde{\Gamma}})$ of $\mathbb{H}^2$ under the action of $\tilde{\Gamma}$. Since $\tilde{\Gamma}$ is a subgroup of $\mathrm{PSL}_2(\mathbb{Z})$, and elements of $\mathrm{PSL}_2(\mathbb{Z})$ act discontinuously on the complete metric space $(\mathbb{H}^2, d_{\mathrm{hyp}})$, the quotient space $(\mathbb{H}^2/\tilde{\Gamma}, \bar{d}_{\tilde{\Gamma}})$ (and therefor also $(\bar{\tilde{X}}, \bar{d}_{\tilde{X}})$) is complete (Lemma A.2). Hence, by Poincaré's polygon theorem 2.15 and the considerations above, $s_1 + s_3 + s_5 = 0$. $\square$

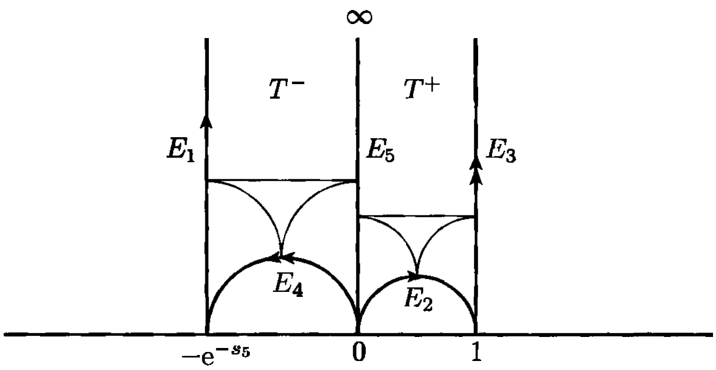

Figure 10: Replacing the triangle T^- by $\varphi_5(T^-)$ with $\varphi_5(z) = e^{-s_5}$, we obtain a new hyperbolic polygon $\tilde{X}$.

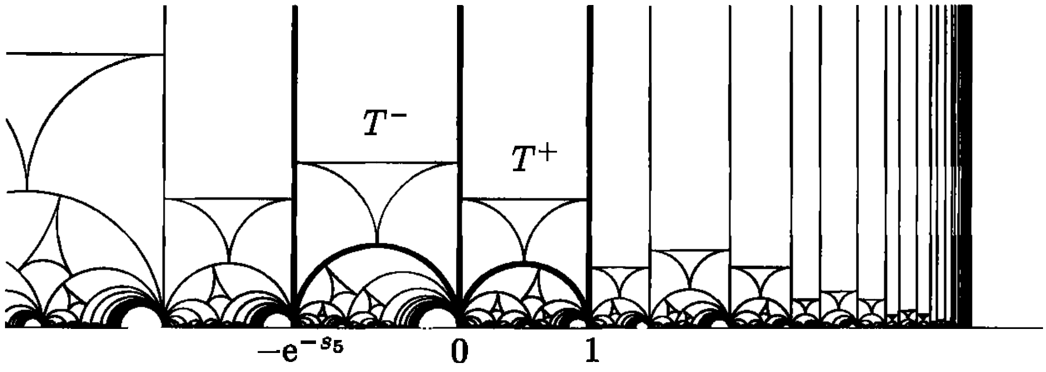

Figure 11: Using shear parameters s_1, s_3, s_5, we obtain another partial tessellation of $\mathbb{H}^2$. Here, $s_1 = 0.25, s_3 = -0.75$ and $s_5 = 0.25$, so by Lemma 4.3 it is not a tessellation.

Let us recall what we just did. We started with the once-punctured torus from Section 1, that gives rise to a tessellation of $\mathbb{H}^2$. We modified the gluing isometries to obtain a partial tessellation $\mathcal{T}_{s_1,s_3}$, but did not know yet if it covered the whole hyperbolic plane. Introducing a third shear parameter s_5, we just showed that the tessellation corresponding to shear parameters s_1, s_3, s_5 is complete if and only if the shear parameters sum up to 0. Hence, we now know infinitely many tessellations of $\mathbb{H}^2$, that can all be obtained from our starting point, the Farey tessellation, by shearing.

A. Appendix

Here, we state (without proof) two properties concerning quotient spaces used above. They can be found in Chapter 7 of [Bon00].

Theorem A.1. *Let the group Γ act by isometries and discontinuously on $(\mathbb{H}^2, d_{\text{hyp}})$, and let Δ be a fundamental domain for the action of Γ, i.e. a connected polygon in $\mathbb{H}^2$ such that all images of Δ under elements of Γ are distinct and form a tessellation of $\mathbb{H}^2$. Then the space $(\bar{\Delta}, \bar{d}_\Delta)$ obtained from Δ by gluing edges is isometric to the quotient space $({\mathbb{H}^2}/{\Gamma}, \bar{d}_{\text{hyp}})$ of $\mathbb{H}^2$ by the action of Γ.*

Lemma A.2. *Let Γ be a group acting discontinuously on the complete metric space X. Then the quotient space $({X}/{\Gamma}, \bar{d}_X)$ is complete.*

References

[Bon00] Francis Bonahon. *Low-dimensional Geometry.* American Mathematical Society, 2000.