Geodesics

Seminar on Riemannian Geometry
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1 Geodesics

1.1 Motivation

The general idea behind the concept of geodesics is the generalisation of straight lines in Euclidian space to Riemannian manifolds. A geodesic will be a constantly paramatrized, smooth curve on the manifold, that is **locally** the shortest curve connecting two points with each other (the latter will be proven in the next talk). They are of great importance in the further study of Riemannian Geometry, as well as in theoretical physics, in particular in General Relativity, where they are the trajectories of test objects moving in an non-trivial spacetime geometry, which replaces the notion of gravitational field in this context.

1.2 Definition and Basics

Throughout, let (M, g) be a Riemannian manifold of dimension n and ∇ the Riemannian connection.

Definition 1. *A parametrized, smooth curve $\gamma : I \to M$ is called* ***geodesic at*** *$t_0 \in I$, if*

$$\frac{D}{dt}(\frac{d\gamma}{dt}) = 0, \tag{1}$$

at $t_0 \in I$. If γ is geodesic at all points $t \in I$, it is said to be ***geodesic****.*
If $[a, b] \subset I$, then the restriction $\gamma\big|_{[a,b]}$ is called ***geodesic segment*** *between $\gamma(a)$ and $\gamma(b)$.*

As outlined in the beginning, the geodesic is parametrized "with constant velocity":

Lemma 1. *Let $\gamma : I \to M$ be a geodesic. Then the lenght of the tangent vecor $\left|\frac{d\gamma}{dt}\right| = \sqrt{\langle \frac{d\gamma}{dt}, \frac{d\gamma}{dt} \rangle_g}$ is constant.*

Proof.

$$\frac{d}{dt}\langle \frac{d\gamma}{dt}, \frac{d\gamma}{dt} \rangle_g = 2\langle \underbrace{\frac{D}{dt}\frac{d\gamma}{dt}}_{=0}, \frac{d\gamma}{dt} \rangle_g = 0 \tag{2}$$

□

Therefore, $\left|\frac{d\gamma}{dt}\right| = c$ for some constant c (assumed to be non-zero). Ultimately, we want to proof, that geodesics minimize the following quantity

Definition 2. *The value*

$$s(t) = \int_{t_0}^{t} \left|\frac{d\gamma(t)}{dt}\right| dt = c(t - t_0) \tag{3}$$

is called ***arc length*** *of* γ*. It is proportional to the parameter of the geodesic. By setting the value of* $c = 1$*,* γ *is said to be* ***normalized****.*

To study geodesics, it turns out to be helpful to analyse their properties in local coordinates $(U, \mathbf{x})$ and setting $\gamma(t) = (x_1(t), ..., x_n(t))$. By application of the definition above one gets

$$\frac{D}{dt}\left(\frac{d\gamma}{dt}\right) = \sum_k \left(\frac{d^2x_k}{dt^2} + \sum_{i,j} \Gamma_{ij}^k \frac{dx_i}{dt}\frac{dx_j}{dt}\right)\frac{\partial}{\partial x^k} = 0, \tag{4}$$

which is only satisfied, if

$$\boxed{\frac{d^2x_k}{dt^2} + \sum_{i,j} \Gamma_{ij}^k \frac{dx_i}{dt}\frac{dx_j}{dt} = 0,} \tag{5}$$

for all k. This ordinary, non-linear, second order Differential Equation (sometimes called "Geodesic Equation") provides use the powerful tool of the whole theory of differential equations to study further properties of geodesics, in particular their existence and uniqueness. To do that, it is convenient to transform the second order equation to a system of two first order equations by going into the tangent bundle TM. Additionally, one can compute the geodesics at least in local coordinates, by computing the Christoffel symbols and solving these equations, however this is often connected to long calculations (for the torus see e.g. http://www.rdrop.com/ half/math/torus/torus.geodesics.pdf). But first we can consider the most simple example:

- Let (M, g) be three dimensional Euklidian space together with the standard metric $(\mathbb{R}^3, g_{euklic.})$

  In local coordinates we have

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{6}$$

  and by using the expression in local coordinates for the Christoffel Symbols

$$\Gamma_{ij}^k = \frac{1}{2} \sum_k \left(\frac{\partial}{\partial x_i} g_{jk} + \frac{\partial}{\partial x_j} g_{ki} - \frac{\partial}{\partial x_k} g_{ij}\right) g^{mk} \tag{7}$$

  we see, that all Christoffel symbols vanish. Therefore the geodesic equation yields

$$\frac{d^2 x_k}{dt^2} = 0 \qquad \forall k \tag{8}$$

with the solution (considering again γ)

$$\gamma(t) = vt + p. \tag{9}$$

This can be generalized to $\mathbb{R}^n$ of course, so the geodesics defined in the above way really have the properties that we want, because in Euklidian space the straight lines are the curves that minimize arc length and are parametrized with constant velocity. The same argument would hold for Minkowskian space and $g_{Mink.}$.

If

$$\gamma : I \to U \subset M \tag{10}$$

$$t \mapsto \gamma(t) \tag{11}$$

is a geodesic, then the curve

$$\gamma : I \to TU \tag{12}$$

$$t \mapsto (x_1(t), ...x_n(t), \frac{dx_1(t)}{dt}, ..., \frac{dx_n(t)}{dt}), \tag{13}$$

satisfies the system of ordinary, first order, non-linear differential equations

$$\frac{dx_k}{dt} = y_k \tag{14}$$

$$\frac{dy_k}{dt} = -\sum_{ij} \Gamma^k_{ij} y_i y_j \tag{15}$$

for all k, where $(x_1, ..., x_n, y_1, ...y_n)$ are local coordinates on TU (remember, that if $dim(M) = n \Rightarrow dim(TM) = 2n$). With this in mind, we get the important Lemma

Lemma 2. *There exists a* ***unique*** *vectorfield $\mathcal{G} : TM \to T(TM)$ on TM, such that the integral curves of $\mathcal{G}$ are given by geodesics γ on TM.*

Proof. Define the vector field on TU for $U \subset M$ a chart, as (remember $(\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_1}, ...\frac{\partial}{\partial y_n})$ is locally a basis for TU)

$$\mathcal{G} = \sum_k \left(y_k \frac{\partial}{\partial x_k} - \Gamma^k_{ij} y_i y_j \frac{\partial}{\partial y_k} \right). \tag{16}$$

For each component, it is satisfying the system of first order equations (14) and (15), therefore the integral curves are geodesics in TU (12), so $\mathcal{G}$ exists. Since it exists, it is unique because of the uniqueness of solutions of differential equations of the type (14) and (15).

□

Definition 3. *The vector field $\mathcal{G}$ is called* ***geodesic field*** *on TM and its flow* ***geodesic flow*** *on TM.*

The next step is, upon use of a theorem from the theory of ordinary differential equations, to proof the uniqueness of the geodesics on M. Remember the theorem from the first talk

Theorem 1. *If X is a C^∞ vector field on the open set $V \subset M$ and $p \in V$ then there exists a open set $V_0 \subset V$, $p \in V_0$, a number $\delta > 0$, and a C^∞ mapping (the flow) $\varphi : (-\delta, \delta) \times V_0 \to V$, such that the curve $t \mapsto \varphi(t, q)$, $t \in (-\delta, \delta)$, is the* ***unique*** *integral curve of X which at $t = 0$ passes through q, for all $q \in V_0$.*

By use of this Theorem and setting $X = \mathcal{G}$, as well as define the composition $\gamma = \pi \circ \varphi$ of the flow and the canonical projection of the bundle, we end up with the uniqueness of geodesics globally on M at least in some intervall $I_v = (-\delta, \delta)$ and for $|v| < \epsilon$ with numbers $\delta > 0$ and $\epsilon > 0$ (for details see: do Carmo). Additionally we have the Lemma

Lemma 3 (Homogeneity of Geodesics)**.** *Let*

$$\gamma : (-\delta, \delta) \to M \tag{17}$$

$$t \mapsto \gamma(t, q, v) \tag{18}$$

be a geodesic. Then

$$\gamma : (-\frac{\delta}{a}, \frac{\delta}{a}) \to M \tag{19}$$

$$t \mapsto \gamma(t, q, av) \qquad a \in \mathbb{R}_> \tag{20}$$

is a geodesic and $\gamma(t, q, av) = \gamma(at, q, v)$.

Proof. Define $h : (-\frac{\delta}{a}, \frac{\delta}{a}) \to M$ as a curve with $t \mapsto \gamma(at, q, v)$. It is

$$h'(t) = a\gamma'(at, q, v), \qquad h(0) = q, \qquad h'(0) = av. \tag{21}$$

By applying the connection

$$\frac{D}{dt}(\frac{dh}{dt}) = \nabla_{h'(t)} h'(t) = a^2 \nabla_{\gamma'(at,q,v)} \gamma'(at, q, v) = 0 \tag{22}$$

Therefore, h is a geodesic which passes through q with velocity av at $t = 0$. It follows from the uniqueness, that $h(t) = \gamma(at, q, v) = \gamma(t, q, av)$. □

The final result can be stated as follows

Theorem 2 (Existence and Uniqueness of Geodesics)**.** *Let (M, g) be a Riemannian Manifold with Riemannian connection ∇.*
For all $p \in M$ and $v \in T_pM$ there exists an open intervall I_v with $0 \in I_v$ and a geodesic $\gamma_v : I_v \to M$, such that $\gamma_v(0) = p$ and $\frac{d\gamma_v}{dt}(0) = v$.
If $\gamma^1 : I^1 \to M$ and $\gamma^2 : I^2 \to M$ with $\gamma^1(t_0) = \gamma^2(t_0)$ and $\frac{d\gamma^1}{dt}(t_0) = \frac{d\gamma^2}{dt}(t_0)$ for some $t_0 \in I^1 \cap I^2$, then $\gamma^1(t) = \gamma^2(t)$ for all $t \in I^1 \cap I^2$.
For each $v \in TM$ there is a maximal intervall I_v and maximal geodesic $\gamma_v : I_v \to M$ with

$\frac{d\gamma_v}{dt}(0) = v$.
If $a \in \mathbb{R} - \{0\}$, $\gamma_v : I_v \to M$ and $\gamma_{av} : I_{av} \to M$, then

$$\gamma_{av}(t) = \gamma_v(at) \qquad I_{av} = \frac{1}{a} I_v \tag{23}$$

Definition 4. *If $\gamma : \mathbb{R} \to M$ is a geodesic defined on whole $\mathbb{R}$, i.e. $I_v = \mathbb{R}$, (M, g) is said to be geodesically complete.*

Remark 1. *If $f : (M, g) \to (N, g')$ is a local isometry, i.e. a local diffeomorphism, such that for two tangent vectors $v, w \in T_pM$ $\langle v, w\rangle_g = \langle df(v), df(w)\rangle_{g'}$, then*

$$\gamma : I \to M \quad \text{is a geodesic} \iff f \circ \gamma : I \to N \quad \text{is a geodesic.} \tag{24}$$

Example 1. *$M = D^2 - \{0\}$ and $N = S^1 \times [0, 1] \Big/ S^1 \times 1 = Cone$*

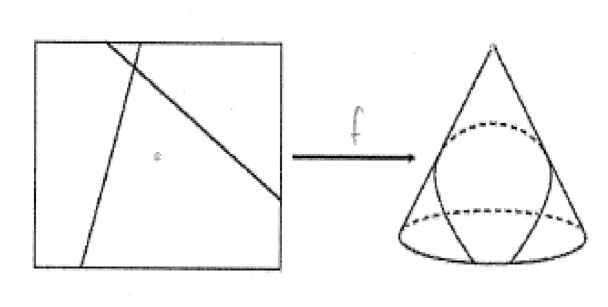

Figure 1: Geodesics on the cone; soure: Lecture notes on Differential Geometry (C.Baer)

There is another useful theorem, which provides the possibility of using isometries in order to determine geodesics globally:

Theorem 3. *Let (M, g) be a Riemannian manifold and $f \in Isom(M, g)$ be an isometry. Then is for $p \in Fix(f)$ (the fixed point set) and $v \in T_pM$ with $df(v)\big|_p = v$, the geodesic γ with starting conditions $\gamma(0) = p$ and $\frac{d}{dt}\gamma(0) = v$ completely contained in $Fix(f)$.*

Proof. Let $\tilde{\gamma}(t) := f \circ \gamma(t)$, which is also a geodesic, since f is in particular a local isometry. The initial conditions are

$$\tilde{\gamma}(0) = f(\gamma(0)) = f(p) = p = \gamma(0) \tag{25}$$

and

$$\frac{d}{dt}\tilde{\gamma}(0) = df(\frac{d}{dt}\gamma(0))\big|_{\gamma(0)} = df(v)\big|_p = v = \frac{d}{dt}\gamma(0), \tag{26}$$

so by uniqueness of geodesics we have

$$\gamma(t) = \tilde{\gamma}(t) \quad \forall t \in I. \tag{27}$$

□

Example 2. *Let $M = S^2$ with the standard metric, $p \in S^2$ and $v \in T_pS^2$ like in the picture and $E \subset \mathbb{R}^3$ like in the picture. Let A be the map, which maps every $p \in \mathbb{R}^3$ to its mirror image with respect to E. The restriction*

$$f = A\big|_{S^2} \tag{28}$$

is clearly an isometry of S^2. Since $Fix(f) = E \cap S^2$ all geodesics γ are fully contained in the great circles of S^2. After a suitable parametrization of the great circles and a look over the initial conditions, one ends up with the following expression for the geodesics on S^2

$$\gamma(t) = p \cdot cos(|v|t) + \frac{\varphi_p(v)}{|v|} \cdot sin(|v|t) \tag{29}$$

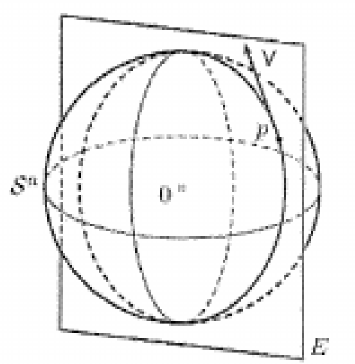

Figure 2: Geodesics on the sphere S^2; source: Lecture notes on Differential Geometry (C. Baer)

1.3 The Exponential Map

Definition 5. *Let $p \in M$ and $\mathcal{U} \subset TM$ with $|v| < \epsilon$ (so that we stay in the maximal intervall on which the geodesic is defined; this can always be achieved by homogeneity).*
The map $exp : \mathcal{U} \to M$ defined by

$$exp(p, v) = \gamma(1, p, v) = \gamma(|v|, p, \frac{v}{|v|}) \qquad (p, v) \in \mathcal{U}, \tag{30}$$

*is called **exponential map** on $\mathcal{U}$.*

It can be viewed as $\pi \circ \varphi_{t=1}$ and is therefore, as a composition of smooth maps, smooth. Intuitively, it takes a tangent vector $v \in T_pM$ on a point $p \in M$ and shifts it along the geodesic, which is uniquely determined by this initial conditions. It moves the vector for a unit "time" (the actual length is determined by the normalisation) or equivalently, by unit "speed" and a certain time, given by homogeneity. It will turn out to have very interesting properties and will be of great importance later.

Most of the time, we will consider the restriction of *exp* to an open subset of the tangent space T_pM for some point $p \in M$. Define:

$$exp_p : B_\epsilon(0) \subset T_pM \to M \tag{31}$$

$$exp_p(v) \mapsto exp(p, v), \tag{32}$$

where $B_\epsilon(0)$ is an open ball around 0 with radius ϵ. It is clear, that $exp_p(0) = p$.

Example 3. $(M, g) = (\mathbb{R}^n, g_{eukl.})$. *The exponential map is given by*

$$exp_p(v) = p + \varphi_p(v) \tag{33}$$

and is defined on the complete tangent space. However, there are also examples for which the latter is not true, e.g. $(M, g) = (\mathbb{R} - \{0\}, g_{eukl.})$.

Proposition 1. *For* $p \in M$ *there is an* $\epsilon > 0$*, s.t.* $exp_p : B_\epsilon(0) \subset T_pM \to M$ *is a diffeomorphism onto an open subset of* M

$$exp_p\big|_{B_\epsilon(0)} : B_\epsilon(0) \to exp_p(B_\epsilon(0)). \tag{34}$$

Proof. Consider the differential of *exp*:

$$d(exp_p)(v)\big|_0 = \frac{d}{dt}(exp_p(0 + tv))\big|_{t=0} = \frac{d}{dt}(\gamma(1, p, tv))\big|_{t=0} = \frac{d}{dt}(\gamma(t, p, v))\big|_{t=0} = v. \tag{35}$$

Therefore, the map $d(exp_p)\big|_0 : T_0T_pM \tilde{\to} T_pM$ is just the identity on T_pM, $d(exp_p)\big|_0 = id_{T_pM}$. The statement of the proposition follows from the inverse function theorem. □

Remark 2. *In order for the inverse function theorem to be applicable, we need* $d(exp_p)$ *to be invertible. Since this is not true in general,* exp_p *is in general* ***not*** *a diffeomorphism.*

Example 4. *Let* $M = S^2$ *with the standard metric. The exponential map is given by*

$$exp_p(v) = p \cdot cos(|v|) + \frac{\varphi_p(v)}{|v|} \cdot sin(|v|) \tag{36}$$

and for all $v \in T_pM$ *with* $|v| = \pi$ *we have*

$$exp_p(v) = p \cdot cos(\pi) = -p \tag{37}$$

where $-p$ *is the antipodal point of* p *on* S^2*. Therefore, exp is not injective and in particular no diffeomorphism.*

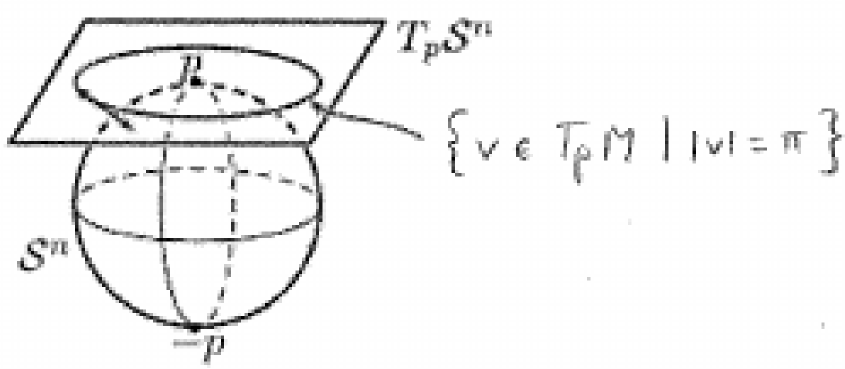

Figure 3: Exponential map on the sphere S^2; source: Lecture notes on Differential Geometry (C. Baer)

1.4 Minimizing Properties of Geodesics (Part I)

In this part we introduce the standard definitions and some tools in order to study the minimizing properties of geodesics, mentioned in the beginning. This chapter will be continued in the next talk.

Definition 6. *Let $[a, b] \subset \mathbb{R}$ be a closed intervall. A continuous mapping $c : [a, b] \to M$ is called* ***piecewise differentiable curve****, if it satisfies the condition:*
There exists a partition

$$a = t_0 < t_1 < ... < t_{k-1} < t_k = b \tag{38}$$

of $[a, b]$, s.t. the restrictions $c\big|_{[t_i,t_{i+1}]}$ are differentiable for $i = 0, ..., k-1$. The points on the curve $c(t_i)$ are called ***vertices*** *of c and the angle between $\lim_{t\nearrow t_i} c'(t)$ and $\lim_{t\searrow t_i} c'(t)$ is called the* ***vertex angle*** *at $c(t_i)$.*

This can be depicted as

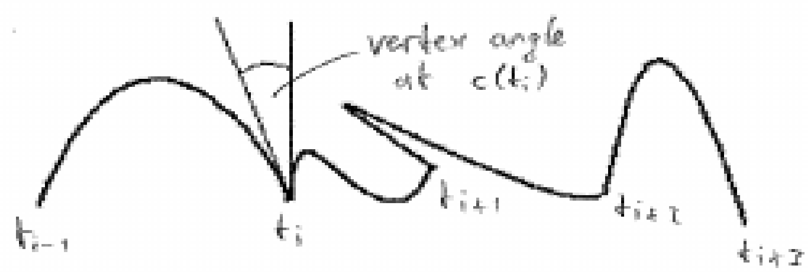

Figure 4: Piecewise differentiable curve and vertex angle

Remark 3. *The definition of geodesics and parallel transport can be easily generalized to piecewise differentiable curves, by just extending the vectorfields along the smooth segments of* c.

Definition 7. *A segment of a geodesic* $\gamma : [a, b] \to M$ *is called* ***minimizing****, if* $s(\gamma) \le s(c)$ *for each piecewise differentiable curve, joining* $\gamma(a)$ *and* $\gamma(b)$.

Definition 8. *Let* A *be a connected, closed set in* $\mathbb{R}^2$ *and* $U \subset A \subset \bar{U}$ *with* U *open in* $\mathbb{R}^2$, *s.t. the boundary* ∂A *is a piecewise differentiable curve with vertex angles different from* π.
A ***parametrized surface*** *in* M *is a differentiable mapping* $s : A \subset \mathbb{R}^2 \to M$.
A ***vectorfield along s*** *is a map* $p \in A \mapsto V(p) \in T_{s(p)}M$, *s.t.* $p \mapsto V(p) \circ f$ *is differentiable, if* f *is differentiable.*

By investigation of this setup in local coordinates, we will get a property of the covariant derivative along certain vector fields, that will be needed in the proof of the very important Gauss Lemma:

Let (u, v) be local (cartesian) coordinates on $\mathbb{R}^2$. The mapping $u \mapsto s(u, v_0)$ for fixed v_0 is a curve in M and the differential $ds(\frac{\partial}{\partial u}) := \frac{\partial s}{\partial u}$ is a vector field along the curve. This can be done for all $v_0 \in \mathbb{R}$ and therefore $\frac{\partial s}{\partial u}$ is a vectorfield along s. Likewise, we can define the vectorfield $\frac{\partial s}{\partial v}$ along s.
For these vectorfields, we define the covariant derivate $\frac{DV}{\partial u}$ (or $\frac{DV}{\partial v}$) like this:
$\frac{DV}{\partial u}(u, v_0)$ is the covariant derivative along the curve $u \mapsto s(u, v_0)$ of the restriction of some vector field along this curve. Again, this defines the covariant derivative for all $(u, v) \in A$.

The Lemma needed for the Gauss Lemma is the following:

Lemma 4 (Symmetry)**.** *Let* M *be a Riemannian manifold with a symmetric connection and* $s : A \to M$ *a parametrized curve. Then*

$$\frac{D}{\partial v}\frac{\partial s}{\partial u} = \frac{D}{\partial u}\frac{\partial s}{\partial v}. \tag{39}$$

Proof. Let $\varphi : U \subset \mathbb{R}^n \to M$ be a chart on M. We can write the local coordinates of this chart in terms of the parametrization of the surface as

$$\varphi^{-1} \circ s(u, v) = (x^1(u, v), ..., x^n(u, v)). \tag{40}$$

Application of the covariant derivative $\frac{D}{\partial v}$ to $\frac{\partial s}{\partial u}$ yields

$$\begin{aligned}\frac{D}{\partial v}\frac{\partial s}{\partial u} = \frac{D}{\partial v}\left(\sum_i \frac{\partial x^i}{\partial u}\frac{\partial}{\partial x^i}\right) &= \sum_i \frac{\partial^2 x^i}{\partial v \partial u}\frac{\partial}{\partial x^i} + \sum_i \frac{\partial x^i}{\partial u}\nabla_{\sum_j \left(\frac{\partial x^j}{\partial v}\right)\frac{\partial}{\partial x^j}}\frac{\partial}{\partial x^i} \\ &= \sum_i \frac{\partial^2 x^i}{\partial v \partial u}\frac{\partial}{\partial x^i} + \sum_{i,j}\frac{\partial x^i}{\partial u}\frac{\partial x^j}{\partial v}\nabla_{\frac{\partial}{\partial x^j}}\frac{\partial}{\partial x^i}.\end{aligned} \tag{41}$$

By using the symmetry of the connection this is equal to the similar evaluation of $\frac{D}{\partial u}\frac{\partial s}{\partial v}$.
□

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