Affine and Riemannian Connections

Seminar Riemannian Geometry
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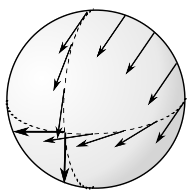
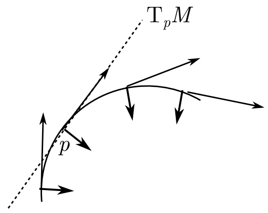
Notation:	$\mathfrak{X}(M)$	*space of smooth vector fields on M*
	$\mathfrak{D}(M)$	*space of smooth functions on M*
	$\partial_i = \frac{\partial}{\partial x^i}$	*natural basis vector fields*
	X^i	*components of the vector field* $X = \sum_i X^i \, \partial_i$

Motivation and Outline

The aim of this talk is to introduce the concept of parallel transport for Riemannian manifolds. In general, the result of a parallel transport will not only depend upon the initial and final point, but also upon the path between them. Therefore, what we are looking for is a way to locally identify vectors at different tangent spaces along a curve (a so-called connection). Conceptionally, this is related to having a way to derive vector fields along curves or, (as we will see) equivalently, with respect to vectors: We could then define a vector field to be locally constant if and only if its derivative is constantly zero.



the result of the parallel transport will in general depend upon the path



the vector obtained by component-wise differentiation will in general not lie in the appropriate tangent space

However, for a general manifold M there is a-priori no canonical way to identify vectors at different tangent spaces and at the same time there is no way to derive vector fields along curves. In the case of submanifolds of $\mathbb{R}^n$ we could just propose to derive each component individually. The problem is, that the resulting vector might not lie in the appropriate tangent space, whereas for general manifolds we do not know what non-tangent vectors shall be. We would solve the problem by projecting the resulting vector onto the tangent space, but there are infinitely many ways to project a vector onto a subspace of a vector space. However, in the case of the euclidean $\mathbb{R}^n$ there is one distinguished projection: the orthogonal projection. At the very end of this talk, we will prove that introducing a Riemannian structure indeed gives rise to a distinguished linear connection, once two natural conditions are imposed:

- One of which deals with the symmetry of the yet-to-define Christoffel symbols. It can be justified by the desire that the locally shortest lines always be straight.

- The other one has to do with the compatibility of the connection with the metric: We expect a pair of vectors to keep its scalar product constant when parallel-transported along a curve.

In order to find a definition for a linear connection, we will borrow some properties from a similar concept – the Lie derivative. We will then examine why the Lie derivative is not what we are looking for and strengthen one condition. Remember

$$\forall X, Y \in \mathfrak{X}(M): \quad L_X Y(p) \equiv [X,Y](p) = \sum_{i,j} \left(X^j \partial_j Y^i - Y^j \partial_j X^i \right)\Big|_p \partial_i(p) \quad .$$

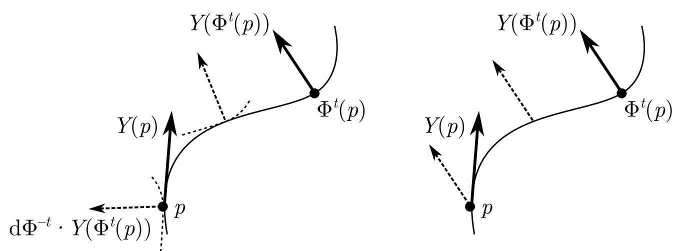
It subjects to the following properties:

- $\mathbb{R}$-linearity in X,
- $\mathbb{R}$-linearity in Y,
- Leibniz rule: $L_X(fY) = X(f)\,Y + f\,L_X Y$.

Note the occurrence of the term $\partial_j X^i$: The Lie derivative applies the differential to compare the values of Y:

$$L_X Y(p) = \lim_{t \to 0} \frac{\mathrm{d}\Phi^{-t}|_{\Phi^t(p)} \cdot Y(\Phi^t(p)) - Y(p)}{t} \quad ,$$

where $\{\Phi^t\}_t$ is the one-parameter local group of diffeomorphisms generated by X. The differential turns the vectors $Y(\Phi^t(p))$ as the representing curves are turned. But we want to carry over the vectors $Y(\Phi^t(p))$ 'unturned' by parallel-transport. Therefore, the value of the derivative $\nabla_X Y(p)$ should only depend upon $X(p)$. We express this by strengthening the first condition to $\mathfrak{D}(M)$-linearity ($\mathfrak{X}(M)$ is at the same time an $\mathbb{R}$-vector space and a $\mathfrak{D}(M)$-module).



Lie derivative: comparison by differential $\mathrm{d}\Phi^t$

covariant derivative: comparison by parallel transport

Affine connections

Definition 1. An *affine connection* ∇ on a smooth manifold M is a mapping *affine connection*

$$\begin{aligned} \nabla: \quad \mathfrak{X}(M) \times \mathfrak{X}(M) &\longrightarrow \mathfrak{X}(M) \\ (X,Y) &\longmapsto \nabla_X Y \end{aligned}$$

subject to the properties:

$$\begin{aligned} \nabla_{fX+gY} Z &= f L_X Z + g \nabla_Y Z \quad (\mathfrak{D}(M)\text{-linearity in the first argument}) \\ \nabla_X(\alpha Y + \beta Z) &= \alpha L_X Y + \beta \nabla_X Z \quad (\mathbb{R}\text{-linearity in the second argument}) \\ \nabla_X(fY) &= f \nabla_X Y + X(f)\,Y \quad (\text{Leibniz rule}) \quad , \end{aligned}$$

where $X, Y, Z \in \mathfrak{X}(M)$, $f, g \in \mathfrak{D}(M)$.

Remark 2. To an affine connection there are associated smooth functions $\Gamma^k_{ij} := (\nabla_{\partial_i}\partial_j)^k \in \mathfrak{D}(M)$, the so called Christoffel symbols. By the linearity properties, they determine the connection ∇ completely:

Christoffel symbols

$$(\nabla_X Y)^k = X(Y^k) + \sum_{i,j}\Gamma^k_{ij}X^iY^j = \sum_i\Big(\partial_j Y^k + \sum_j \Gamma^k_{ij}Y^j\Big)X^i \quad .$$

The Christoffel symbols are *not* the components of a tensor field! It can be shown[1] that under a coordinate change they obey the transformation law

$$\Gamma'^{k'}_{i'j'} = \sum_{i,j,k}\underbrace{\frac{\partial x'^{k'}}{\partial x^k}\frac{\partial x^i}{\partial x'^{i'}}\frac{\partial x^j}{\partial x'^{j'}}}_{\text{transf. law for tensors}}\Gamma^k_{ij} + \sum_l \frac{\partial x'^{k'}}{\partial x^l}\frac{\partial^2 x^l}{\partial x'^{i'}\partial x'^{j'}} \quad .$$

In the sequel, let M denote a smooth manifold with a given affine connection ∇.

Looking for a way to derive vector fields V along a curve $c : t \longmapsto c(t)$, one would like to define $\frac{DV}{dt} := \nabla_{\dot c}V$. However, globally not every vector field along c is the restriction of a vector field on M. Nevertheless, if *linearity* is required, this defines $\frac{D}{dt}$ uniquely, as by a choice of coordinates every vector field along c can be written as a *linear combination* of vector fields on M:

Proposition 3. *There is a unique way of associating to a vector field V along a differentiable curve $c\colon I \longrightarrow M$ another vector field $\frac{DV}{dt}$ along c such that*

covariant derivative

$$\frac{D}{dt}(\alpha V + \beta W) = \alpha\,\frac{DV}{dt} + \beta\,\frac{DW}{dt} \qquad (\mathbb{R}\text{-}linearity)$$

$$\frac{D}{dt}(fV) = \dot f V + f\,\frac{DV}{dt} \qquad (Leibniz\ rule)$$

$$\text{if } V(t) \equiv Y(c(t))\text{:} \quad \frac{DV}{dt} = \nabla_{\dot c}Y \quad .$$

Proof. Uniqueness: Introduce coordinates around every point of $c(I)$ and write $(c(t))^i = x^i(t)$, $V = \sum_j V^j\partial_j$ with V^j, ∂_j regarded as depending on the curve parameter t. By the above properties, write this as

$$\frac{DV}{dt} = \sum_j\Big(\dot V^j\partial_j + \sum_i \dot x^i V^j\,\nabla_{\partial_i}\partial_j\Big) \quad .$$

Use this local expression to show existence; by uniqueness, this does not depend upon the choice of coordinates. ☐

Definition 4. A vector field V along a differentiable curve $c :\, I \longrightarrow M$ is called *parallel* iff $\frac{DV}{dt} \equiv 0$.

Proposition 5. *Let $c :\, I \longrightarrow M$ be a differentiable curve, $V_0 \in T_{c(t_0)}M$. Then there exists a unique parallel vector field V along c such that $V(t_0) = V_0$; $V(t)$ is called the parallel transport of $V(t_0)$ along c.*

parallel transport

Lemma 6 (global Picard–Lindelöf theorem)**.** *E Banach space, $f : [a,b]\times E \to E$ continuous and globally Lipschitzian in the second variable; Then for each $y_0 \in E$ there exists a global solution to the Cauchy problem*

$$\dot y(t) = f(t, y(t)), \quad y(0) = y_0 \quad ,$$

and there are no further local solutions.

[1] Yvonne Choquet-Bruhat, Introduction to General Relativity, Black Holes and Cosmology, Oxford University Press 2015, p. 19

Proof of the proposition. By a compactness argument, it suffices to show existence and uniqueness within the domain of a chart. Adapt the above notation and write $V_0 = \sum_j (V_0)^j \partial_j(c(t_0))$.

To show uniqueness, suppose that there exists a V with the desired property. It follows that

$$0 = \frac{DV}{dt} = \sum_j \left(\dot{V}^j \partial_j + \sum_i V^j \dot{x}^i \nabla_{\partial_i} \partial_j \right) = \sum_k \left(\dot{V}^k + \sum_{i,j} \Gamma^k_{ij} \dot{x}^i V^j \right) \partial_k$$

$$\implies \boxed{\forall k: \quad 0 = \dot{V}^k + \sum_{i,j} \Gamma^k_{ij} \dot{x}^i V^j} \quad .$$

By the global Picard–Lindelöf theorem, this system of differential equations possesses a global unique solution which satisfies the initial conditions $V^k(t_0) = (V_0)^k$:

$$\mathbf{f}(t, \mathbf{V}) := -\sum_{i,j,k} \Gamma^k_{ij}(c(t))\, \dot{x}^i(t)\, V_k\, \mathbf{e}_k$$

is Lipschitzian in the second argument since $t \longmapsto \Gamma^k_{ij}(c(t))\, \dot{x}^i(t)$ is bounded. □

Riemannian Connections

In the sequel, M is assumed to be Riemannian.

Definition 7. ∇ is said to be *compatible* with $\langle \cdot, \cdot \rangle$ iff $\langle P, P' \rangle = \text{const}$ for any two parallel vector fields P, P' along a smooth curve c.

Proposition 8. *∇ is compatible with $\langle \cdot, \cdot \rangle$ iff for all vector fields V, W along $c : I \to M$*

$$\frac{d}{dt}\langle V, W \rangle \equiv \left\langle \frac{DV}{dt}, W \right\rangle + \left\langle V, \frac{DW}{dt} \right\rangle \quad .$$

Corollary 9. *∇ is compatible with $\langle \cdot, \cdot \rangle$ iff $\forall\, X, Y, Z \in \mathfrak{X}(M)$:*

$$X\langle Y, Z \rangle = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle \quad .$$

Proof of the proposition. $\Longleftarrow$: obvious; $\Longrightarrow$: By $t_0 \in I$ choosing an orthonormal basis $\{P_i(t_0)\}_{i=1}^n$ of $T_{c(t_0)}M$ and extending it to an orthonormal basis $\{P_i(t)\}$ of $T_{c(t)}M$ for each t, $V = \sum_i V^i P_i$ and $DV/dt = \sum_i \dot{V}^i P_i$ (the same for W) and thus

$$\frac{d}{dt}\langle V, W \rangle = \frac{d}{dt} \sum_i V^i W^i = \sum_i \left(\dot{V}^i W^i + V^i \dot{W}^i \right) = \left\langle \frac{DV}{dt}, W \right\rangle + \left\langle V, \frac{DW}{dt} \right\rangle \quad .$$

□

Definition 10. ∇ is said to be *symmetric* iff

$$\forall X, Y \in \mathfrak{X}(M) : \ \nabla_X Y - \nabla_Y X \equiv [X, Y] \quad \Longleftrightarrow \quad \Gamma^k_{ij} \equiv \Gamma^k_{ji} \quad .$$

Theorem 11 (Levi-Cività)**.** *There exists a unique linear connection ∇ on M – the* Riemannian or Levi-Cività connection *– s.t. ∇ is symmetric and compatible with $\langle \cdot, \cdot \rangle$.*

Levi-Cività connection

Proof. Uniqueness:

$$
\begin{aligned}
X\langle Y,Z\rangle &= \langle \nabla_X Y, Z\rangle + \underline{\langle Y, \nabla_X Z\rangle} \quad ,\\
Y\langle Z,X\rangle &= \underline{\underline{\langle \nabla_Y Z, X\rangle}} + \langle Z, \nabla_Y X\rangle \quad ,\\
-Z\langle X,Y\rangle &= -\underline{\langle \nabla_Z X, Y\rangle} - \underline{\underline{\langle X, \nabla_Z Y\rangle}} \quad ,\\
X\langle Y,Z\rangle + Y\langle Z,X\rangle - Z\langle X,Y\rangle &= \underline{\langle [X,Z],Y\rangle} + \underline{\underline{\langle [Y,Z],X\rangle}}\\
&\quad +\langle [X,Y],Z\rangle + 2\langle Z,\nabla_Y X\rangle \quad .
\end{aligned}
$$

It follows the *Koszul formula*:

$$
\begin{aligned}
\langle Z, \nabla_Y X\rangle &= \frac{1}{2}\{X\langle Y,Z\rangle + Y\langle Z,X\rangle - Z\langle X,Y\rangle\\
&\quad -\langle [X,Z],Y\rangle - \langle [Y,Z],X\rangle - \langle [X,Y],Z\rangle\} \quad .
\end{aligned}
$$

Existence: Use the Koszul formula as definition. □

From the Koszul formula follows

$$
\langle \partial_k, \nabla_{\partial_i}\partial_j\rangle = \frac{1}{2}\{\partial_i\langle \partial_j,\partial_k\rangle + \partial_j\langle \partial_k,\partial_i\rangle - \partial_k\langle \partial_i,\partial_j\rangle \pm \langle 0,\dots\rangle \dots\}
$$

and thus

$$
\boxed{\Gamma_{ij}^k = \frac{1}{2}\sum_k g^{km}\{\partial_i g_{jk} + \partial_j g_{ki} - \partial_k g_{ij}\}} \quad .
$$