Minimizing properties of geodesics and convex neighborhoods

Seminar Riemannian Geometry

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1. Normal spheres

In the sequal, we understand the tangent space to $T_P M$ at $v \in T_p M$ as $T_P M$ itself.

Lemma 1 (Gauss). *Let* $p \in M$, $v \in T_p M$ *s. t.* $exp_p v$ *is defined. Let* $w \in T_v T_p M$. *Then*

$$\langle (dexp_p)_v(v), (dexp_p)_v(w) \rangle = \langle v, w \rangle.$$

Proof. Recall that

$(dexp_p)_v(v): T_p M \to T_v exp_p M$, whereas, $T_v B_\varepsilon(0) \subset T_v T_p M \cong T_P M$.

We shall prove this lemma in three steps.

(i) $\langle (dexp_p)_v(v), (dexp_p)_v(v) \rangle = \langle v, v \rangle$.

(ii) $\langle (dexp_p)_v(v), (dexp_p)_v(w_N) \rangle = 0$, where $\mathrm{w} = w_T + w_N$.

(iii) $\frac{\partial}{\partial t}\langle \frac{\partial f}{\partial t}, \frac{\partial f}{\partial s} \rangle = 0$.

(i) $(dexp_p)_v(v) = v$.

Let u be a curve in $\mathrm{T_p M}$, $\mathrm{u} : \mathbb{R} \supset \mathrm{I} \to T_p\mathrm{M}$, such that $u(0) := v \in T_v T_p M \cong T_P M$.

Choose $u(t) = tv$ among its equivalent classes, namely $\tilde{v}(t) = v(t + \alpha), where\ \alpha \in \mathbb{R}$. With this construction,

$$\langle (dexp_p)_v(v), (dexp_p)_v(v) \rangle = \langle \frac{d}{dt}(exp_p(v))\Big|_{t=0}, \frac{d}{dt}(exp_p(v))\Big|_{t=0} \rangle$$
$$= \langle \frac{d}{dt}(\gamma(t,p,v))\Big|_{t=0}, \frac{d}{dt}(\gamma(t,p,v))\Big|_{t=0} \rangle = \langle v, v \rangle.$$

We now separate w into its tangential, and normal component to v.

(ii) $\langle (dexp_p)_v(v), (dexp_p)_v(w_N)\rangle = 0$

We define the curve u as $u: [-\varepsilon, \varepsilon] \times [0, 1] \to T_pM, (s,t) \mapsto tv + tsw_N$.

It follows, $u(0,1) = v$, $\frac{\partial u}{\partial t}(s,t) = v + sw_N$, $\frac{\partial u}{\partial s}(0,t) = tw_N$.

Furthermore, we consider the parametrized surface

$$f: [-\varepsilon, \varepsilon] \times [0, 1] \to M, (s,t) \mapsto exp_p(tv + tsw_N).$$

Under this construction;

$$(dexp_p)_v(v) = (dexp_p)_{u(0,1)}\left(\frac{\partial u}{\partial t}(0,1)\right) = \frac{\partial}{\partial t} exp_p \circ u(s,t)\Big|_{t=1,s=0}$$
$$= \frac{\partial f}{\partial t}(0,1),$$

$$(dexp_p)_v(w_N) = (dexp_p)_{u(0,1)}\left(\frac{\partial u}{\partial s}(0,1)\right) = \frac{\partial}{\partial s} exp_p \circ u(s,t)\Big|_{t=1,s=0}$$
$$= \frac{\partial f}{\partial s}(0,1),$$

Hence,

$$\langle (dexp_p)_v(v), (dexp_p)_v(w_N)\rangle = \langle \frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}\rangle(0,1).$$

(iii) $\frac{\partial}{\partial t}\langle \frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}\rangle\Big|_{(0,t)} = 0$

We are verifying that the above scalar product is actually independent on the variable t, so that we can conclude that $\langle \frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}\rangle(0,1) = 0$ through the property $\lim_{t\to 0}\frac{\partial f}{\partial s}(0,t) = \lim_{t\to 0}(dexp_p)_{tv}(tw_N) = 0$.

Since the map $t \mapsto f(s,t)$ is geodesic for all s and t, $\frac{\partial}{\partial t}\langle \frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}\rangle = \langle \frac{D}{\partial t}\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}\rangle + \langle \frac{\partial f}{\partial t}, \frac{D}{\partial t}\frac{\partial f}{\partial s}\rangle = \langle \frac{\partial f}{\partial t}, \frac{D}{\partial s}\frac{\partial f}{\partial t}\rangle = \frac{1}{2}\frac{\partial}{\partial s}\langle \frac{\partial f}{\partial t}, \frac{\partial f}{\partial t}\rangle$.

And since the map $t \longmapsto f(s,t)$ is geodesic, the function $t \mapsto \langle \frac{\partial f}{\partial t}, \frac{\partial f}{\partial t}\rangle$ is constant. Thus $\frac{\partial}{\partial s}\langle \frac{\partial f}{\partial t}, \frac{\partial f}{\partial t}\rangle\Big|_{s=0} = \frac{\partial}{\partial s}\langle v + sw_N, v + sw_N\rangle\Big|_{s=0} = 2\langle v, w_N\rangle = 0$.

(i),(ii), and (iii) yield us to complete the proof of the lemma with the bilinearity of the scalar product.

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It is said that $\mathrm{U} = exp_p V$ is a normal neighborhood of p, if exp_p is a diffeomorphism on a neighborhood V of the origin in $T_p M$. $B_\varepsilon(p) = exp_p B_\varepsilon(0)$, a normal ball (or a geodesic ball) if the closure of epsilon ball is in such V. Moreover, the Gauss lemma gives us that the normal sphere $S_\varepsilon(p) = \partial B_\varepsilon(p)$ is a hypersurface in the manifold M, which is orthogonal to the geodesics starting from p. Further, the geodesics in $B_\varepsilon(p)$ with starting point p are called radial geodesics.

2. The minimizing property of geodesics.

In this section we will observe that geodesics locally minimize the arc length. Namely,

Proposition 2. *Let* $p \in M, U = exp_p V, B = B_\varepsilon(p)$. *Let* $\gamma: [0,1] \to B$ *be a geodesic segment with* $\gamma(0) = p, and\ c: [0,1] \to M$ *be any piecewise differentiable curve with* $c(0) = \gamma(0), c(1) = \gamma(1)$. *Then it holds for* γ *and* c, $\ell(\gamma) \leq \ell(c)$. *The equality holds if and only if* $c([0,1]) = \gamma([0,1])$.

Proof. w.l.o.g. suppose that $\mathrm{c}([0,1]) \subset \mathrm{B}$. Otherwise cut c into parts along the boundary of B and be parametrized by [0,1]. To apply the Gauss lemma we shall find a parametrized surface $f, such\ that\ f(\gamma(t), t) = exp_p(\gamma(t) \cdot v(t)), where\ t \to v(t)\ a\ curve\ in\ T_p M, |v(t)| = 1,$ $and\ \gamma: (0,1] \to \mathbb{R}$ is a positive piecewise differentiable function. Simple calculus gives us, except for some singularities,

$$\frac{dc}{dt} = \frac{\partial f}{\partial \gamma}\frac{d\gamma}{dt} + \frac{\partial f}{\partial t}.$$

Applying the Gauss lemma yields, $\langle \frac{\partial f}{\partial \gamma}, \frac{\partial f}{\partial t} \rangle = 0$. For $\left|\frac{\partial f}{\partial \gamma}\right| = 1$,

$$\left|\frac{dc}{dt}\right|^2 = \left|\frac{d\gamma}{dt}\right|^2 + \left|\frac{\partial f}{\partial t}\right|^2 \geq \left|\frac{d\gamma}{dt}\right|^2,$$

as,

$$\int_\varepsilon^1 \left|\frac{dc}{dt}\right| dt \geq \int_\varepsilon^1 \left|\frac{d\gamma}{dt}\right| dt \geq \int_\varepsilon^1 \frac{d\gamma}{dt} dt \geq \gamma(1) - \gamma(\varepsilon).$$

Taking $\varepsilon \to 0$ shows what it was claimed. If the equality holds for some c, then $\left|\frac{\partial f}{\partial \gamma}\right| = 0$, which let $v(t)$ be constant, meaning that the curve is rescaled form of γ.

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We have now reached that there is a piecewise differentiable curve which locally minimizes the arc length. We shall further prove that this curve is a geodesic by showing the existence of normal neighborhoods.

Theorem 3. *For any p in M, there exist a neighborhood W of p, which is a normal neighborhood of each of its points. In other words,* $\forall p \in M, \exists W \subset M, \delta > 0, such\ that\ \forall q \in W, exp_q\ is\ a\ diffeomorphism\ on\ B_\delta(0) \subset T_qM\ and\ exp_q\big(B_\delta(0)\big) \supset W.$

Proof. We shall prove this theorem in following manner.

(i) Construct a local diffeomorphism

(ii) Applying the inverse function theorem. For proof of this theorem, see the script of Analysis 2, Prof. Knüpfer, SS2014.

Let $\varepsilon > 0$, $V\ a\ neighborhood\ of\ p \in M$, $\Omega = \{(q,v) \in TM; q \in V, v \in T_qM, |v| < \varepsilon\}$. Define $F: \Omega \to M \times M, (q,v) \mapsto (q, exp_q v)$. F is then a local diffeomorphism around *(p,0)*. Indeed, $(dexp_p)_0 = I, and\ the\ matrix\ dF_{(p,0)}\ is\ \begin{pmatrix} I & I \\ 0 & I \end{pmatrix}$. Thus we have a local diffeomorphism. From the inverse function theorem, there exists a neighborhood $\Omega' \subset \Omega\ of\ (p,0)$, and a neighborhood $W'of\ (p,p)$, so that $F: \Omega' \to W'$ is bijective and differentiable. By choosing $\Omega' = \{(q,v) \in TM; q \in V' \subset V, v \in T_qM, |v| < \delta\}$ and $W\ such\ that\ W \times W \subset W'$, we get the assertion.

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Remark: It is said that W is a totally normal neighborhood of $p \in M$. As we have seen that there is a unique minimizing geodesic γ with $\ell(\gamma) < \delta$, for given two points of W. By having this theorem we can conclude that there is a unique $v\ in\ T_\sigma M$, for $(\sigma, \tau)\ in\ W$, such that $\gamma'(0) = v$.

Corollary 4. *A piecewise differentiable curve* $\gamma: [a,b] \to M$ *with parameter proportional to arc length is a geodesic, if for any other piecewise differentiable curve* $c: [a', b'] \to M, c(a') = \gamma(a)\ and\ c(b') = \gamma(b)$, *holds* $\ell(\gamma) \leq \ell(c)$.

Proof. We shall prove with the compactness argument. There exists a pathwise connected compactum K, with $K^\circ \neq \emptyset$. Let W be a totally normal neighborhood of $\gamma(t)$, $t \in K$. Then the inclusion $\gamma(K) \subset W$ holds. Consider the restriction of $\gamma_K: K \to W$. Since $\gamma_K(K)$ is in a normal ball, it has equal length of a radial geodesic in W. From the minimizing property, γ_K is a geodesic. Furthermore, since it holds for all $t \in [a,b]$ that $\frac{d}{dt}\langle\frac{d\gamma}{dt}, \frac{d\gamma}{dt}\rangle = 0$, which implies $C \coloneqq \langle\frac{d\gamma}{dt}, \frac{d\gamma}{dt}\rangle = const.$ Thus, if C is non-zero, γ must be a regular curve. (If C is equal to zero, γ degenerates to a point in M.)

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In following example, we are using this corollary above, to determine the geodesics in so called the Lobatschevski plane, whilst the geodesics is being mapped to geodesics by isometries of a Riemannian manifold.

Example: Given that G is the upper half-plane in two dimensional Euclid space with the Riemannian metric $g_{11} = g_{22} = \frac{1}{y^2}$, $g_{12} = g_{21} = 0$. We are claiming that the segment $\gamma: [a,b] \to G, a > 0, \gamma(t) = (0,t)$ is the image of a geodesic under an isometry. [Indeed, for any $c: [a,b] \to G, t \mapsto \big(x(t), y(t)\big)$ $with\ c(a) = (0,a)\ and\ c(b) = (0,b),$ it holds that

$$\ell(c) = \int_a^b \left|\frac{dc}{dt}\right| dt = \int_a^b \sqrt{(\frac{dx}{dt})^2 + (\frac{dx}{dt})^2}\, \frac{dt}{y} \geq \int_a^b \left|\frac{dy}{dt}\right| \frac{dt}{y} \geq \int_a^b \frac{dy}{dt}\frac{dt}{y} \geq \int_a^b \frac{dy}{y} = \ell(\gamma).$$

Hence γ is the minimizer of arc length, by the corollary it is a geodesic.]

To this purpose, we take the m$\ddot{o}$bius transformations $\Phi_{abcd}: \mathbb{C} \cong \mathbb{R}^2 \to \mathbb{C}$ as our isometries, given by $z \mapsto \frac{az+b}{cz+d}, ad - bc = 1$. For details, see the script of Funktionentheorie, Prof. Kohnen. These transformations give us rays or half circles, which vary on the values of *a*,*b*,*c and d.*

3. Convex neighborhoods

We have observed that every point in a Riemannian manifold has a totally normal neighborhood. Still, there are cases that our geodesics are not completely lying in a certain totally normal neighborhood. In this section we will see that a totally normal neighborhood W can be chosen on a Riemannian manifold, so that W is becoming strongly convex.

Definition 5. *A subset $S \subset M$ is <u>strongly convex</u> if for any two points $q_1, q_2 \in \bar{S}$ there exists unique minimizing geodesic γ such that $(\gamma([q_1,q_2]))^\circ \subset S$.*

Lemma 6. *For any $p \in M$, there is an upper boundary c for radius r, such that any geodesic in M that is tangent at $q \in M$ to $S_r(p) = \partial B_r(p)$ stays out of the $B_r(p) = exp_p B_r(0)$.*

Proof. Let W be a totally normal neighborhood of $p \in M$. Since all geodesics in W can be considered to have the velocity one through the natural parametrization, it is enough to prove with the unit tangent bundle $T_1W = \{(q,v); q \in W, v \in T_qM, |v| = 1\}$.

Consider the differentiable mapping $\gamma: (-\varepsilon,\varepsilon) \times T_1W \to M$, where $t \to \gamma(t,q,v)$ be the geodesic with the initial condition at $t = 0, \gamma\ passes\ q = \gamma(0,q,v)$, $|v| = 1$. Let $u(t,q,v) = exp_p^{-1}(\gamma(t,q,v))$ and define $F: (-\varepsilon,\varepsilon) \times T_1W \to \mathbb{R}, F(t,q,v) = |u(t,q,v)|^2$. From the constructions follows $\frac{\partial F}{\partial t} = 2\langle\frac{\partial u}{\partial t}, u\rangle, \frac{\partial^2 F}{\partial t^2} = 2\langle\frac{\partial^2 u}{\partial t^2}, u\rangle + 2\left|\frac{\partial u}{\partial t}\right|^2$. Further, we choose our radius $r > 0$, so that $B_r(p) = exp_p B_r(0) \subset W$. Observe that if γ is tangent to $S_r(p) =$

$\partial B_r(p)$, at $q = \gamma(0,q,v)$, then $\left.\frac{\partial F}{\partial t}\right|_{(0,q,v)} = \left.\langle\frac{\partial u}{\partial t}, u\rangle\right|_{(0,q,v)} = 0$, from the Gauss lemma. We are showing that F has a strict minimum at $(0,q,v)$, for some enough small radius r. For starters we observe for $q = p$, we have $u(t,p,v) = exp_p^{-1}(\gamma(t,p,v)) = tv$, as well as $\left.\frac{\partial^2 F}{\partial t^2}\right|_{(0,p,v)} = 2\left|\frac{\partial}{\partial t}(tv)\right|^2 = 2|v|^2 = 2 > 0$. Thus there exist a neighborhood $\Omega' \subset$ W of p such that $\left.\frac{\partial^2 F}{\partial t^2}\right|_{(0,q,v)} > 0, for\ all\ q \in \Omega', v \in T_qM, |v| = 1$. Now choose an *upper boundary* c *s.t.* $exp_p B_c(0) \subset \Omega'$. Since a strict minimum gives us the minimizer for distance between p and to a point p_γ that is moving along the geodesic γ, with minimizing property there is another geodesic in $B_c(p) = exp_p B_c(0)$, joining p and p_γ, which is tangent to $S_r(p) = \partial B_r(p)\ at\ q = \gamma(0,q,v)$, as claimed.

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Proposition 7. (Convex neighborhoods). *For any* $p \in M$, *there is a number* $\beta > 0$ *such that the normal ball* $B_\beta(p)$ *is strongly convex.*

Proof. Let c be the upper boundary given in the above lemma for a $p \in M$. Choose $\frac{c}{2} > \delta > 0$ and $W\ s.t.\ \forall q \in W,\ exp_q\ is\ a\ diffeomorphism\ on\ B_\delta(0) \subset T_qM\ and\ exp_q(B_\delta(0)) \supset W$. We shall prove that $B_\beta(p)$ is strongly convex for $\beta < \delta : B_\beta(p) \subset W$. Given that two random points $q_1, q_2 \in \overline{B_\beta(p)}$, let γ be the geodesic with $\ell(\gamma) < 2\delta < c$, joining $q_1\ and\ q_2$. Assume that $(\gamma([q_1, q_2]))^\circ \not\subset B_\beta(p)$ i.e. $B_\beta(p)$ is not strongly convex. Then there is a point x in $(\gamma([q_1, q_2]))^\circ$, with the maximum distance r from q. Hence for enough small $\varepsilon > 0, B_\varepsilon(x) \cap \overline{B_r(p)} \neq \emptyset$, which is contradiction to the lemma 6.

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