Seminar: ”Gemoetric Structures on Manifolds”
Orbifolds

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1 Introduction

Orbifolds can be viewed as a slight generalisation of manifolds. Where manifolds often arise as the quotient of a space X by a group G acting *freely* (so where for any compact subset K of X, $\{g \in G | gK \cap K\}$ is trivial), orbifolds tend to arise as the quotient by a group acting *properly discontinuously* (where $\{g \in G | gK \cap K\}$ is instead only required to be finite).

This will allow us, for example, to talk about the quotient of $\mathbb{R}^2$ by a triangle group. This is a group of isometries generated by reflections in the sides of a triangle which tiles the plane. It is fairly easy to show that if Δ_{n_1,n_2,n_3} is such a triangle with angles $\frac{\pi}{m_1}, \frac{\pi}{m_2}, \frac{\pi}{m_3}$ where the m_i are integers, then the m_i are one of the triplets $(3,3,3), (2,3,6)$ or $(2,4,4)$. In some sense these quotients are covered by $\mathbb{R}^2$, and later we will make sense of this notion.
The triple m_1, m_2, m_3 was constrained by the relation $\frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} = 1$. When we look at triples satisfying $\frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} > 1$, it is possible to find corresponding tilings of S^2 by congruent triangles. Similary, tilings of H^2 can be found when

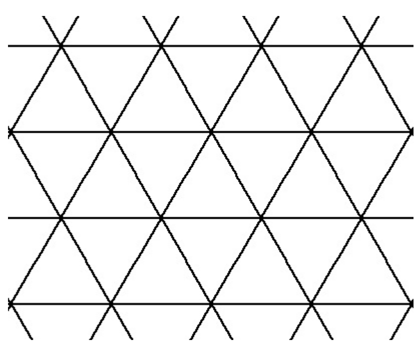


Figure 1: A tiling of $\mathbb{R}^2$ by $\Delta_{3,3,3}$

$\frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} < 1$. In both of these cases, it is once again possible to quotient the appropriate space by a group generated by reflections in the sides of the triangles.

Looking at simpler isometries of the plane, we can get a firmer notion of what orbifolds look like. Consider a group G generated by a rotation about the origin by an angle of $2\pi/n$ for integer n. G certainly acts properly discontinuously, and the space obtained by quotienting by G looks much like a cone. We can induce a metric on this space by lifting paths back up to $\mathbb{R}^2$ and defining the distance between two points as the infimum of lengths of paths between them. Note that while paths passing through the origin may not have unique lifts, this length is still well defined. The space $\mathbb{R}^2/G$ is homeomorphic to $\mathbb{R}^2$, but this unusual geometric behaviour about the origin should not be forgotten, and it is often important to distinguish between orbifolds and their underlying spaces.

Another simple space to understand is the quotient of $\mathbb{R}^2$ by a reflection. Here we observe a line of points that behave unusually, and that topologically this space is a surface with boundary. As we shall see, this orbifold doesn't have a boundary.

We can consider a more general finite subgroup of $O(2)$: D_{2m}, the dihedral group generated by reflections in lines an angle $2\pi/n$ apart. Quotienting by this group gives a space with properties similar to the last example, but here there are two lines of 'unusual' points.

These three examples will turn out to be very important, as they classify most of the non-manifold like behaviour of 2-dimensional orbifolds. With the examples in mind, we now give a rigourous definition of orbifolds.

Definition 1.1 *An orbifold $\mathcal{O}$ is a Hausdorff space $X_{\mathcal{O}}$ (called the underlying space), together with a covering by open sets U_i closed under finite intersections. For each U_i there must exist an open set $\tilde{U}_i$ in $\mathbb{R}^n$ acted upon equivariantly by some finite group Γ_i such that there exists a homeomorphism $\varphi : \tilde{U}_i/\Gamma_i \to U_i$, called a chart.*

For each $U_i \subseteq U_j$, we require an injective homomorphism $f_{ij} : \Gamma_i \to \Gamma_j$ and an embedding $\tilde{\varphi_{ij}} : \tilde{U}_i \to \tilde{U}_j$, invariant under action by Γ_i, such that the following diagram commutes

$$
\begin{tikzcd}
\tilde{U}_i \arrow[rr, "\tilde{\varphi}_{ij}"] \arrow[d] & & \tilde{U}_j \arrow[d] \\
\tilde{U}_i/\Gamma_i \arrow[rr, "\varphi_{ij} = \tilde{\varphi}_{ij}/\Gamma_i"] & & \tilde{U}_j/\Gamma_i \arrow[d, "f_{ij}"] \\
 & & \tilde{U}_j/\Gamma_j \\
U_i \arrow[uu, "\varphi_i"] & \subset & U_j \arrow[u, "\varphi_j"']
\end{tikzcd}
$$

By imposing conditions on the gluing maps we can define differentiable, smooth and other categories of orbifolds.

In the introduction, we observed points with unusual behaviour on orbifolds, and it will be very useful to give a formal definition of these.

Definition 1.2 *Let $\mathcal{O}$ be an orbifold with underlying space $X_{\mathcal{O}}$. For any point x in $\mathcal{O}$, take a coordinate system $\psi : U \to \tilde{U}/\Gamma$. Let Γ_x be the stabiliser of x in Γ (sometimes called the* local group *of x). It is easy to see the Γ_x is well-defined up to isomorphism. Let $\Sigma_{\mathcal{O}} = \{x \in X_{\mathcal{O}} | \Gamma_x \neq \{1\}\}$, called the singular locus of $\mathcal{O}$.*

An orbifold with empty singular locus is called a manifold.

Note that the singular locus is closed, since every intersection of it with the coordinate maps is closed.

An orbifold is said to be connected when its underlying surface is, and compact when its underlying surface is.

With these definitions in place, we briefly revisit the previous examples. The cone like behaviour of quotienting by a rotation group can be quite simplt put on a compact orbifold by taking an atlas for the sphere as a manifold and replacing one of the charts with a map from the quotient of an open disc by a rotation group. The resulting space is called the teardrop orbifold, as seen in figure 2. This has singular locus of a single

Figure 2: The teardrop orbifold

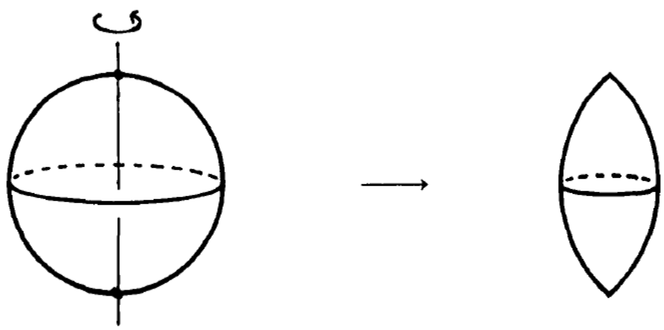


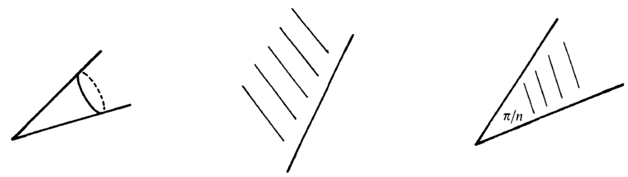
Figure 3: The quotient of S^2 by a finite cyclic rotation group

point, with local group isomorphic to $\mathbb{Z}_n$ for some n. Such a point is called a cone point of order n.

We may similarly add multiple cone points to S^2. In particular, the orbifold with underlying surface S^2 and two cone points of different order is called a spindle orbifold. When the orders are the same the orbifold can be seen as a direct quotient of S^2 by a group generated by a rotation of order n, as seen in figure **??**.

We classify other types of singularity similarly. Since dihedral groups and their subgroups are the only finite subgroups of $O(3)$, the only types of singularity on 2-orbifolds are those arising from discrete rotation groups, a single reflection, or at points where the lines of two reflections meet. These singularities are called cone points, reflector lines and corner reflectors respectively.

Orbifolds with reflector lines are easy to construct. Any manifold with boundary can be viewed instead as an orbifold by replacing appropriate charts. Perhaps the simplest



(a) (b) (c)

Figure 4: The three types of singularity on 2-orbifolds

such construction is for D^2, which can also be viewed as the quotient of S^2 by a reflection in a great circle.

A good example of corner reflectors are the triangle groups we have already seen. The corners of a triangle are corner reflectors, while the sides are reflector lines. If a point x in the singular locus $X_{\mathcal{O}}$ of an orbifold has local group isomorphic to D_{2n} for some n, it is called a corner reflector of order n.

We can construct new orbifolds by identifying the boundaries (of the underlying surfaces of) two copies of a triangle orbifold. It's not hard to see that a corner reflector of order n in the original triangle group will become a cone point of order n in the new orbifold.

2 Coverings

As with manifolds, it will be important to consider possible coverings of orbifolds. The classical notion of covering is, however, too restrictive for working with orbifolds. For example, we would like the orbifold $S^2/\mathbb{Z}_2$, where $\mathbb{Z}_2$ acts on S^2 by reflection in a great circle, to be covered by S^2. With the classical notion of covering this is not the case, since no open neighbourhood of a point on the reflector line in $S^2/\mathbb{Z}_2$ is homeomorphic to an open set in S^2.

Definition 2.1 *Let $\mathcal{O}$ be an orbifold. An orbifold cover of $\mathcal{O}$ is an orbifold $\mathcal{O}$ together with a projection $p : X_{\tilde{\mathcal{O}}} \to X_{\mathcal{O}}$ on the underlying spaces such that every x in $X_{\mathcal{O}}$ has a neighbourhood U isometric to $\tilde{U}/\Gamma$ such that each component V_i of $p^{-1}(U)$ is homeomorphic to $\tilde{U}/\Gamma_i$, where $\Gamma_i \leq \Gamma$.*

In most of our discussions of orbifold coverings, we will assume any covering is connected.

When Γ acts properly discontinuously on an orbifold $\mathcal{O}$, $\mathcal{O}$ covers $\mathcal{O}/\Gamma$. This agrees with our earlier notion of the covering of a triangle group by $\mathbb{R}^2$ Any 2-orbifold $\mathcal{O}$ whose singular locus is composed entirely of reflector lines is covered by $m\mathcal{O}$, the manifold given by gluing a second copy of $\mathcal{O}$ along the reflector lines in the obvious way. More generally, any orbifold with relfector lines has a proper covering.

Recall the teardrop given by S^2 with a single cone point. As we shall see later, the teardrop has no proper coverings and thus cannot be covered by a manifold.

Definition 2.2 *An orbifold which can be covered by a manifold is called good, and is otherwise called bad.*

Definition 2.3 *A universal covering of an orbifold $\mathcal{O}$ is an orbifold covering $p : \tilde{\mathcal{O}} \to \mathcal{O}$ such that gven any other orbifold covering $\tilde{\mathcal{O}}'$, there exists an orbifold covering $q : \tilde{\mathcal{O}} \to \mathcal{O}$ with $p' \circ q = p$.*

It is possible to copy the classical approaches to universal covers and fundamental groups. To do this, we would need to refine the notions of paths and homotopies, or else we can only obtain the fundamental group of the underlying structure.

Definition 2.4 *An orbifold path is a continuous map $\alpha : I \to \mathcal{O}$ such that there are only finitely many singular points on α, and for each t such that $\alpha(t)$ is singular there is a triple (φ, V, l). Here, $\varphi : \tilde{U}/\Gamma \to U$ is a chart, V is a neighbourhood of t in I such that $\forall u \in V - \{t\}$, $\alpha(u)$ is non-singular and lies in U, and l is a lift of $\alpha|_V$ to $\tilde{U}$.*

Definition 2.5 *Let α be a path in an orbifold $\mathcal{O}$, U be an open subset and $\varphi : \tilde{U}/\Gamma \to U$ a chart. For $[a, b]$ a subinterval of I with image contained in U, let β be a lift of $\alpha|_{[a,b]}$ to $\tilde{U}$. Replacing $\alpha|_{[a,b]}$ by the projection of any path in $\tilde{U}$ which is homotopic to β (relative to the endpoints) is called an elementary homotopy of α. An orbifold homotopy of paths is a collection of elementary homotopies on an orbifold path α.*

With these defined, it is possible to carefully define the fundamental group as the group of *orbifold* homotopy classes of orbifold paths, and the universal cover through the space of orbifold homotopy classes of orbifold paths with a fixed starting point. See [3] for a more thorough discussion of this approach. In dimension 2 there is a simpler method, outlined below.

Proposition 2.1 *Every 2-orbifold $\mathcal{O}$ has a universal cover.*

Proof As remarked earlier, every orbifold $\mathcal{O}$ with reflector lines is covered by $m\mathcal{O}$, so we need only show the result for orbifolds whose singular locus is a set of cone points.

Let $\mathcal{O}$ be a 2- orbifold whose singular locus $\Sigma_{\mathcal{O}}$ consists only of cone points, and N be the surface given by removing small discs around each cone point of $\mathcal{O}$. For any covering, $\tilde{N} = p^{-1}(N)$ is a covering in the classical sense.

Take a circle C_i in the boundary of N, which bounds a cone with cone angle $2\pi/n_i$. Since $\tilde{\mathcal{O}}$ was an orbifold covering, a component of the preimage of C_i must then be an

m_i-sheeted covering of C_i, where $m_i|n_i$. If α_i is an element of $\pi_1(N)$ with representative the circle C_i, then $\pi_1(\tilde{N})$ must contain lifts of all conjugates of $\alpha_i^{n_i}$. If we let G be the group given by adding the relations $\alpha_i^{n_i}$ to $\pi_1(N)$, then by the above argument $\pi_1(\tilde{N})$ contains the kernel K of the natural map $\pi_1(N) \to G$.

Thus the covering N_K corresponding to the kernel is universal amongst coverings of N which extend to orbifold coverings of $\mathcal{O}$. The orbifold covering $\mathcal{O}_K$ given by adding the appropriate cones back to N_K must then be universal. $\square$

Definition 2.6 *The orbifold fundamental group of an orbifold $\mathcal{O}$ is the group of deck transformations of its universal cover, written $\pi_1^{orb}(\mathcal{O})$.*

The construction used in the proof of proposition 2.1 can be very useful in calculating the fundamental group of an orbifold, using the orbifold equivalent of the Seifert-van Kampen theorem. The group G constructed is in fact the fundamental group of an orbifold with no reflector lines, and in particular this shows that this definition of fundamental group truly extends the one for manifolds - if the singular locus is empty then $G = \pi_1(N)$.

For calculating fundamental groups of orbifolds that may have reflector lines, a similar approach is used. Remove a small neighbourhood of each component of the singular locus, and use the orbifold Seifert-van Kampen theorem to build up the orbifold fundamental group from $\pi_1(\Sigma_{\mathcal{O}})$ and the fundamental groups of basic orbifolds.

We've already seen how to treat cone points: the fundamental group of a neighbourhood of a cone point is isomorphic to $\mathbb{Z}_n$ where n is the order of the cone point.On a compact orbifold without boundary, the only other possible connected component in the singular locus is a circle lying in $\partial\Sigma_{\mathcal{O}}$. A closed neighbourhood of such a circle has underlying surface homeomorphic to $S^1 \times I$ and orbifold fundamental group isomorphic to $\mathbb{Z} \times \mathbb{Z}_2$. On non-compact manifolds, there is a third type of component with underlying surface $\mathbb{R} \times I$ and orbifold fundamental group isomorphic to $\mathbb{Z}_2$. The fundamental groups of each of these types of components can be demonstrated by careful use of the definitions of orbifold paths and homotopies.

With this approach, it is not too hard to see that if $\mathcal{O}$ is an orbifold with underlying surface a genus g torus and with cone points of orders $n_1, \ldots, n_l$ (noting that there can only ever be finitely many cone points, since $\Sigma_{\mathcal{O}}$ is compact), then $\pi_1^{orb}(\mathcal{O})$ has presentation

$$\left\{ a_1, b_1, \ldots, a_g, b_g, x_1, \ldots, x_l : x_i^{n_i} = 1, \prod_{i=1}^{g} [a_i, b_i] x_1, \ldots, x_l \right\}.$$

Another simple example is the orbifold $\mathcal{O}$ with underlying surface X a torus with some open disc removed with the boundary ∂X a reflector circle. Then by the Seifert-van Kampen theorem with the orbifold split into two parts as in figure 5,

$$\pi_1^{orb}(\mathcal{O}) \cong \pi_1(X) *_{\mathbb{Z}} (\mathbb{Z} * \mathbb{Z}_2) \cong \pi_1(X) * \mathbb{Z}_2.$$

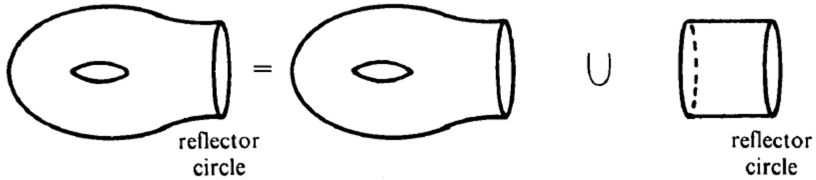


Figure 5: Decomposition of a torus with a reflector circle

More significantly to our classification of 2-orbifolds, we can now calculate the orbifold fundamental group of a teardrop or spindle orbifold. The fundamental group of the teardrop is easily seen to be trivial when we observe that any loop not passing through the cone point is null homotopic. An argument using Seifert-van Kampen is not much more complex. For the spindle, we use the Seifert-van Kampen theorem to obtain the representation $\{x : x^{n_1} = x^{n_2} = 1\}$ and observe that since n_1 and n_2 are coprime, the group is trivial. This gives us a proof that these two orbifolds are bad, and indeed that any orbifolds covered by them must also be bad (namely those with underlying surface D^2 and one corner reflector, or two corner reflectors with distinct orders). In fact, we can show that these are the only bad 2-orbifolds.

Proposition 2.2 *A bad 2-orbifold is of one of the following types:*

Underlying surface	*Cone points*	*Corner reflectors*
S^2	$(n), n \neq 1$	-
S^2	$(n_1, n_2), n_1 \neq n_2$	-
D^2	-	$(n), n \neq 1$
D^2	-	$(n_1, n_2), n_1 \neq n_2$

Proof As we have seen, each of these orbifolds are bad. It remains to show that there are no other bad 2-orbifolds. Suppose $\mathcal{O}$ is a bad orbifold with no proper orbifold coverings. Clearly it can have no reflector curves, or else $m\mathcal{O}$ would be a proper covering. So $X_{\mathcal{O}}$ is a surface without boundary. If $X_{\mathcal{O}}$ weren't simply connected, a classical proper covering of it would induce a proper orbifold cover of $\mathcal{O}$. Therefore $X_{\mathcal{O}}$ must be S^2 or $\mathbb{R}^2$.

Suppose for contradiction that $\mathcal{O}$ has at least three cone points. Then there is a subset U of $X_{\mathcal{O}}$ homeomorphic to D^2 and containing exactly three cone points, of orders n_1, n_2, n_3, say. As we showed in our discussion of triangle groups on $\mathbb{R}^2$, S^2 and H^2, the orbifold $\mathcal{O}'$ with cone points of orders n_1, n_2, n_3 and underlying surface S^2 has a proper covering by one of $\mathbb{R}^2$, S^2 and H^2, which depends on the sum $\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3}$. If we remove a disc containing no cone points from $\mathcal{O}'$ we obtain a space homeomorphic to U and with the same cone points. Taking the preimage of this space under the covering map gives a proper orbifold cover of it, and thus a proper orbifold covering of U. We may then glue a copy of $\mathcal{O}\backslash U$ to each boundary component of this cover to obtain a proper orbifold covering of $\mathcal{O}$. Thus $\mathcal{O}$ must have at most two cone points.

Noting that if $\mathcal{O}$ had underlying surface $\mathbb{R}^2$ then it would be easy to find a proper covering of it, we conclude that $\mathcal{O}$ must have underlying surface S^2 and one or two cone points. If $\mathcal{O}$ had two cone points with the same orders n, then it would be the quotient of S^2 by a finite cyclic group of rotations. Thus $\mathcal{O}$ must be of one of the types listed in the statement of the proposition. □

3 Euler Characteristics on Orbifolds

The Euler characteristic $\chi(X)$ of a triangulable space X is a very useful topological invariant. We'll use it to motivate a similar definition for orbifolds, which will allow us to further classify compact 2-orbifolds. Recall that the Euler characteristic of a triangulable space can be defined as $\chi(X) = \sum_{\sigma_i}(-1)^{dim(\sigma_i)}$, where X is triangulated by some finite simplicial complex with simplices σ_i.

Definition 3.1 *Let $\mathcal{O}$ be a compact orbifold triangulated by a finite simplicial complex such that Γ_x is constant (up to isomorphism) on the images of interiors of simplices. Write $\Gamma(\sigma)$ for the group on the interior of a simplex σ. Then let*

$$\chi^{orb}(\mathcal{O}) := \sum_{\sigma_i}(-1)^{dim(\sigma_i)}\frac{1}{\Gamma(\sigma_i)}.$$

Since this definition clearly extends that of the classical Euler characteristic, it will not always be neccessary to distinguish between the two and so χ will be written for both. It is not immediately obvious that the orbifold Euler characteristic is independent of the triangulation. To demonstrate this fact for 2-orbifolds, we show that χ can be expressed in a triangulation independent manner.

Proposition 3.1

$$\chi^{orb}(\mathcal{O}) = \chi(X_{\mathcal{O}}) - \sum_{i=1}^{r}\left(1-\frac{1}{n_i}\right) - \frac{1}{2}\sum_{j=1}^{s}\left(1-\frac{1}{m_j}\right),$$

where $\mathcal{O}$ has r cone points $u_1,\ldots,u_r$ with orders $n_1,\ldots,n_r$ and s reflector corners $v_1,\ldots,v_s$ with orders $m_1,\ldots,m_s$.

Proof Let $\sigma_1,\ldots,\sigma_k$ be a suitable triangulation of $\mathcal{O}$. Then we have

$$\chi(X_{\mathcal{O}}) - \chi^{orb}(\mathcal{O}) = \sum_{\sigma_i}(-1)^{dim(\sigma_i)}\left(1-\frac{1}{|\Gamma(\sigma_i)|}\right).$$

The terms of the sum corresponding to simplices σ_i where $\Gamma(\sigma_i) = \{1\}$ vanish, and we are left with a calculation on each connected component of $\Sigma_{\mathcal{O}}$.

For a cone point u_i which lies in the image of some 0-simplex σ, the corresponding contribution to the sum is simply $1 - \frac{1}{|\Gamma(\sigma)|} = 1 - \frac{1}{n_i}$.

A connected component corresponding to a reflector circle in $\Sigma_{\mathcal{O}}$ is triangulated by a cycle of 1-simplices. The points corresponding to 0-simplices in this cycle have local group isomorphic to either $\mathbb{Z}_2$ or D_{2m} for some m, so contribute $1 - 1/2$ or $1 - 1/(2m) = (1 - 1/2) + (1/2)(1 - 1/m)$ respectively. If the cycle has l 1-simplices and covers the corner reflectors $v_{j_1}, \ldots, v_{j_p}$, then the overall contribution to $\chi(X_{\mathcal{O}}) - \chi^{orb}(\mathcal{O})$ is

$$-\sum_{i=1}^{l}\left(1-\frac{1}{2}\right)+\sum_{i=1}^{l-p}\left(1-\frac{1}{2}\right)+\sum_{i=1}^{p}\left(\left(1-\frac{1}{2}\right)+\frac{1}{2}\left(1-\frac{1}{m_{j_i}}\right)\right)$$

$$=\sum_{i=1}^{p}\frac{1}{2}\left(1-\frac{1}{m_{j_i}}\right).$$

Putting these pieces together, we obtain the desired formula

$$\chi^{orb}(\mathcal{O}) = \chi(X_{\mathcal{O}}) - \sum_{i=1}^{r}\left(1-\frac{1}{n_i}\right) - \frac{1}{2}\sum_{j=1}^{s}\left(1-\frac{1}{m_j}\right).$$

□

We are now in a position to prove a very useful result, which follows immediately from the following lemma.

Lemma 3.1 *Let $\tilde{\mathcal{O}} \to \mathcal{O}$ be an orbifold covering. Then the number of sheets of the cover (defined as the number of preimages of a non-singular point) is given by*

$$\sum_{\tilde{x}|p(\tilde{x})=x}\frac{|\Gamma_x|}{|\Gamma_{\tilde{x}}|}$$

for any point x in $\mathcal{O}$.

Proof The result is trivially true when x is non-singular. Consider a singular point x. Then there exists a coordinate system $\psi : U \to \tilde{U}/\Gamma$ such that $x \in U$. Take some non-singular $y \in U$. For each $\tilde{x}$ in the fiber of x, y has $|\Gamma_x|/|\Gamma_{\tilde{x}}|$ preimages. Since y is non-singular it has the same number of preimages as the number of sheets of the cover, and the result follows. □

Proposition 3.2 *Let $\tilde{\mathcal{O}} \to \mathcal{O}$ be a k-sheeted orbifold covering. Then*

$$\chi^{orb}(\tilde{\mathcal{O}}) = k\chi^{orb}(\mathcal{O}).$$

This allows us to prove some interesting results. For example, the Euler characteristic of a triangle orbifold Δ_{n_1,n_2,n_3} is $-\frac{1}{2}\left(1 - \frac{1}{n_i}\sum\right)$. Thus $\chi(\Delta_{2,3,5}) = 1/60$, so it can be covered by neither $\mathbb{R}^2$ nor H^2. Similarly $\chi(\Delta_{2,3,6}) = 0$ and $\chi(\Delta_{2,3,7}) = -1/84$ show that each of these triangle orbifolds can have only one type of gemoetric structure (parabolic and hyperbolic respectively).

Proposition 3.2 also leads to an alternative proof that the teardrop and spindle orbifolds are bad. It is possible to define Riemannian metrics on orbifolds away from their singularities by defining a Γ invariant metric on $\tilde{U}$ for each chart $\phi : U \to \tilde{U}/\Gamma$, with certain gluing conditions. Then the orbifolds in question cna be shown to have Riemannian metrics of strictly positive curvature, and thus any covering manifold must be compact. Since their Euler characteristic is one of $1 + 1/n$ or $1/n_1 + 1/n_2$, any manifold covering them must have Euler characteristic strictly greater than 2.

We now also have the tools to prove most of the theorem completely classifying closed 2-orbifolds.

Theorem 3.2 *Let $\mathcal{O}$ be a closed 2-orbifold. Then $\mathcal{O}$ has an elliptic, parabolic or hyperbolic structure iff it is good. $\mathcal{O}$ has a hyperbolic structure iff $\chi(\mathcal{O}) < 0$, a parabolic structure iff $\chi(\mathcal{O}) = 0$ and is bad or has an elliptic structure iff $chi(\mathcal{O}) > 0$.*

The equation given in proposition 3.1 can be used to show there are only finitely many families of elliptic and parabolic orbifolds, so it is possible to explicitly demonstrate the coverings by S^2 or $\mathbb{R}^2$. The hyperbolic orbifolds require a rather more technical approach. See [2] theorem 13.3.6 for the details, as well as a tabulation of the families of bad, elliptic and parabolic orbifolds.

References

[1] Peter Scott, *The Geometries of 3-Manifolds*, Bull. London Math. Soc., 1983

[2] William P. Thurston, *The Geometry and Topology of Three-Manifolds*, msri.org/publications/books/gt3m, 2002

[3] Michel Boileau, Sylvain Maillot, and Joan Porti, *Three-Dimensional Orbifolds and their Geometric Structures*, Panoramas et Syntheses, 2003