0 Review of Differential Geometry

Definition 1 Let $\mathcal{M}$ be a smooth manifold and $p \in \mathcal{M}$ be an arbitrary point in $\mathcal{M}$. A linear map $X : \mathcal{C}^\infty(\mathcal{M}) \to \mathbb{R}$ is called a *derivation* at $p \in \mathcal{M}$ if and only if:

$$\forall f, g \in \mathcal{C}^\infty(\mathcal{M}) : \quad X(fg) = f(p)Xg + g(p)Xf$$

The set of all derivations at $p \in \mathcal{M}$ is called the tangent space to $\mathcal{M}$ at p and is denoted by $T_p\mathcal{M}$.

Definition 2 Let $\mathcal{M}$ and $\mathcal{N}$ be two smooth manifolds and $F : \mathcal{M} \to \mathcal{N}$ be a smooth map. Then define for each $p \in \mathcal{M}$ the differential of F in the following way:

$$dF_p : T_p\mathcal{M} \to T_{F(p)}\mathcal{N}, \quad X \mapsto dF_p(X)$$

whereby we have $\forall f \in \mathcal{C}^\infty(\mathcal{N})$, $dF_p(X)(f) := X(f \circ F)$. Note that $X(f \circ F)$ is well defined because $\mathcal{M} \xrightarrow{F} \mathcal{N} \xrightarrow{f} \mathbb{R} \implies f \circ F \in \mathcal{C}^\infty(\mathcal{M})$

Proposition 3 Let $\mathcal{M}$, $\mathcal{N}$ and $\mathcal{P}$ be smooth manifolds and let $F : \mathcal{M} \to \mathcal{N}$ and $G : \mathcal{N} \to \mathcal{P}$ be smooth maps. We have the flowing properties for all $p \in \mathcal{M}$

(a) $dF_p : T_p\mathcal{M} \to T_{F(p)}\mathcal{N}$ is linear.

(b) $d(G \circ F)_p = dG_{F(p)} \circ dF_p : T_p\mathcal{M} \to T_{G\circ F(p)}\mathcal{P}$

(c) If F is a diffeomorphism, then dF_p is an isomorphism and we have for the inverse $(dF_p)^{-1} = d(F^{-1})_{F(p)}$

Definition 4 Let $F : \mathcal{M} \to \mathcal{N}$ be a smooth map. We define the rank of F at point $p \in \mathcal{M}$ to be the rank of the linear map $dF_p : T_p\mathcal{M} \to T_{F(p)}\mathcal{N}$. We say that F has a constant rank if the rank of F is the same for all $p \in \mathcal{M}$.

Definition 5 A smooth map $F : \mathcal{M} \to \mathcal{N}$ is called

(a) a submersion if and only if $\forall p \in \mathcal{M} : \operatorname{rank} F = \dim \mathcal{N}$ if and only if dF_p is surjective $\forall p \in \mathcal{M}$

(b) an immersion if and only if $\forall p \in \mathcal{M} : \operatorname{rank} F = \dim \mathcal{M}$ if and only if dF_p is injective $\forall p \in \mathcal{M}$

(c) an embedding if and only if F is a smooth immersion and a topolgical embedding. ie. $F : \mathcal{M} \to F(\mathcal{M})$ is a homeomorphism if $F(\mathcal{M}) \subseteq \mathcal{N}$ is endowed with the subspace topology.

Theorem 6 (Global Rank Theorem) *Let $F : \mathcal{M} \to \mathcal{N}$ be a smooth map of constant rank.*

(a) If F is surjective, then F is a submersion.

(b) If F is injective, then F is an immersion.

(c) If F is bijective, then F is a diffeomorphism.

Definition 7 Let $\mathcal{M}$ be a smooth manifold. We define the subset $\mathcal{N} \subseteq \mathcal{M}$ to be an embedded submanifold of $\mathcal{M}$ if and only if $\mathcal{N}$ has the subspace topology with a smooth structure such that the inclusion map $\mathcal{N} \hookrightarrow \mathcal{M}$ is a smooth embedding.

Definition 8 Let $\mathcal{M}$ be a smooth manifold. We define the immersed submanifold $\mathcal{N} \subseteq \mathcal{M}$ if and only if

(i) $\mathcal{N}$ has a topological structure such that $\mathcal{N}$ is a topological manifold

(ii) $\mathcal{N}$ has a smooth structure such that the inclusion map $\mathcal{N} \hookrightarrow \mathcal{M}$ is a smooth immersion.

Lemma 9 Let $\mathcal{M}$ be a smooth manifold and $\mathcal{S} \subseteq \mathcal{M}$ an embedded submanifold of $\mathcal{M}$. Then every smooth map $F : \mathcal{N} \to \mathcal{M}$ with the property that $F(\mathcal{N}) \subseteq \mathcal{S}$. Then $F : \mathcal{N} \to \mathcal{S}$ is also smooth.

Theorem 10 *Let $\mathcal{M}$ and $\mathcal{N}$ be smooth manifolds and let $F : \mathcal{M} \to \mathcal{N}$ be a smooth map of constant rank. Then $\forall p \in \mathcal{N}$ the preimage of p, $F^{-1}(p)$ is an embedded submanifold of $\mathcal{M}$.*

1 Lie Groups

Definition 11 A Lie Group $\mathcal{G}$ is a smooth manifold without boundary, which is also a group. That is there exists two smooth maps

$$m : \mathcal{G} \times \mathcal{G} \to \mathcal{G},\ (g,h) \mapsto m(g,h) = gh$$

$$i : \mathcal{G} \to \mathcal{G},\ g \mapsto i(g) = g^{-1}$$

The identity element $e \in \mathcal{G}$ is defined as in the usual algebraic sense: $\forall g \in \mathcal{G} : ge = eg = g$

Proposition 12 If $\mathcal{G}$ is a smooth manifold with a group structure such that the map $\mathcal{G} \times \mathcal{G} \to \mathcal{G}$, $(g,h) \mapsto gh^{-1}$ is smooth, then $\mathcal{G}$ is a Lie Group. **q.e.d.**

Definition 13 Let $\mathcal{G}$ be a Lie group. Then each element $g \in \mathcal{G}$ defines left and right translation maps, $L_g, R_g : \mathcal{G} \to \mathcal{G}$ by,

$$\forall h \in \mathcal{G} : \quad L_g(h) = gh \quad R_g(h) = hg$$

Proposition 14 The left and right translation maps are diffeomorphisms for all $g \in \mathcal{G}$.

Proof. Define a map $\iota_g : \mathcal{G} \to \mathcal{G} \times \mathcal{G}$, $h \mapsto (g,h)$, which is trivially a smooth map. Then we can write L_g as the composition of two smooth functions, namely we have,$L_g = m \circ \iota_g$. Since the composition of smooth functions is again smooth, we have thus shown that $L_g \in \mathcal{C}^\infty(\mathcal{G}, \mathcal{G})$. Note also that $L_{g^{-1}}$ is also a smooth function and the inverse function of L_g. Thus it is also bijective, which shows that L_g is a diffeomorphism. The proof for the right translation map is analogous. **q.e.d.**

Example 15 Each of the following is a Lie group.

(a) The set of real invertible matrices $\mathrm{GL}(n, \mathbb{R})$ is a group with group action as the matrix multiplication. $\mathrm{GL}(n, \mathbb{R})$ is open in $\mathrm{M}(n, \mathbb{R})$ because it is defined as the preimage of $\mathbb{R} \setminus \{0\}$ under det, which is a continuous function. Thus $\mathrm{GL}(n, \mathbb{R})$ is a submanifold of $\mathrm{M}(n, \mathbb{R})$. Since the matrix multiplication is a polynomial function of the matrix entries, it is smooth. The inverse map is also due to Cramer's rule smooth. **q.e.d.**

(b) The set of matrices with positive determinant $\mathrm{GL}^+(n,\mathbb{R})$ is an open subset of $\mathrm{GL}(n,\mathbb{R})$ because of the same argument with determinant. Thus it is a submanifold of $\mathrm{GL}(n,\mathbb{R})$. Since $\det(AB) = \det A \det B$ and $\det(A^{-1}) = 1/\det(A)$, it is also a subgroup of $\mathrm{GL}(n,\mathbb{R})$, which makes it with the restriction of the group operator in $\mathrm{GL}(n,\mathbb{R})$ to a Lie group. **q.e.d.**

(c) In general each open subgroup $\mathcal{H} \subseteq \mathcal{G}$ of a Lie group $\mathcal{G}$ is a Lie group with the group operator in $\mathcal{H}$ as the restriction of the group operator in $\mathcal{G}$.

(d) Similarly $\mathrm{GL}(n,\mathbb{C})$ is a Lie group under matrix multiplication.

(e) $(\mathbb{R}^n,+)$ and $(\mathbb{C},+)$ are trivially Lie groups since the group operation is linear.

(f) The circle $\mathbb{S}^1 \subseteq \mathbb{C}^*$ is a smooth manifold and a group under complex multiplication. In the polar representation the group operation is given as $(\theta,\phi) \mapsto \theta + \phi$ and the inversion map i is given as $\theta \mapsto -\theta$, which are both smooth. The Lie group $\mathbb{S}^1$ is also called the *circle group*.

(g) If $\mathcal{G}_1, \dots, \mathcal{G}_k$ are Lie groups, then their direct product $\mathcal{G}_1 \times \cdots \times \mathcal{G}_k$ is also a Lie group with componentwise multiplication

$$(g_1, \dots, g_k)(h_1, \dots, h_k) = (g_1 h_1, \dots, g_k h_k)$$

In particular the n-Torus $\mathbb{T}^n := \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$ is an abelian Lie group.

2 Lie Group Homomorphisms

Definition 16 Let $\mathcal{G}$ and $\mathcal{H}$ be Lie groups and $F : \mathcal{G} \to \mathcal{H}$ a smooth map, which is also a group morphism. Then F is called a Lie group morphism. If F is a diffeomorphism from $\mathcal{G}$ to $\mathcal{H}$ then F is called a Lie group isomorphism. In this case we say $\mathcal{G}$ and $\mathcal{H}$ are isomorphic Lie groups.

Example 17 (a) The inclusion map $\mathbb{S}^1 \hookrightarrow \mathbb{C}^*$ is trivially a Lie group homomorphism.

(b) $\exp : (\mathbb{R},+) \to (\mathbb{R}^*,\cdot)$ is a Lie group homomorphism. Similarly $\exp : (\mathbb{R},+) \to (\mathbb{R}^+,\cdot)$ is a Lie group isomorphism with inverse $\log : \mathbb{R}^+ \to \mathbb{R}$

(c) Define the map $\varepsilon : \mathbb{R} \to \mathbb{S}^1$, $\theta \mapsto e^{2\pi i\theta}$ is a Lie group homomorphism. Similarly $\mathbb{R}^n \to \mathbb{T}^n$ is also a Lie group homomorphism.

(d) The determinant function $\det : \mathrm{GL}(n,\mathbb{F}) \to \mathbb{F}^*$ is a smooth function and is a Lie group homomorphism because $\det(AB) = \det A \cdot \det B$

Theorem 18 *Every Lie group homomorphism $F : \mathcal{G} \to \mathcal{H}$ has constant rank.*

Proof. Let $e \in \mathcal{G}$ and $\tilde{e} \in \mathcal{H}$ denote the identity elements. Let $g_0 \in \mathcal{G}$ be an arbitrary element. It is sufficient to show that dF_{g_0} has the same rank as dF_e. We have for all $g \in \mathcal{G}$:

$$F \circ L_{g_0}(g) = F(L_{g_0}(g)) = F(g_0 g) = F(g_0)F(g) = L_{F(g_0)}F(g) = L_{F(g_0)} \circ F(g)$$

Thus we have $F \circ L_{g_0} = L_{F(g_0)} \circ F$. Taking the differentials of both sides at the identity $e \in \mathcal{G}$ and using Proposition 3 we get:

$$dF_{g_0} \circ d(L_{g_0})_e = d(L_{F(g_0)})_{\tilde{e}} \circ dF_e$$

Since $L_\bullet$ is a diffeomorphism, $d(L_\bullet)_g$ is an isomorphism for all $g \in \mathcal{G}$. From linear algebra lectures, we know that composing a linear function with an isomorphism does not change the rank of the function. Thus we find that $\operatorname{rank} dF_{g_0} = \operatorname{rank} dF_e$ **q.e.d.**

Corollary 19 A Lie group homomorphism is a Lie group isomorphism if and only if it is bijective.

Proof. This follows directly from Global Rank Theorem. **q.e.d.**

3 Lie Subgroups

Definition 20 Let $\mathcal{G}$ be a Lie group. We call a subset $\mathcal{H} \subseteq \mathcal{G}$ a Lie subgroup if and only if

(i) (algebraic property) $\mathcal{H}$ is a subgroup of $\mathcal{G}$.

(ii) (topological property) $\mathcal{H}$ has a topological and smooth structure such that $\mathcal{H}$ is a Lie group and an immersed submanifold of $\mathcal{G}$.

Proposition 21 Let $\mathcal{G}$ be a Lie group and $H \subseteq \mathcal{G}$ a subgroup of $\mathcal{G}$, such that $\mathcal{H}$ is also an embedded submanifold of $\mathcal{G}$. Then $\mathcal{H}$ is a Lie subgroup.

Proof. Note that we only need to check that $\mathcal{H}$ is a Lie group, as the other properties are fulfilled. The multiplication map $m : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ is a smooth map since $\mathcal{G}$ is a Lie group. Thus the restriction $m|_{\mathcal{H}\times\mathcal{H}} : \mathcal{H} \times \mathcal{H} \to \mathcal{G}$ is also a smooth map. Since $\mathcal{H}$ is a subgroup it is closed with respect to multiplication thus $m_{\mathcal{H}} : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ is well-defined. The fact that this map is smooth is provided by Lemma 9. The proof for the inverse map is similar. **q.e.d.**

Example 22 The circle $\mathbb{S}^1 \subseteq \mathbb{C}^*$ is embedded in $\mathbb{C}^*$ and also a subgroup of $\mathbb{C}^*$. Thus $\mathbb{S}^1$ is an embedded Lie subgroup of $\mathbb{C}^*$.

Lemma 23 Let $\mathcal{G}$ be a Lie group and $\mathcal{H} \subseteq \mathcal{G}$ an open subgroup. Then $\mathcal{H}$ is an embedded Lie group and $\mathcal{H}$ is closed in the topological sense. Thus it is a union of connected components of $\mathcal{G}$.

Proof. If $\mathcal{H}$ is open in $\mathcal{G}$, then it is also an embedded submanifold of $\mathcal{G}$. Thus by Proposition 21 it is a Lie subgroup. We define the left coset of $\mathcal{H}$ to be $g\mathcal{H} := \{gh \,|\, h \in \mathcal{H}\}$. Note that $g\mathcal{H} = L_g(\mathcal{H})$ and that L_g is a diffeomorphism. Thus $g\mathcal{H}$ is open $\forall g \in \mathcal{G}$. We thus have:

$$\mathcal{H}^C = \mathcal{G} \setminus \mathcal{H} = \bigcup_{g \in \mathcal{G}\setminus\mathcal{H}} g\mathcal{H}$$

which is a union of open sets and is thus open. Thus $\mathcal{H}$ is closed. Since $\mathcal{H}$ is a clopen set it is a union of connected components. **q.e.d.**

Example 24 $\mathrm{GL}^+(n,\mathbb{R}) \subseteq \mathrm{GL}(n,\mathbb{R})$ is an open subgroup. Thus an embedded Lie subgroup of $\mathrm{GL}(n,\mathbb{R})$.

Proposition 25 Let $F : \mathcal{G} \to \mathcal{H}$ be a Lie group homomorphism. Then $\ker F$ is an embedded Lie subgroup of $\mathcal{G}$.

Proof. It is clear from linear algebra lectures that $\ker F \neq \emptyset$ and that $\ker F$ is a subgroup of $\mathcal{G}$. From Theorem 10 we know that $\ker F = F^{-1}(\tilde{e})$ is an embedded submanifold of $\mathcal{G}$. Thus it follows from Proposition 21 that $\ker F$ is a Lie group. **q.e.d.**

Example 26 $\mathrm{SL}(n,\mathbb{F})$ is the set of real $\mathbb{F} = \mathbb{R}$ (or complex $\mathbb{F} = \mathbb{C}$) $(n \times n)$ matrices with determinant equal to 1. Since it is the kernel of the smooth determinant function $\det : \mathrm{GL}(n,\mathbb{F}) \to \mathbb{F}^*$ it flows from the above proposition that $\mathrm{SL}(n,\mathbb{F})$ is a Lie subgroup of $\mathrm{GL}(n,\mathbb{F})$.

4 Group Actions

Definition 27 The left action of a Lie group $\mathcal{G}$ on a smooth manifold $\mathcal{M}$ is defined as the map $\theta : \mathcal{G} \times \mathcal{M} \to \mathcal{M}$, $(g,p) \mapsto g \cdot p$ such that the following holds:

(i) $\forall g, g' \in \mathcal{G}, \forall p \in \mathcal{M} : \quad g \cdot (g' \cdot p) = (gg') \cdot p$

(ii) $\forall p \in \mathcal{M} : \quad e \cdot p = p$

We'll sometimes use $\theta_g(p)$ instead of $g \cdot p$. With this notation the above requirements become for the left action

(i) $\forall g, g' \in \mathcal{G} : \quad \theta_g \circ \theta_{g'} = \theta_{gg'}$

(ii) $\theta_e = \mathrm{id}_{\mathcal{M}}$

Definition 28 We define right action in a similar way. Let again $\mathcal{G}$ be a Lie group and $\mathcal{M}$ be a smooth manifold. Then the right action of $\mathcal{G}$ on $\mathcal{M}$ is defined by the map $\theta : \mathcal{M} \times \mathcal{G} \to \mathcal{M}$, $(p,g) \mapsto (p \cdot g)$ such that the following holds:

(i) $\forall g, g' \in \mathcal{G} : \quad \theta_g \circ \theta_{g'} = \theta_{g'g}$

(ii) $\theta_e = \mathrm{id}\,\mathcal{M}$

Note that the composition rule for right and left action is different. In particular every left action can be converted into a right action if we define $g \cdot p := g^{-1} \cdot p$. Thus everything we prove using left actions also apply for right actions. Note that the notational convenience of writing $g \cdot p$ may lead to misunderstandings since only the composition rule defines wheteher an action is a left or a right action an not how we write it.

Definition 29 If $\mathcal{M}$ is a topological space and $\mathcal{G}$ is a topological group, then the left/right action of $\mathcal{G}$ on $\mathcal{M}$ is said to be continuous if the defining map is continuous. In this case we say that $\mathcal{M}$ is a left/right $\mathcal{G}$-space Similarly if $\mathcal{M}$ is a smooth manifold and $\mathcal{G}$ is a Lie group, then the left/right action is said to be smooth if the defining map is a smooth map.

Note that $\theta_g : \mathcal{M} \to \mathcal{M}$ is a diffeomorphism because it is by definition a smooth function and has the inverse $\theta_{g^{-1}}$ which is also by definition smooth.

Definition 30 Let $\mathcal{M}$ be a set, $\mathcal{G}$ a group and $\theta : \mathcal{G} \times \mathcal{M} \to \mathcal{M}$ a left action on $\mathcal{M}$.

(a) The orpit of an arbitrary $p \in \mathcal{M}$ is defined as the set

$$\mathcal{G} \cdot p := \{g \cdot p \,|\, g \in \mathcal{G}\}$$

(b) The istropy group of an arbitrary $p \in \mathcal{M}$ is defined as the set of group elements, which map p to p or in mathematical language:

$$\mathcal{G}_p := \{g \in \mathcal{G} \,|\, g \cdot p = p\}$$

Note that $\mathcal{G}_p \subseteq \mathcal{G}$ is a subgroup of $\mathcal{G}$.

(c) The action is called transitive if and only if

$$\forall p, q \in \mathcal{M},\ \exists g \in \mathcal{G} : \quad g \cdot p = q \iff \forall p \in \mathcal{M} : \quad \mathcal{G} \cdot p = \mathcal{M}$$

(d) The action is called free if and only if

$$\forall p \in \mathcal{M} : \quad \Big[\big(g \cdot p = p \implies g = e\big) \iff \big(\#\mathcal{G}_p = 1\big)\Big]$$

Example 31 (0) Let $\mathcal{G}$ be any Lie group and let $\mathcal{M}$ be a smooth manifold. Define the left action as $g \cdot p = p$ for all $p \in \mathcal{M}$ and $g \in \mathcal{G}$.

(a) We call the action of $\mathrm{GL}(n, \mathbb{R})$ on $\mathbb{R}^n$ by left matrix multiplication the natural action if we interpret $\mathbb{R}^n$ as column vectors. This is a lie group action because $\mathbb{1}_n x = x$ for all $x \in \mathbb{R}^n$ and the matrix multiplication is associative. Furthermore the action is smooth because it is a polynomial function of the coordinates. **q.e.d.**

(b) Let $\mathcal{H}$ be a Lie subgroup of $\mathcal{G}$. Then the action $\mathcal{H} \times \mathcal{G} \to \mathcal{G}$ is a smooth action

5 Equivarent Maps

Definition 32 Let $\mathcal{G}$ be a Lie group which acts both of the smooth manifolds $\mathcal{M}$ and $\mathcal{N}$. We say that a map $F : \mathcal{M} \to \mathcal{N}$ is equivarent if and only if:

$$\forall g \in G,\ \forall p \in \mathcal{M} : \quad F(g \cdot p) = g \cdot F(p)$$

if and only if the flowing diagram commutes:

$$\begin{tikzcd} \mathcal{M} \arrow[r, "F"] \arrow[d, "\theta_g"'] & \mathcal{N} \arrow[d, "\varphi_g"] \\ \mathcal{M} \arrow[r, "F"'] & \mathcal{N} \end{tikzcd}$$

Example 33 Let $v = (v_1 \dots v_n) \in \mathbb{R}^n$ be a fixed vector. Define the smooth action of $\mathbb{R}$ on $\mathbb{R}^n$ and $\mathbb{T}^n$ by

$$t \cdot (x_1 \dots x_n) = (x_1 + v_1 t, \dots, x_n + v_n t)$$
$$t \cdot (z_1 \dots z_n) = (e^{2\pi i t v_1} z_1 \dots e^{2\pi i t v_n} z_n)$$

Then the smooth map $\varepsilon^n : \mathbb{R}^n \to \mathbb{T}^n$ is equivarent with respect to these action.