Theorem of Gauss-Bonnet

In this chapter M will be a compact, oriented, differentiable manifold of dimension two.
X will be a differentiable vector field on M.

1. The Riemannian Metric on M

First, we will define a Riemannian metric on M. We start off with a finite differentiable structure (f^α, U^α) of M - which exists since M is compact - and define arbitrarily an inner product $\langle , \rangle^\alpha$ on each coordinate neighbourhood $f^\alpha(U^\alpha)$. Now we choose a differentiable partition of unit $\{\varphi_\alpha\}$ subordinate to the covering $\{f^\alpha(U^\alpha)\}$ and set for each $p \in M$

$$\langle , \rangle_p = \sum_\alpha \varphi_\alpha(p) \langle , \rangle_p^\alpha$$

φ_α and $\langle , \rangle_p^\alpha$ vary differentiably with p, so $\langle , \rangle_p$ as well. $\langle , \rangle_p^\alpha$ is positive definite and $0 \leq \varphi_\alpha \leq 1$, so $\langle , \rangle_p$ is positive definite. $\blacksquare$

2. The Index of X at p

Definition A point $p \in M$ is **a singular point of** X if $X(p) = 0$. The singular point p is **isolated** if there exists a neighbourhood $V \subset M$ of p which contains no singular point other than p.

Let $p \in M$ be an isolated singular point of X. Now we approach the next definition

1. Choose a Riemannian metric on M
2. Choose a neighbourhood $V \subset M$ of p which contains no singular point of X other than p and is homeomorphic to an open disk in the plane.
3. Let $\{\overline{e}_1, \overline{e}_2\}$ be the moving frame, where $\overline{e}_1 = \frac{X}{|X|}$ and $\overline{e}_2$ is a unit vector field orthogonal to $\overline{e}_1$ and in the orientation of M. Thus $\overline{\omega}_1, \overline{\omega}_2, \overline{\omega}_{12}$ is defined in $V - \{p\}$.
4. Choose a moving frame $\{e_1, e_2\}$ which is defined throughout V and has again the same orientation as M. We obtain $\omega_1, \omega_2, \omega_{12}$ in V.
5. Choose a simple closed Curve C bounding a compact region of V, oriented as ∂V and containing p in its interior.
6. As in the chapter before, we name the difference $\tau := \overline{\omega}_{12} - \omega_{12}$ which is defined in $V - \{p\}$. From Lemma 5 of Chapter 5 we know that restriction of τ to C is the differential of the angle $\varphi(t)$ between e_1 and $\overline{e}_1$ along C.

Definition Using the setup above, we define **index** I **of** X **at** p as following:

$$\int_C \tau = \int_C d\varphi = 2\pi \cdot I$$

3. Well-defindness of I

Lemma 1. *The definition of I does not depend on the chosen Curve C*

Proof Consider first the case in which we have two simple non-intersecting closed curves C_1 and C_2 around p, as in the definition of I which bound an annular region $\triangle$. We show that the computed indices I_1 and I_2 coincide, by using Stokes-Theorem and the fact, that τ is closed:

$$0 = \frac{1}{2\pi}\int\limits_{\triangle} d\tau = \frac{1}{2\pi}\int\limits_{C_1} \tau + \frac{1}{2\pi}\int\limits_{-C_2} \tau = \frac{1}{2\pi}\int\limits_{C_1} \tau - \frac{1}{2\pi}\int\limits_{C_2} \tau = I_1 - I_2$$

If in general, C_1 and C_2 intersect we choose the curve C_3 which does not intersect both C_2 and C_1 and apply the argumentation above and conclude $I_1 = I_2 = I_3$ ∎

Lemma 2. *Consider $S_r := \partial B_r$, the boundary of a disk of radius r around p and the moving frame $\{\bar{e}_1, \bar{e}_2\}$ of the definition and the corresponding $\overline{\omega}_{12}$. Then*

$$\lim_{r\to 0} \frac{1}{2\pi}\int\limits_{S_r} \overline{\omega}_{12} =: \overline{I}$$

exists.

Proof Let $r_2 < r_1$ and $\triangle$ the annular region bounded by S_{r_1} and S_{r_2}. As r_1, r_2 go to zero, $\triangle$ vanishes and by stokes theorem we get:

$$\int\limits_{S_{r_1}} \overline{\omega}_{12} - \int\limits_{S_{r_2}} \overline{\omega}_{12} = \int\limits_{\triangle} d\overline{\omega}_{12} \xrightarrow{r_i \to 0} 0 \tag{1}$$

So any sequenze

$$\int\limits_{S_{r_1}} \overline{\omega}_{12}, \int\limits_{S_{r_2}} \overline{\omega}_{12}, \ldots$$

with $\{r_n\} \to 0$, is a Cauchy sequenze. Thus there exists a limit

$$\lim_{r\to 0} \frac{1}{2\pi}\int\limits_{S_r} \overline{\omega}_{12} = \overline{I}$$ ∎

Lemma 3. *The definition of I does not depend on the choice of the frame $\{e_1, e_2\}$.*

Proof Let $r_2 < r_1$ and S_{r_1}, S_{r_2} as before. We know $\overline{\omega}_{12} = \omega_{12} + \tau$ and conclude:

$$\begin{aligned}\int\limits_{S_{r_1}} \overline{\omega}_{12} &= \int\limits_{S_{r_1}} \omega_{12} + \int\limits_{S_{r_1}} \tau = \int\limits_{B_{r_1}} d\omega_{12} + \int\limits_{S_{r_1}} \tau = -\int\limits_{B_{r_1}} K\omega_1 \wedge \omega_2 + 2\pi I \\ &= -\int\limits_{B_{r_1}} K\sigma + 2\pi I\end{aligned}$$

In (1) we let r_2 go to zero and by Lemma 2 we get some real number $\overline{I}$ with:

$$\int\limits_{S_{r_1}} \overline{\omega}_{12} - 2\pi\overline{I} = \int\limits_{B_{r_1}} d\overline{\omega}_{12} = -\int\limits_{B_{r_1}} K\overline{\omega}_1 \wedge \overline{\omega}_2$$
$$= -\int\limits_{B_{r_1}} K\sigma$$

Thus $\overline{I} = I$. Since $\overline{I}$ in Lemma 2 was independent from $\{e_1, e_2\}$, so I is. ∎

Lemma 4. *The definition of I does not depend on the chosen metric.*

Proof Let $\langle,\rangle^0$ and $\langle,\rangle^1$ be two Riemannian metrics on M. Let , for $t \in [0,1]$

$$\langle,\rangle^t := t \cdot \langle,\rangle^1 + (1-t)\langle,\rangle^0$$

Since $\langle,\rangle^0$ and $\langle,\rangle^1$ are Riemannian metrics on M, $\langle,\rangle^t$ is also a Riemannian metric on M. Thus $\langle,\rangle^t$ is a one-parameter familiy of metrics on M which starts with $\langle,\rangle^0$ and ends with $\langle,\rangle^1$. Let I_0, I_t, I_1 be the corresponding indices. I_t is a continuous function of t. To proove this, we see that $\overline{e}_{1_t}$ and $\overline{e}_{2_t}$ of the corresponding moving frame $\{\overline{e}_{1_t}, \overline{e}_{2_t}\}$ of $\langle,\rangle^t$ varies continuosly with t, since

$$\overline{e}_{1_t} = \frac{X}{\sqrt{\langle X, X\rangle^t}} = \frac{X}{\sqrt{t \cdot \langle X, X\rangle^1 + (1-t)\langle X, X\rangle^0}}$$

Following the definitions $\overline{\omega}_{1_t}$ and $\overline{\omega}_{2_t}$ vary linearly with $\overline{e}_{1_t}$ and $\overline{e}_{2_t}$ and $\overline{\omega}_{12_t}$ varies linearly with $\overline{\omega}_{1_t}$ and $\overline{\omega}_{2_t}$. Hence $\overline{\omega}_{12_t}$ varies continuosly with t. By Lemma 2 we conclude that I_t varies continuously with t. Since I_t is an Integer, it follows I_t is constant and $I_0 = I_1$. ∎

4. The Theorems

Theorem 5. *Let $p_1, \ldots, p_k$ be the isolated singular points of X and $I_1, \ldots, I_k$ their corresponding indices. For any Riemannian metric on M,*

$$\int_M K\sigma = 2\pi \sum_{i=1}^{k} I_i =: 2\pi \cdot \chi(M)$$

K is the Gaussian curvature of the metric, σ its element of area. $\chi(M)$ is called the ***Euler-Poincaré characteristic of M****.*

Proof Let $\{\overline{e}_1, \overline{e}_2\}$ be the moving frame in $M - \bigcup_i \{p_i\}$ where $\overline{e}_1 = \frac{X}{|X|}$ and $\overline{e}_2$ is a unit vector field orthogonal to $\overline{e}_1$ in the orientation of M. Furthermore B_i will be a ball with center p_i an a radius r_i, as small as it contains no singular point other than p_i. Note that ∂B_i has the orientation induced by B_i which is opposite to the orientation of $M - B_i$, so by Lemma 2

$$\int\limits_{M-\bigcup_i B_i} K\sigma = \int\limits_{M-\bigcup_i B_i} K\overline{\omega}_1 \wedge \overline{\omega}_2 = -\int\limits_{M-\bigcup_i B_i} d\overline{\omega}_{12} = \int\limits_{\bigcup_i(\partial B_i)} \overline{\omega}_{12} = \sum_i \int\limits_{\partial B_i} \overline{\omega}_{12} \xrightarrow{r_i \to 0} 2\pi \sum_i I_i$$ ∎

Theorem 6. *Given an oriented, compact, two-dimensional differentiable manifold $\mathcal{M}$ with boundary $\partial\mathcal{M}$, a differnetiable vector field X on $\mathcal{M}$ that is nowhere tangent to $\partial\mathcal{M}$, isolated singular points $p_1, \ldots, p_k \in \mathcal{M}\backslash\partial\mathcal{M}$ of X and their indices $I_1, \ldots I_k$. Then for any Riemannian metric on $\mathcal{M}$*

$$\int\limits_{\mathcal{M}} K\sigma + \int\limits_{\partial M} k_g ds = 2\pi\chi(\mathcal{M})$$

k_g is the geodesic curvature of $\partial\mathcal{M}$, ds the arc element of $\partial\mathcal{M}$.

Proof Choose a Riemannian metric on M and let $\{\overline{e}_1, \overline{e}_2\}$ be again the orthonormal oriented moving frame with $\overline{e}_1 = \frac{X}{|X|}$. Choose in a neighbourhood $V \subset M$ of ∂M another oriented orthonormal moving frame $\{e_1, e_2\}$ such that, restricted to ∂M, e_1 is tangent to ∂M. Let $i : \partial M \to M$ be the inclusion map and φ the angle between $\overline{e}_1$ and e_1 along ∂M. Since e_1 is parallel to ∂M we get from the proof of Proposition 4 in Chapter 5

$$i^*\overline{\omega}_{12} = i^*\omega_{12} + d\varphi \tag{2}$$

Let B_i be a ball of center p_i, such that B_i contains no other point than p_i. From Stokes Theorem we get:

$$\int_{M-\bigcup B_i} K\overline{\omega}_1 \wedge \overline{\omega}_2 = -\int_{M-\bigcup B_i} d\overline{\omega}_{12} = \int_{\bigcup \partial B_i} \overline{\omega}_{12} - \int_{\partial M} i^*\overline{\omega}_{12} \tag{3}$$

By (2) and the definition of k_g, we get

$$\int_{\partial M} i^*\overline{\omega}_{12} = \int_{\partial M} i^*\omega_{12} + \int_{\partial M} d\varphi = \int_{\partial M} k_g ds + \int_{\partial M} d\varphi = \int_{\partial M} k_g ds \tag{4}$$

$\int_{\partial M} d\varphi$ has to be a 2π multiple and due to the fact that $\overline{e}_1 = \frac{X}{|X|}$ is nowhere tangent to ∂M, it is 0. Taking the limit when the radii of B_i goes to zero and combining (3) and (4), we obtain

$$\int\limits_{\mathcal{M}} K\sigma + \int\limits_{\partial M} k_g ds = 2\pi\chi(\mathcal{M}) \quad \blacksquare$$