Spaces of Constant Curvature

1. Theorem of Cartan

Consider two $n-$dimensional manifolds $M, \tilde{M}$ with curvature R and $\tilde{R}$ respectively and points $p \in M$ and $p \in \tilde{M}$. Since M and $\tilde{M}$ have the same dimension we find linear isometries between $T_P M$ and $T_{\tilde{p}}\tilde{M}$. Choose a linear isometry $i : T_p(M) \to T_{\tilde{p}}(\tilde{M})$.

Let $V \subset M$ be a normal neighbourhood of p such that $\exp_{\tilde{p}}$ is defined at $i \circ \exp_p^{-1}(V)$. This is always possible. To find such neighbourhood, we know from Proposition 2.9 in Chapter 3 that we always find a normal neighbourhood V of p . Let U be the neighbourhood in T_pM , such that $\exp_p U = V$. It is $0 \in U$ and since i is linear, we have $0 \in i \circ \exp^{-1}(V) = i(U)$. Now $\exp_{\tilde{p}}$ is defined in some set $0 \in \tilde{U} \subset T_{\tilde{p}}\tilde{M}$. Finally, we choose V small enough, such that $i \circ \exp_p^{-1}(V) \subset \tilde{U}$ and get the desired $V \subset M$.

We define

$$\begin{aligned} f : V &\to \tilde{M} \\ q &\mapsto \exp_{\tilde{p}} \circ i \circ \exp_p^{-1}(q) \end{aligned}$$

For every point $q \in V$ there exists a unique geodesic passing through p and q since V is a normal neighbourhood of p. Due to Homogeinity of the geodesic we can consider this geodesic to be normalized. Let $\lambda : [0, t] \to M$ with $\lambda(0) = p$ and $\lambda(t) = q$ be that geodesic. Denote by P_t the parallel transport along λ from $\lambda(0)$ to $\lambda(t)$.

Consider now the normalized geodesic $\tilde{\lambda} : [0, 1] \to \tilde{M}$ given by $\tilde{\lambda}(0) = \tilde{p}$, $\tilde{\lambda}'(0) = i(\lambda'(0))$. Denote by $\tilde{P}_t$ the parallel transport along $\tilde{\lambda}$ from $\tilde{\lambda}(0)$ to $\tilde{\lambda}(t)$. We define:

$$\begin{aligned} \phi_t : T_q(M) &\to T_{f(q)}(\tilde{M}) \\ v &\mapsto \tilde{P}_t \circ i \circ P_t^{-1}(v) \end{aligned}$$

Theorem 1 (Cartan)**.** *With the setting above, if we have for every $q \in V$ and all $x, y, u, v \in T_qM$:*

$$\langle R(x, y)u, v\rangle = \langle \tilde{R}\left(\phi_t(x), \phi_t(y)\right)\phi_t(u), \phi_t(v)\rangle,$$

then $f : V \to f(V)$ is a local isometry and $df_p = i$.

Proof. f is a local isometry iff for all $q \in V$ and all $v, w \in T_qM$:

$$\langle v, w\rangle_q = \langle df_q(v), df_q(w)\rangle_{f(q)}$$

holds. Due to the identity

$$\langle v, w\rangle = \frac{1}{2}\left(\|v\|^2 + \|w\|^2 - \|v - w\|^2\right)$$

it is sufficient to show that for every $q \in V$ and every $v \in T_qM$

$$\|v\|_q = \|df_q(v)\|_q \tag{1}$$

holds. For that we choose $q \in V$. Let $\lambda : [0, l] \to M$ be a normalized geodesic with $\lambda(0) = p$ and $\lambda(l) = q$.

Let $v \in T_q(M)$. By Proposition 3.9 in Chapter 5 there is a unique Jacobi field along λ satisfying $J(0) = 0$ and $J(l) = v$. Now we choose an orthonormal basis $\{e_1, ..., e_n\}$ of T_pM with $e_n = \lambda'(0)$. Denote by $e_i(t)$, $i \in \{1, ..., n\}$ the parallel transport of e_i along λ. Now we can write

$$J(t) = \sum_i^n a_i(t) e_i(t)$$

where a_i is some differentiable function. Using the Jacobi equation at $t = 0$, we conclude that for $j \in \{1, ..., n\}$

$$a_j''(0) + \sum_i^n \langle R\left(\lambda'(0), e_i(0)\right) e_n(0), e_j(0)\rangle a_i(0) = a_j''(0) + \sum_i^n \langle R\left(e_n, e_i\right) e_n, e_j\rangle a_i(0) = 0$$

holds. Now we consider $\tilde{M}$. Let $\tilde{\lambda} : [0, l] \mapsto \tilde{M}$ be a normalized geodesic given by $\tilde{\lambda}(0) = \tilde{p}$ and $\tilde{\lambda}'(0) = i(\lambda'(0))$. Let $\tilde{e_j}(t) := \phi_t(e_j(t))$ and define a Vector field on $\tilde{M}$

$$\tilde{J}(t) = \sum_i a_i(t) \tilde{e}_i(t)$$

with $t \in [0, l]$. $\tilde{J}$ is the field along $\tilde{\lambda}$ given by $\tilde{J} = \phi_t \circ J$ By hypothesis

$$\langle R\left(e_n, e_i\right) e_n, e_j\rangle = \langle \tilde{R}(\tilde{e}_n, \tilde{e}_i)\tilde{e}_n, \tilde{e}_j\rangle$$

holds and we conclude for $j \in \{1, ..., n\}$

$$a_j'' + \sum_i \langle \tilde{R}(\tilde{e}_n, \tilde{e}_i)\tilde{e}_n, \tilde{e}_j\rangle a_i = 0$$

So $\tilde{J}$ is a Jacobi field along $\tilde{\lambda}$ with $\tilde{J}(0) = \phi_t \circ J(0) = 0$. Since parallel transport is an isometry, $\|\tilde{J}(l)\| = \|J(l)\| = \|v\|$.

Due to (1) it is sufficient to proove

$$\tilde{J}(l) = df_q(v) = df_q(J(l)) \tag{2}$$

Since J and $\tilde{J}$ are Jacobi fields along λ and $\tilde{\lambda}$ respectively and vanish at 0, we have from Corollary 2.5 of Chapter 5 for $t \in [0, l]$

$$\begin{aligned} J(t) &= (d\exp_p)_{t\lambda'(0)}(tJ'(0)), \\ \tilde{J}(t) &= (d\exp_{\tilde{p}})_{t\tilde{\lambda}'(0)}(t\tilde{J}'(0)). \end{aligned}$$

So $J(l) = (d\exp_p)_{l\lambda'(0)}(lJ'(0))$ and therefore $lJ'(0) = (d\exp_p)^{-1}_{l\lambda'(0)}(J(l))$ Consider $\tilde{J} = \phi_t \circ J$ for $t = 0$. We have then $\phi_0 = i$ since there is no parallel transport. We conclude $\tilde{J}'(0) = i(J'(0))$ and finally

$$\begin{aligned} \tilde{J}(l) &= (d\exp_p)_{l\tilde{y}'(0)}(l\tilde{J}'(0)) = (d\exp_p)_{l\tilde{y}'(0)} li(J'(0)) = (d\exp_p)_{l\tilde{y}'(0)} i(lJ'(0)) \\ &= (d\exp_p)_{l\tilde{y}'(0)} \circ i \circ \left((d\exp_p)_{l\lambda'(0)}\right)^{-1}(J(l)\big) = df_q(J(l)) \end{aligned}$$

Finally we have from (2) that $(df)_p(v) = \tilde{J}(l) = \phi_0 \circ J(l) = \tilde{P}_0 \circ i \circ P_0^{-1} \circ J(l) = \tilde{P}_0 \circ i \circ P_0^{-1}(v) = i(v)$ $\square$

2. Hyperbolic Space

Definition The Riemannian Manifold (H^n, g_{ij}) is called the **hyperbolic space** of dimension n. The **upper half space** is given by

$$H^n = \{(x_1, ..., x_n) \in \mathbb{R}^n; x_n > 0\}$$

and the metric by

$$g_{ij}\left(x_1, ..., x_n\right) = \frac{\delta_{ij}}{x_n^n}$$

Proposition 2. *The hyperbolic space H^n is a complete simply connected Riemannian Manifold with constant sectional curvature -1.*

Proof. In the book. □

Proposition 3. *The straight lines perpendicular to the hyperplane $x_n = 0$, and the circles of H^n whose planes are perpendicular to the hyperplane $x_n = 0$ and whose centers are in this hyperplane are the geodesics of H^n.*

3. Space Forms

Definition A **space form** is a complete Riemannian manifold with constant sectional curvature.

Theorem 4. *Let M^n be a space form of Curvature K. Then the universal covering $\tilde{M}$ of M, with the covering metric, is isomorphic to*

1. *H^n, if $K = -1$,*
2. *$\mathbb{R}^n$, if $K = 0$,*
3. *S^n, if $K = 1$*

Lemma 5. *Let M and N be connected Riemannian manifolds and $f_1, f_2 : M \to N$ two local isometries. Suppose that there is $p \in M$ such that $f_1\left(p\right) = f_2(p)$ and $\left(df_1\right)_p = \left(df_2\right)_p$. Then $f_1 = f_2$.*

Proof. Let V be a normal neighbourhood of p such that the restriction $f_1|_V$ and $f_2|_V$ are diffeomorphisms. We define $\varphi := f_1^{-1} \circ f_2 : V \to V$ and have $\varphi(p) = p$ and $d\varphi_p = id$. It is $f_1 = f_2$. In order to proove that, let $q \in V$. Since V is a normal neighbourhood of p, there exists a unique $v \in T_pM$ such that $\exp_p(v) = q$. So it is $\varphi(q) = q$.

Choose $r \in M$ now and let $\alpha : [0, 1] \to M$, such that $\alpha(0) = p$, $\alpha(1) = r$. Since M is connected this is well-defined. Consider

$$A := \{t \in [0, 1]; f_1(\alpha(t)) = f_2(\alpha(t)) \text{ and } (df_1)_{\alpha(t)} = (df_2)_{\alpha(t)}\}$$

Since $f_1 = f_2$ on V, $\sup A > 0$. If $t_0 := \sup A \neq 1$, we start over the proof with the point $p = \alpha(t_0)$ but get a contradiction because then there is no normal neighbourhood around p. Therefore $\sup A = 1$, hence $f_1 = f_2$ on M. □

Proof of the Theorem. By the covering map of the universal covering, we know that for each point in $\tilde{M}$ we have some neighbourhood that is homeomorphic to some neighbourhood in M. So $\tilde{M}$ is a simply connected, complete Riemannian manifold, with constant sectional curvature K, since these properties are all local. Firstly, we consider the first two cases of the theorem an denote by Δ both H^n and $\mathbb{R}^n$. Choose some $p \in \Delta$, $\tilde{p} \in \tilde{M}$ and a linear isometry $i : T_p(\Delta) \to T_{\tilde{p}}(\tilde{M})$. From the fact that Δ is a complete, simply connected Riemannian manifold with non-positive sectional curvature, we conclude by the Theorem of Hadamard, that $\exp_p : T_p(\Delta) \to \Delta$ is a diffeomorphism and so $\exp_p^{-1}$ is well defined on Δ. Since $\tilde{M}$ is complete $\exp_{\tilde{p}}$ is defined on all $T_{\tilde{p}}(\tilde{M})$. It follows, that also

$$\begin{aligned} f : \Delta &\to \tilde{M} \\ q &\mapsto \exp_{\tilde{p}} \circ i \circ \exp_p^{-1}(q) \end{aligned}$$

is well-defined. As we apply the Theorem of Cartan, f is a local isometry and $df_p(v) = i(v)$ for all $v \in T_p\Delta$ and since i is an isometry, we have $|df_p(v)| = |v|$ and conclude by Lemma 3.3 of Chapter 7 that f is an diffeomorphism which proves the first two cases.

For the spherical case, we fix $p \in S^n$, $\tilde{p} \in \tilde{M}$ and a linear isometry $i : T_p(S^n) \to T_{\tilde{p}}(\tilde{M})$. Define

$$f := \exp_{\tilde{p}} \circ i \circ \exp_p^{-1} : S^n - \{q\} \to \tilde{M}$$

where $q \in S^n$ is the antipodal point of p. Now choose $p' \in S^n - \{p, q\}$ and set $\tilde{p}' = f(p')$, $i' = df_{p'}$ and

$$f' := \exp_{\tilde{p}'} \circ i' \circ \exp_{p'}^{-1} : S^n - \{q'\} \to \tilde{M}$$

where q' is the antipodal point of p'. Since $S^n - (\{q\} \cup \{q'\}) = W$ is connected and

$$f(p') = \tilde{p}' = f'(p'), \qquad df_{p'} = i' = df'_{p'}$$

we conclude from the Lemma, that $f = f'$ on W. Define

$$g : S^n \to \tilde{M}, r \mapsto \begin{cases} f(r) & , r \in S^n - \{q\} \\ f'(r) & , r \in S^n - \{q'\} \end{cases}$$

By the Theorem of Cartan g is a local isometry and by that a local diffeomorphism. g is a covering map and from Algebraic Topology, we know that g is a diffeomorphism and therefore g is an isometry. □

Definition Let G be a group and M be a set. G acts on M, if there is a map $G \times M \to M$, $(g, x) \mapsto gx$, satisfying for $x \in M$ and $g_1, g_2 \in G$

$$ex = e, \qquad (g_1 g_2)x = g_1(g_2 x)$$

The orbit of $x \in M$ is defined as $Gx := \{gx, g \in G\}$. Denote by $M/G := \{Gx | x \in M\}$ the set of all orbits. There exists a natural surjective $\pi : M \to M/G, x \mapsto Gx$ The action of G is transitive if $Gx = M$.

As from now, we consider M to be an Riemannian manifold and $G := \Gamma$ is supposed to be a subgroup of $isom(M)$ which acts in a totally discontinuos manner, meaning, for every $x \in M$ there is a neighbourhood

$U \subset M$ such that $i(U) \cap U = \emptyset$ for all $i \in \Gamma - \{id\}$. In this case, we know from algebraic topology, the natural projection $\pi : M \mapsto M/\Gamma$ is a regular map, so Γ acts transitive on $\pi^{-1}(p)$ and Γ is the group of covering transformations. From Chapter 0 we know that M/Γ can be given a differentiable structure such that the natural projection is a local diffeomorphism.

Additionally, we can define for every $p \in M/\Gamma$ and every $u, v \in T_p(M/\Gamma)$ an inner product

$$\langle u, v \rangle := \langle d\pi^{-1}(u), d\pi^{-1}(v) \rangle_{\tilde{p}}$$

where $\tilde{p} \in \pi^{-1}(p)$. By definition π is a local isometry. This Riemannian metric is called the metric on M/Γ induced by the covering π. Since we have that local isometry, M/Γ is complete if and only if M is complete and M/Γ has constant curvature K if and only if M has constant curbvature K. So M/Γ is a for $M = S^n/\mathbb{R}^n/H^n$ a complete manifold of constant curvature $1/0/-1$. The following proposition implies that there are no more manifolds of that kind.

Proposition 6. *Let M be a complete Riemannian manifold with constant sectional curvature $K = 1/0/-1$. Then M is isometric to $\tilde{M}/\Gamma$, where $\tilde{M} = S^n/\mathbb{R}^n/H^n$ and Γ is a subgroup of $isom(\tilde{M})$ which acts in a totally disonctinuos manner on $\tilde{M}$. The metric on $\tilde{M}/\Gamma$ is induced from the covering $\pi : \tilde{M} \to \tilde{M}/\Gamma$.*

Proof. Let $p : \tilde{M} \to M$ be the universal covering and provide $\tilde{M}$ with the covering metric, that is, given a $q \in M$, choose $q' \in p^{-1}(q)$ and set for $u, v \in T_q\tilde{M}$

$$\langle u, v \rangle = \langle dp^{-1}(u), dp^{-1}(v) \rangle_{q'}$$

Let Γ be the group of covering transformations of the covering p. Then $\Gamma \subset isom(\tilde{M})$ is a subgroup and acts on $\tilde{M}$ in a totally discontinuos manner. So, as described before, we can introduce on $\tilde{M}/\Gamma$ the Riemannian metric induced by the natural projection $\pi : \tilde{M} \to \tilde{M}/\Gamma$. From topology, we know, that p is regular and

$$p(\tilde{x}) = p(\tilde{y}) \Longleftrightarrow \Gamma\tilde{x} = \Gamma\tilde{y} \Longleftrightarrow \pi(\tilde{x}) = \pi(\tilde{y}).$$

for $\tilde{x}, \tilde{y} \in \tilde{M}$. So, the equivalence classes we get from p and π on $\tilde{M}$ are the same and we get some bijection $\psi : M \to \tilde{M}/\Gamma$ such that $\pi = \psi \circ p$. ψ is a local isometry, since π and p are local isometries. Since ψ is a bijection, it is an isometry of M onto $\tilde{M}/\Gamma$. $\square$

Remark With the last proposition one can proove that every compact orientable surface of genus $p > 1$ can be provided with a metric of constant negative curvature.