A quotient space of representations of triangle reflection groups

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The present paper contains notes of my talk given in the seminar *Hyperbolic geometry, symmetry groups, and more* organized by Prof. Dr. Anna Wienhard and Dr. Gye-Seon Lee in the winter term 2013/2014. It is based on publications of Xin Nie [Nie] and Yves Benoist [Ben01], [Ben02].

Let Λ be a group generated by reflections with respect to the faces of a triangle $P \subset \mathbb{H}^2$. The group Λ shall admit the presentation

$$< r_1, r_2, r_3 \mid \{(r_i r_j)^{m_{ij}} : i, j \in \{1,2,3\}\} >,$$

where $m_{ji} = m_{ij} \in \{2, 3, 4, ...\}$ with $m_{ii} = 1$ and $\frac{1}{m_{12}} + \frac{1}{m_{13}} + \frac{1}{m_{23}} < 1$. Denote this presentation by Γ_0. We want to study the space

$$\mathcal{F}_{\Gamma_0} = \{\rho \in \mathrm{Hom}(\Gamma_0, G) \text{ faithful with discrete image } \Gamma := \rho(\Gamma_0) \\ \text{dividing a properly convex open set } \Omega_\rho \subset \mathbb{S}^2\},$$

where G is the group of projective transformations of the projective sphere $\mathbb{S}^2$. The group Γ divides Ω if its action on Ω is proper and cocompact. The group G acts on $\mathcal{F}_{\Gamma_0}$ by conjugation and we can define the quotient

$$X_{\Gamma_0} = {}_{G}\diagdown^{\mathcal{F}_{\Gamma_0}}.$$

Our goal is to sketch a proof of the following proposition:

Proposition 1. *If all m_{ij} are not equal to 2 then X_{Γ_0} is homeomorphic to $\mathbb{R}_+$. Otherwise it is homeomorphic to a point.*

If X_{Γ_0} is homeomorphic to $\mathbb{R}_+$ then there is a one-parameter family of representations $\{\rho_t\}_{t \in \mathbb{R}_+}$ such that, if Ω_t is the convex open set associated to ρ_t, then Ω_t converges to P when t tends to 0 or ∞. This fact is being illustrated in Figure 1.

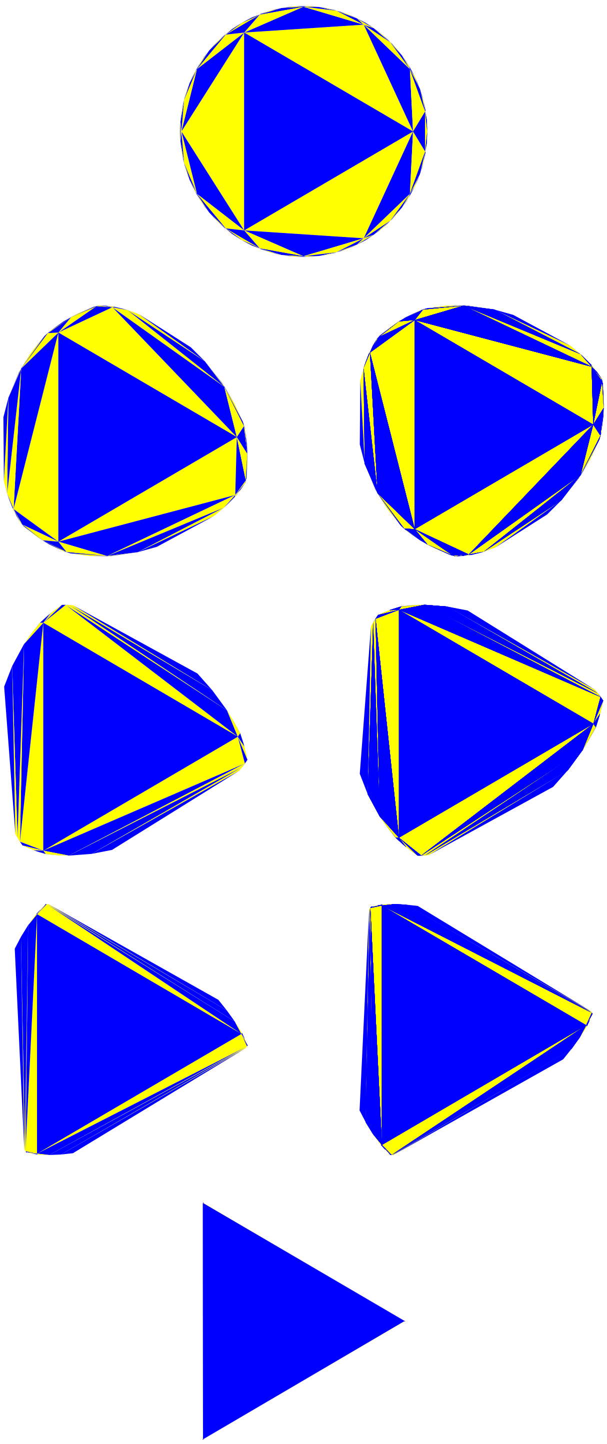

Figure 1: Deformation of Ω_t, where Γ_0 is the $(4,4,4)$-triangle group. The plots were created with SAGE [S$^+$09]. The source code is included in the appendix.

In order to study the space X_{Γ_0}, we need to know more about projective reflections and find conditions that tell us when the translates of a triangle under the action of the group generated by the reflections with respect to its faces tile a convex subset of $\mathbb{S}^2$.

Definition. Let $V := \mathbb{R}^3$.

1. The **projective sphere** is the set $\mathbb{S}^2 = \mathbb{S}(V) := (V - \{0\})/\mathbb{R}_+$.
2. The group of its **projective transformations** is $G := \mathrm{SL}^{\pm}(3, \mathbb{R})$.
3. A **reflection** σ is an element of order 2 in G which is the identity on a hyperplane.

All projective reflections are of the form $\sigma = \sigma_{\alpha,v} := Id - \alpha \otimes v$ for some linear form $\alpha \in V^*$ and $v \in V$ with $\alpha(v) = 2$. We have $\sigma|_{\ker(\alpha)} = Id_{\ker(\alpha)}$, and $\sigma(v) = -v$. The action on $\mathbb{S}^2$ is defined by $\sigma(x) = x - \alpha(x)v$.

Definition. A subset $\Omega \subseteq \mathbb{S}^2$ is **convex** if its intersection with any great circle is connected. It is **properly convex** if it is convex and its closure $\overline{\Omega}$ does not contain two opposite points.

Lemma 2. *Let $\sigma_1 = \sigma_{\alpha_1,v_1}$ and $\sigma_2 = \sigma_{\alpha_2,v_2}$ be distinct projective reflections, let Δ be the group they generate, and define $a_{12} := \alpha_1(v_2)$ and $a_{21} := \alpha_2(v_1)$. Let L be the intersection of the two half-spheres $\alpha_1 \leq 0$ and $\alpha_2 \leq 0$.*

1. *If $a_{12} > 0$ or $a_{21} > 0$ then the $\delta(L)$, $\delta \in \Delta$, do not tile any subset of $\mathbb{S}^2$.*
2. *Suppose now $a_{12} \leq 0$ and $a_{21} \leq 0$. Consider the following cases:*
   (a) *$a_{12}a_{21} = 0$. If both a_{12} and a_{21} are equal to 0 then the product is of order 2, and the $\delta(L)$ tile $\mathbb{S}^2$. Otherwise they do not tile.*
   (b) *$0 < a_{12}a_{21} < 4$. The product $\sigma_1\sigma_2$ is a rotation of angle θ given by $4\cos(\frac{\theta}{2})^2 = a_{12}a_{21}$. If $\theta = 2\pi/m$ for some integer $m \geq 2$ then $\sigma_1\sigma_2$ is of order m, and the $\delta(L)$ tile $\mathbb{S}^2$. Otherwise they do not tile.*

Proof. See Proposition 6 in [Vin]. □

Let $\sigma_i := Id - \alpha_i \otimes v_i$, $i = 1, 2, 3$, be projective reflections, let Γ be the group they generate and define $a_{ij} := \alpha_i(v_j)$ for all $i, j = 1, 2, 3$. According to Lemma 2, if we want the images $\gamma(P)$ of the triangle P, which is the intersection of the half-spheres $\alpha_i \leq 0$, to tile some subset of $\mathbb{S}^2$, the following conditions are necessary: For all $i \neq j$ we have

1. a_{ij} and a_{ji} are either both negative or both 0,
2. $a_{ij}a_{ji} = 4\cos(\frac{\pi}{m_{ij}})^2$ with an integer $m_{ij} \geq 2$.

The next theorem due to Tits and Vinberg is the key to understand X_{Γ_0}. It says that the conditions given above are not only necessary, but also sufficient. Further information on this theorem can be found in [Ben02].

Theorem 3. *Let $P \subset \mathbb{S}^2$ be a triangle, and for each face i of P, let $\sigma_i = Id - \alpha_i \otimes v_i$ be a projective reflection fixing the face i. Suppose that P is the intersection of the half-spheres $\alpha_i \leq 0$ and that the projective reflections satisfy*

1. *a_{ij} and a_{ji} are either both negative or both 0, and*
2. *$a_{ij}a_{ji} = 4\cos(\frac{\pi}{m_{ij}})^2$ with an integer $m_{ij} \geq 2$.*

Then

1. *the group Γ generated by the reflections σ_i is discrete,*
2. *the triangles $\gamma(P)$, $\gamma \in \Gamma$, tile a convex subset $\Omega \subset \mathbb{S}^2$, and*
3. *the morphism $\sigma\colon \Gamma_0 \to \Gamma$ given by $\sigma(r_i) = \sigma_i$ is an isomorphism.*

We outline the idea of the proof of Theorem 3. Define an abstract space X by glueing copies of P indexed by Γ_0 together along their edges and show that this space is convex. A bijection from X into $\Omega = \bigcup_{\gamma\in\Gamma} \gamma(P)$ yields the desired properties.

Define $X := \Gamma_0 \times P/_{\sim}$. The equivalence relation is generated by

$$(\gamma, p) \sim (\gamma', p') \text{ if } p' = p \text{ and } \gamma^{-1}\gamma' \in \Gamma_p,$$

where Γ_p is the group generated by the σ_i such that p is contained in the face i. Furthermore, define

$$\pi : X \to \mathbb{S}^2, \quad \pi(\gamma, p) = \gamma p.$$

We need the notion of a segment in order to determine, whether a space is convex.

Definition.

1. A subset $S \subset \mathbb{S}^2$ is called a **segment** if $\mathring{S}$ is a 1-dimensional convex subset.
2. For every $x, y \in X$, a **segment** $[x, y]$ is a compact subset of X such that the restriction of π to $[x, y]$ is a homeomorphism onto some segment of $\mathbb{S}^2$ with endpoints $\pi(x)$ and $\pi(y)$.

Lemma 4. *For every $x, x' \in X$ there exists at least one segment $[x, x']$.*

Lemma 5. *The map $\pi\colon X \to \Omega$ is bijective and Ω is convex.*

Proof. According to the previous lemma, there is a segment $[x, x']$ for all $x, x' \in X$. Hence, if $\pi(x) = \pi(x')$, then $x = x'$ (see the definition of a segment in X). This proves that $\pi\colon X \to \Omega$ is bijective (the map is surjective because of the definition of Ω). Since $\pi\colon X \to \Omega$ is bijective, all pairs of points in Ω can be joined by a segment. Therefore Ω is convex. □

This concludes the proof of Theorem 3, since the statements 2. and 3. follow from Lemma 5, and 1. follows from 2.

Lemma 6. *The following statements are equivalent:*

1. *For every vertex x of P, the group Γ_x is finite.*

2. The convex set Ω is open.

Note that under these conditions Γ divides Ω. Hence, to be sure that the translates $\gamma(P)$ tile some open convex set, it is enough to check local conditions around each vertex of the triangle.

According to [Ben01] the set Ω is properly convex if the vectors v_i generate V and the linear forms α_i generate V^*.

Now, we will identify the quotient space X_{Γ_0} with a quotient space of matrices $\bar{M}/_\sim$. Let $\rho \in \mathcal{F}_{\Gamma_0}$ and $\Gamma = \rho(\Gamma_0)$, then, according to Theorem 3, Γ is generated by projective reflections σ_i, satisfying the conditions stated in the theorem. Hence, ρ can be identified with a 3×3 matrix $A = (a_{ij})$, with $a_{ii} = 2$ for $i = 1, 2, 3$ and $a_{ij} \leq 0$ for $i \neq j$. Two representations $\Gamma_1 = \rho_1(\Gamma_0)$ and $\Gamma_2 = \rho_2(\Gamma_0)$, that are given by A_1 and A_2, are conjugate by a projective transformation if and only if A_1 and A_2 are conjugate by a matrix $\mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$ with $\lambda_i > 0$.

Conversely, let $\bar{M}$ be the set of all 3×3 matrices A such that its entries a_{ij} satisfy the conditions given in Theorem 3. Define an equivalence relation as follows: A_1 and A_2 in $\bar{M}$ are equivalent if they are conjugate by a matrix $\mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$ with $\lambda_i > 0$. Let $M := \bar{M}/_\sim$. As every $A \in \bar{M}$ yields a representation $\rho : \Gamma_0 \to G$ in $\mathcal{F}_{\Gamma_0}$, we have

$$X_{\Gamma_0} \cong M.$$

Now, it suffices to prove Proposition 1 for the quotient space M. Therefore, we need to introduce cyclic products.

Definition. Let $A = (a_{ij})$ be an $n \times n$ matrix and let $1 \leq i_1, ..., i_k \leq n$ with $k \geq 1$ be an ordered set of pairwise distinct indices. Then $a_{i_1 i_2} a_{i_2 i_3} ... a_{i_{k-1} i_k} a_{i_k i_1}$ is a **cyclic product** of length k.

Lemma 7. *Let $A = (a_{ij})$ be an 3×3 matrix satisfying the condition that for any i we have $a_{ii} \neq 0$ and for any $i \neq j$, $a_{ij} = 0$ if and only if $a_{ji} = 0$. Let B satisfy the same condition. We say $A \sim B$ if there are $\lambda_i \neq 0$ such that*

$$\mathrm{diag}(\lambda_1, .., \lambda_3) A \,\mathrm{diag}(\lambda_1, .., \lambda_3)^{-1} = B.$$

Then, $A \sim B$ if and only if for any ordered subset $\{i_1, ..., i_k\} \subseteq \{1, ..., 3\}$ we have

$$a_{i_1 i_2} ... a_{i_{k-1} i_k} a_{i_k i_1} = b_{i_1 i_2} ... b_{i_{k-1} i_k} b_{i_k i_1}.$$

Proof. See Lemma 1 in [Nie]. Suppose $A \sim B$. Then

$$\begin{pmatrix} a_{11} & \lambda_1 a_{12} \lambda_2^{-1} & \lambda_1 a_{13} \lambda_3^{-1} \\ \lambda_2 a_{21} \lambda_1^{-1} & a_{22} & \lambda_2 a_{23} \lambda_3^{-1} \\ \lambda_3 a_{31} \lambda_3^{-1} & \lambda_1 a_{32} \lambda_2^{-1} & a_{33} \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$$

and all cyclic products coincide.

Now, suppose that all cyclic products with the same indices coincide. We say that a matrix A is reducible if A can be put into block-diagonal form (one may have to reorder the basis). Otherwise A is irreducible.

The hypothesis on A and B implies that $a_{ii} = b_{ii}$ and $a_{ij} = 0$ if and only if $b_{ij} = 0$ for any $i \neq j$ (consider cyclic products of length 1 or 2). Hence, if necessary after a reordering of basis, the matrices A and B can be put into block-diagonal form with irreducible blocks, and the rth block of A has the same size as the rth block of B. Therefore, A and B are conjugate via diagonal matrices if and only if their blocks are. Hence, in the following we can assume that A and B are irreducible.

Let $\lambda_1 = 1$. Irreducibility implies that for λ_2 and λ_3 we can choose

$$\lambda_2 = \begin{cases} \frac{b_{23}b_{31}}{a_{23}a_{31}} & \text{or} \\ \frac{b_{21}}{a_{21}} \end{cases} \quad \text{and} \quad \lambda_3 = \begin{cases} \frac{b_{32}b_{21}}{a_{32}a_{21}} & \text{or} \\ \frac{b_{31}}{a_{31}} \end{cases}$$

Consider λ_2: If $a_{12} = 0$ then $a_{13} \neq 0$ and $a_{32} \neq 0$, because otherwise A can be put into block-diagonal form with more than one block. If $a_{13} = 0$ or $a_{32} = 0$, then $a_{12} \neq 0$ for the same reason.

It remains to check that the value of λ_2 and λ_3 does not depend on our choice. For λ_2 we have

$$\frac{b_{23}b_{31}}{a_{23}a_{31}} = \frac{a_{12}}{b_{12}} = \frac{b_{21}}{a_{21}},$$

where we used the fact that cyclic products of the same set of indices coincide. The proof for λ_3 works in the same way. It is a straightforward calculation to show that $A \sim B$ for this choice of λ_i. □

Proof of Proposition 1. Confer [Nie]. Let $A, B \in \bar{M}$. We differ two cases:

1. There are indices i, j such that $m_{ij} = 2$. Then, cyclic products of length
   - one are diagonal entries, which are equal to 2,
   - two are determined by the conditions given in Theorem 3, and
   - three are equal to 0.

   Hence, according to Lemma 7, it follows that $A \sim B$.

2. For all i, j we have $m_{ij} \neq 2$. Again, cyclic products of length one or two coincide. There are two cyclic products of length three:

   $$\phi(A) = a_{12}a_{23}a_{31} \quad \text{and} \quad \tilde{\phi}(A) = a_{13}a_{32}a_{21}$$

   The product

   $$\phi(A)\tilde{\phi}(A) = 4^3 \cos\left(\frac{\pi}{m_{12}}\right)^2 \cos\left(\frac{\pi}{m_{23}}\right)^2 \cos\left(\frac{\pi}{m_{13}}\right)^2,$$

   which determined by Theorem 3, is constant. By Lemma 7, we have $A \sim B$ if and only if $\phi(A) = \phi(B)$. The value of $\phi(A)$ is always negative. We show that the map

   $$\psi\colon M \to \mathbb{R}_+, \quad [A] \mapsto |\phi(A)|.$$

   is a homeomorphism. Because of the argument above, the map is well-defined. It is clear that ψ is continuous. Let

   $$A_t := \begin{pmatrix} 2 & -t\cos(\frac{\pi}{m_{12}}) & -\cos(\frac{\pi}{m_{12}})\cos(\frac{\pi}{m_{23}})\cos(\frac{\pi}{m_{13}})^2 \\ -t^{-1}\cos(\frac{\pi}{m_{12}}) & 2 & -\cos(\frac{\pi}{m_{23}}) \\ -\frac{1}{\cos(\frac{\pi}{m_{12}})\cos(\frac{\pi}{m_{23}})} & -\cos(\frac{\pi}{m_{23}}) & 2 \end{pmatrix}.$$

Then $\phi(A_t) = -t$. Define

$$\tilde{\psi}\colon \mathbb{R}_+ \to M, \quad t \to [A_t].$$

The function $\tilde{\psi}$ is continuous and we have $\psi\tilde{\psi} = id_{\mathbb{R}_+}$ and $\tilde{\psi}\psi = id_M$. Hence, ψ is a homeomorphism.

$\square$

References

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[Ben01] Y. Benoist, *A survey on divisible convex sets*, Geometry, analysis and topology of discrete groups, Adv. Lect. Math. (ALM), vol. 6, Int. Press, Somerville, MA, 2008, pp. 1-18.

[Ben02] Y. Benoist, *Five lectures on lattices in semisimple Lie goups*, Géometries à courbure négative ou nulle, groupes discretes et rigidité, Sémin. Congr., vol. 18, Soc. Math. France, Paris, 2009, pp. 117-176.

[Vin] È. B. Vinberg, *Discrete linear groups generated by reflections*, Mathematics of the USSR-Izvestiya, vol. 5, 1971, `http://stacks.iop.org/0025-5726/5/i=5/a=A07`.

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Listing 1: SAGE Code

```
### Set Parameters

s = 2

p = 4
q = 4
r = 4

N = 1000

def is_in_interval(x):

    temp_var = 0
    temp_var_2 = false

    m = 0
    k = var('k')


    while temp_var_2 is false:

        if sum(3^k,k,0,m) <= x < sum(3^k,k,0,m+1):

            temp_var_2 = true


            if mod(m,2) == 1:

                temp_var = 1


        m = m + 1

    return temp_var

xRotation =
        matrix([[1,0,0],
                [0,cos(pi/4),-sin(pi/4)],
                [0,sin(pi/4),cos(pi/4)]])
yRotation =
        matrix([[cos(pi/4),0,sin(-pi/4)],
                [0,1,0],
                [-sin(-pi/4),0,cos(pi/4)]])

Rotation = yRotation * xRotation


e_1 = vector([1,0,0])
e_2 = vector([0,1,0])
e_3 = vector([0,0,1])


e1 = matrix([1,0,0]).T
e2 = matrix([0,1,0]).T
e3 = matrix([0,0,1]).T


I = matrix([[1,0,0],[0,1,0],[0,0,1]])
```

```
List_of_Unit_Vectors = [e1,e2,e3]


### Define Reflections

Bilinear_Form_s =
        matrix([[1,-s*cos(pi/p),-cos(pi/q)/s],
                [-cos(pi/p)/s,1,-s*cos(pi/r)],
                [-cos(pi/q)*s,-cos(pi/r)/s,1]])


Generators = [I] * 3


for i in range(3):

    Generators[i] =
        I - 2 * Bilinear_Form_s *
        List_of_Unit_Vectors[i] *
        List_of_Unit_Vectors[i].T



Reflections = [I] * N

Reflections[0] = Generators[0]
Reflections[1] = Generators[1]
Reflections[2] = Generators[2]


for i in range(3,N):

    Reflections[i] =
        Generators[mod(i,3)] * Reflections[int(i/3)-1]


Remember_Repetition = [0] * N


for i in range(0,N):

    for j in range(i+1,N):

        if Reflections[i] == Reflections[j]:

            Remember_Repetition[j] = 1


### Plot

Convex_Set = polygon(
                [[Rotation[0][0],Rotation[1][0]],
                [Rotation[0][1],Rotation[1][1]],
                [Rotation[0][2],Rotation[1][2]]],
                color='blue')

for i in range(N):
```

```
    if Remember_Repetition[i] == 0:

        L_1 = Rotation * Reflections[i] * e_1
        L_2 = Rotation * Reflections[i] * e_2
        L_3 = Rotation * Reflections[i] * e_3

        L_1 = L_1 / sqrt(L_1*L_1)
        L_2 = L_2 / sqrt(L_2*L_2)
        L_3 = L_3 / sqrt(L_3*L_3)


        select_color = is_in_interval(i+1)


        if select_color == 0:
            Convex_Set +=
                polygon([[L_1[0],L_1[1]],
                        [L_2[0],L_2[1]],
                        [L_3[0],L_3[1]]],
                        color='yellow')


        if select_color == 1:
            Convex_Set +=
                polygon([[L_1[0],L_1[1]],
                        [L_2[0],L_2[1]],
                        [L_3[0],L_3[1]]],
                        color='blue')


show(Convex_Set,axes=0)
```