Seminar: Introduction to Riemannian Geometry
Differentiable Manifolds

Carlos Sotillo Rodríguez

In the following Chapter we are going to define the main concepts as well as the basic results. From now on, differentiable will always signify of class C^∞

Definition 1: A differentiable manifold of dimension n is a set M and a family of injective mappings $x_\alpha : U_\alpha \subset \mathbb{R}^n \to M$ (where U_α is an open set of $\mathbb{R}^n$), such that:

1. $\bigcup\limits_\alpha x_\alpha(U_\alpha) = M$
2. for any pair α, β with $x_\alpha(U_\alpha) \bigcap x_\beta(U_\beta) = W \neq \emptyset$, the sets $x_\alpha^{-1}(W)$ and $x_\beta^{-1}(W)$ are open sets of $\mathbb{R}^n$ and the mappings $x_\beta^{-1} \circ x_\alpha$ are differentiable.
3. The family $\{(U_\alpha, x_\alpha)\}$ is maximal relative to conditions 1 and 2.

The pair (U_α, x_α) (or the mapping x_α) with $p \in x_\alpha(U_\alpha)$ is called a parametrization (or system of coordinates) of M at p, $x_\alpha(u_\alpha)$ is then called a coordinated neighborhood at p. A family satisfying 1 and 2 is called a differentiable structure.
The condition 3 is included for technical reasons. Indeed, given a differentiable structure, we can complete it to a maximal one. Therefore, with a certain abuse of lenguage, we can say that a differentiable manifold is a set with a differentiable structure.

Definition 2: Let M_1^n and M_2^m (where n, m are the dimensions of M_1, M_2 respectively). A mapping $\phi : M_1 \to M_2$ is differentible at $p \in M_1$ if given a parametrization $y : V \subset \mathbb{R}^m \to M_2$ at $\varphi(p)$ there exists a parametrization $x : U \subset \mathbb{R}^n \to M_1$ at p such that $\varphi(x(U)) \subset y(V)$ and the mapping

$$y^{-1} \circ \varphi \circ x : U \subset \mathbb{R}^n \to \mathbb{R}^m$$

is differentiable at $x^{-1}(p)$. φ is differentiable on an open set of M_1 if it is differentiable at every point of this open set (it follows from condition 2 that this definition does not depend on the parametrization).The expression $y^{-1} \circ \varphi \circ x$ is called the expression of φ in the parametrizations x and y.

Definition 3: Let M be a differentiable manifold. A differentiable function $\alpha : (-\epsilon, \epsilon) \to M$ is called a (differentiable) curve in M. Suppose that $\alpha(0) = p \in M$ and let D be the set of functions on M that are differentiable at p. The tangent vector to the curve α at $t = 0$ is a function $\alpha'(0) : D \to \mathbb{R}$ given by

$$\alpha'(0)(f) = \left.\frac{d(f \circ \alpha)}{dt}\right|_{t=0}, \; f \in D$$

A tangent vector at p is the tangent vector at $t = 0$ of some curve $\alpha : (-\epsilon, \epsilon) \to M$ with $\alpha(0) = p$. The set of all tangent vectors to M at p will be indicated by T_pM.
Further, the set T_pM is a vector space with assiciated basis $\{(\frac{\partial}{\partial x_1})_0, \cdots, (\frac{\partial}{\partial x_n})_0\}$. It is immediate that the basis depends on the parametrization, but not the linear structure.

Proposition 4: Let M_1^n and M_2^m be differentiable manifolds and let $\varphi : M_1 \to M_2$ be a differentiable mapping. For every $p \in M_1$ and for each $v \in T_pM_1$, choose a differentiable curve $\alpha : (-\epsilon, \epsilon) \to M_1$ with $\alpha(0) = p, \alpha'(0) = v$. Take $\beta = \varphi \circ \alpha$. The mapping $d\varphi_p : T_pM_1 \to T_{\varphi(p)}M_2$ given by $d\varphi_p(v) = \beta'(0)$ is a linear mapping that does not depend on the choice of α.

Definitions 5:

- The linear mapping $d\varphi_p$ is called the differential of φ at p.
- Let M_1 and M_2 be differential manifolds. A mapping $\varphi : M_1 \to M_2$ is a diffeomorphism if it is differentiable, bijective and its inverse is differentiable. φ is said to be a local diffeomisphism at $p \in M_1$ if there exist neighborhoods U of p and V of $\varphi(p)$ such that $\varphi : U \to V$ is a diffeomorphism.

Theorem 6: Let $\varphi : M_1^n \to M_2^n$ be a differentiable mapping and let $p \in M_1$ be. $d\varphi_p : T_pM_1 \to T_{\varphi(p)}M_2$ is an isomorphism if and only if φ is a local diffeomorphism at p.

Definition 7: Let M^m and N^n be differentiable manifolds. A differentiable mapping $\varphi : M \to N$ is said to be an immersion if $d\varphi_p : T_pM \to T_{\varphi(p)}N$ is injective for all $p \in M$. If in addition, φ is a homeomorphism onto $\varphi(M) \subset N$, where $\varphi(M)$ has the subspace topology induced from N, we say that φ is an embedding. If $M \subset N$ and the inclusion $i : M \subset N$ is an embedding, we say that M is a submanifold of N.

Proposition 8: Let $\varphi : M_1^n \to M_2^m$, $n \le m$, be an immersion of the differentiable manifold M_1 into the differentiable manifold M_2. For every point $p \in M_1$, there exists a neighborhood $V \subset M_1$ of p such that the restriction $\varphi_{|V} : V \to M_2$ is an embedding.

Definition + Example 9 (The tangent bundle): Let M^n be a differentiable manifold and let $TM := \{(p, v) : p \in M, v \in T_pM\}$. We are going to provide the set TM with a differentiable structure.
Let $\{(U_\alpha, x_\alpha)\}$ be a maximal differentiable structure on M. Denote by $(x_1^\alpha, \cdots, x_n^\alpha)$ the coordinates of U_α and by $\{\frac{\partial}{\partial x_1^\alpha}, \cdots, \frac{\partial}{\partial x_n^\alpha}\}$ the associated bases to the tangent spaces of $x_\alpha(U_\alpha)$. For every α, define

$$y_\alpha : U_\alpha \times \mathbb{R}^n \to TM$$

by

$$y_\alpha(x_1^\alpha, \cdots, x_n^\alpha, u_1, \cdots, u_n) = (x_\alpha(x_1^\alpha, \cdots, x_n^\alpha), \sum_{i=1}^n u_i \frac{\partial}{\partial x_i^\alpha}), \qquad (u_1, \cdots, u_n) \in \mathbb{R}^n$$

We are going to show that $\{(U_\alpha \times \mathbb{R}^n, y_\alpha\})$ is a differentiable structure on TM. Since $\bigcup_\alpha x_\alpha(U_\alpha) =$

M and $(dx_\alpha)_q(\mathbb{R}^n) = T_{x_\alpha(q)}M,\ q \in U_\alpha$, we have only to prove the condition 2. Now let

$$(p, v) \in y_\alpha(U_\alpha \times \mathbb{R}^n) \bigcap y_\beta(U_\beta \times \mathbb{R}^n)$$

Then

$$(p, v) = (x_\alpha(q_\alpha), dx_\alpha(v_\alpha)) = (x_\beta(q_\beta), dx_\beta(v_\beta)),$$

where $q_\alpha \in U_\alpha, q_\beta \in U_\beta, v_\alpha, v_\beta \in \mathbb{R}^n$. Therefore,

$$y_\beta^{-1} \circ y_\alpha(q_\alpha, v_\alpha) = y_\beta^{-1}(x_\alpha(q_\alpha), dx_\alpha(v_\alpha)) = ((x_\beta^{-1} \circ x_\alpha)(q_\alpha), d(x_\beta^{-1} \circ x_\alpha)(v_\alpha)).$$

Since $x_\beta^{-1} \circ x_\alpha$ is differentiable, $d(x_\beta \circ x_\alpha)$ is as well. It follows that condition 2 is verified.

Example 10 (Regular surfaces in $\mathbb{R}^n$): A subset $M^k \subset \mathbb{R}^n$ is a regular surface of dimension $k \leq n$ if for every point $p \in M^k$ there exists a neighborhood V of p in $\mathbb{R}^n$ and a mapping $x : U \to M \cap V$ of an open set $U \subset \mathbb{R}^k$ onto $M \cap V$ such that:

- x is a differentiable homeomorphism.
- $(dx)_q : \mathbb{R}^k \to \mathbb{R}^n$ is injective for all $q \in U$.

It can be proved that if $x : U \to M^k$ and $y : V \to M^k$ are two parametrizations with $x(U) \cap y(V) = W \neq \emptyset$, then the mapping $h = x^{-1} \circ y : y^{-1}(W) \to x^{-1}(W)$ is a diffeomorphism. We give a sketch of the proof: Let $(u_1, \cdots, u_k) \in U$ and $(v_1, \cdots, v_n) \in \mathbb{R}^n$, and write x in these coordinates as

$$x(u_1, \cdots, u_k) = (v_1(u_1, \cdots, u_k), \cdots, v_n(u_1, \cdots u_k))$$

From condition 2 we can suppose that

$$\frac{\partial(v_1, \cdots, v_k)}{\partial(u_1, \cdots, u_k)}(q) \neq 0$$

Extend x to a mapping $F : U \times \mathbb{R}^{n-k} \to \mathbb{R}^n$ given by

$$\begin{aligned} F(u_1, \cdots, u_k, t_{k+1}, \cdots, t_n) =& (v_1(u_1, \cdots, u_k), \cdots, v_k(u_1, \cdots, u_k), v_{k+1}(u_1, \cdots, u_k) \\ &+ t_{k+1}, \cdots, v_n(u_1, \cdots, u_k) + t_n) \end{aligned}$$

where $(t_{k+1}, \cdots, t_n) \in \mathbb{R}^{n-k}$. It is clear that F is differentiable and the restriction of F to $U \times \{(0, \cdots, 0)\}$ coincides with x. By a calculation, we obtain that

$$det(dF_q) = \frac{\partial(v_1, \cdots, v_k)}{\partial(u_1, \cdots, u_k)}(q) \neq 0$$

Now we are under the conditions of the inverse function theorem, which guarantees the existence of a neighborhood Q of $x(q)$ where F^{-1} exists and is differentiable. By the continuity of y, there exists a neighborhood $R \subset V$ of r such that $y(R) \subset Q$. Note that the restriction of h to R, $h_{|R} = F^{-1} \circ y_{|R}$ is a composition of differentiable mappings. Thus h is differentiable at r, hence in $y^{-1}(W)$. A similar argument would prove that h^{-1} is differentiable as well, proving the assertion.

Definition 11: Let M be a differentiable manifold. We say that M is orientable if M admits a differentiable structure $\{(U_\alpha, x_\alpha)\}$ such that:

- for every pair α, β with $x_\alpha(U_\alpha) \cap x_\beta(U_\beta) = W \neq \emptyset$, the differential of the change of coordinates $x_\beta^{-1} \circ x_\alpha$ has positive determinant.

In the opposite case, we say that M is non-orientable. If M is orientable, a choice of a differentiable structure satisfying th condition from above is called as orientation of M.

Example 12 (Discontinuous action of a group): We say that a group G acts on a differentiable manifold M if there exists a mapping $\varphi : G \times M \to M$ such that:

1. For each $g \in G$, the mapping $\varphi_g : M \to M$ given by $\varphi_g(p) = \varphi(g, p),\ p \in M$, is a diffeomorphism, and $\varphi_e = Id_M$.

2. If $g_1, g_2 \in G$, $\varphi_{g_1 g_2} = \varphi_{g_1} \circ \varphi_{g_2}$.

Frequently, when dealing with a single action, we set $gp := \varphi(g, p)$.
We say that the action is properly discontinuous if every $p \in M$ has a neighborhood $U \subset M$ such that $U \cap g(U) = \emptyset$ for all $g \neq e$. When G acts on M, the action determines an equivalence relation $\sim$ on M, in which $p_1 \sim p_2$ if and only if $p_2 = gp_1$ for some $g \in G$. Denote the quotient space of M by this equivalence relation by M/G. The mapping $\pi : M \to M/G, p \mapsto Gp$ will be called the projection of M onto M/G.
Now let M be a differentiable manifold and let $G \times M \to M$ be a properly discontinuous action of a group G on M. We are going to show that there is a differentiable structure on M/G such that $\pi : M \to M/G$ is a local diffeomorphism.
For each point $p \in M$ choose a parametrization $x : V \to M$ at p so that $x(V) \subset U$, where $U \subset M$ is a neighborhood of p such that $U \cap g(U) = \emptyset, g \neq e$. Because of that $\pi_{|U}$ is injective, hence $y = \pi \circ x : V \to M/G$ is injective. The family $\{(V, y)\}$ clearly covers M/G. Now we are going to show that this family is a differentiable structure; let $y_1 = \pi \circ x_1 : V_1 \to M/G$ and $y_2 = \pi \circ x_2 : V_2 \to M/G$ be two mappings with $y_1(V_1) \cap y_2(V_2) \neq \emptyset$ and let π_i be the restriction of π to $x_i(V_i),\ i = 1, 2$. Let $q \in y_1(V_1) \cap y_2(V_2)$ and let $r = x_2^{-1} \circ \pi_2^{-1}(q)$. Let $W \subset V_2$ be a neighborhood of r such that $(\pi_2 \circ x_2)(W) \subset y_1(V_1) \cap y_2(V_2)$. Then the restriction to W is given by

$$(y_1^{-1} \circ y_2)_{|W} = x_1^{-1} \circ \pi_1^{-1} \circ \pi_2 \circ x_2$$

Therefore, it is enough to show that $\pi_1^{-1} \circ \pi_2$ is differentiable at $p_2 = \pi_2^{-1}(q)$. Let $p_1 = \pi_1^{-1} \circ \pi_2(p_2)$. Then p_1 and p_2 are equivalent in M, hence there is a $g \in G$ such that $gp_2 = p_1$. It follows easily that the restriction $(\pi_1^{-1} \circ \pi_2)_{|x_2(W)}$ coincides with the diffeomorphism $\varphi_{|x_2(W)}$, which proves that $\pi_1^{-1} \circ \pi_2$ is differentiable at p_2 and so we have constructed a differentiable structure on M/G.

Proposition 13: With the notation from the previous example the manifold M/G is orientable if and only if there exists an orientation of M that is preserved by all the diffeomorphisms of G.

Definition 14: A vector field X on a differentiable manifold M is a correspondence that asociates to each point $p \in M$ a vector $X(p) \in T_pM$. In terms of mappings, X is a mapping of M into the tangent bundle TM. The field is differentiable if the mapping $X : M \to TM$ is differentiable.

Considering the parametrization $x : U \subset \mathbb{R}^n \to M$ we can write

$$X(p) = \sum_{i=1}^{n} a_i(p) \frac{\partial}{\partial x_i}$$

where each $a_i : U \to \mathbb{R}$ is a function on U and $\{\frac{\partial}{\partial x_i}\}$ is the basis asociated to x, $i = 1, \cdots, n$. It is clear that X is differentiable if and only if the functions a_i are differentiable for some (and, therefore, for any) parametrization. Hence if f is a differentiable function on M, we have the following expression

$$(Xf)(p) = \sum_i a_i(p) \frac{\partial}{\partial x_i}(p)$$

where f denotes the expression of f in the parametrization x. It is immediate that X is differentiable if and only if $Xf \in D$ for any $f \in D$, that is when $X : D \to D$. Observe that if $\varphi : M \to M$ is a diffeomorphism, $v \in T_pM$ and f is a differentiable function in a neighborhood of $\varphi(p)$, we have

$$(d\varphi(v)f)\varphi(p) = \frac{d}{dt}(f \circ \varphi \circ \alpha)\bigg|_{t=0} = v(f \circ \varphi)(p)$$

Now, we are able to see some results of vector fields.

Lemma 15: Let X and Y be differentiable vector fields on a differentiable manifold M. Then there exists a unique vector field Z such that, for every differentiable function f, $Zf = (XY - YX)f$.

The vector field Z is called the bracket of X and Y ($Z = [X,Y] = XY - YX$) and has the following properties.

Proposition 16: If X, Y and z are differentiable vector fields on M, $a, b \in \mathbb{R}$ and f, g are differentiable functions, then:

1. $[X,Y] = -[Y,X]$
2. $[aX + bY, Z] = a[X,Z] + b[Y,Z]$
3. $[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0$
4. $[fX, gY] = fg[X,Y] + fX(g)Y - gY(f)X$

Theorem 17: Let X be a differentiable vector field on a differentiable manifold M, and let $p \in M$. Then there exist a neighborhood $U \subset M$ of p, an interval $(-\delta, \delta)$, $\delta > 0$, and a curve $t \mapsto \varphi(t,q)$, $t \in (-\delta,\delta)$, $q \in U$, is the unique curve which satisfies $\frac{\partial \varphi}{\partial t} = X(\varphi(t,q))$ and $\varphi(0,q) = q$. The function $\varphi_t : U \to M, q \mapsto \varphi(t,q)$ is called the local flow of X.

Proposition 18: Let X, Y be differentiable vector fields on a differentiable manifold M, let $p \in M$, and let φ_t be the local flow of X in a neighborhood U of p. Then

$$[X,Y](p) = \lim_{t \to 0} \frac{1}{t}[Y - d\varphi_t Y](\varphi_t(p))$$

Definition 19:

1. Locally finite: A family of open sets $V_\alpha \subset M$ with $\bigcup_\alpha V_\alpha = M$ is said to e locally finite if every point $p \in M$ has a neighborhood W such that $W \cap V_\alpha \neq \emptyset$ for only a finite number of indices.

2. Support: The support of a function $f : M \to \mathbb{R}$ is the closure of the set of points where f is different from zero.

3. Differentiable partition of unity: We say that a family $\{f_\alpha\}$ of differentiable functions $f_\alpha : M \to \mathbb{R}$ is a differentiable partition of unity if:
   - For all α, $f_\alpha \geq 0$ and the support of f_α is contained in a coordinate neighborhood $V_\alpha = x_\alpha(U_\alpha)$ of a differentiable structure $\{(U_\beta, x_\beta)\}$ of M.
   - The family $\{V_\alpha\}$ is locally finite.
   - $\sum_\alpha f_\alpha(p) = 1$, for all $p \in M$ (this condition makes sense, because for each $p \in M$, $f_\alpha(p) \neq 0$ only for a finite number of indices).

Theorem 20: A differentiable manifold M has a partition of unity if and only if every connected component of M is Hausdorff and has a countable basis.