Seminar “Differential forms and their use”

Integration on manifolds

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1 Integration of Differential Forms

Our goal is to define the integral of a differential n-form on an n-dimensional differentiable Manifold. For that, we will first start with the case of $\mathbb{R}^n$, then move to a Manifold with a straightforward structure and, after proving the existence of the Partition of Unity, generalize it for all differentiable manifolds.

Let ω be a differential n-form defined on the open set $U \subset M^n$. It's support K is the closure of the set A defined by:

$$A = \{p \in M^n;\ w(p) \neq 0\}$$

1.1 The real case (n-form on $\mathbb{R}^n$)

To handle the real case let us assume that $M^n = \mathbb{R}^n$ with

$$\omega = a(x_1, \ldots, x_n)\, dx_1 \wedge \ldots \wedge dx_n$$

Assume furthermore, that the support K of ω is compact and $K \subset U$.

We now define:

$$\int_U \omega = \int_K a\, dx_1 \ldots dx_n$$

Notice that the right side is the usual multiple integral in $\mathbb{R}^n$.

1.2 The general case (n-form on M^n)

Let now M be compact (K, the support of ω will also be compact) and orientable (the coordinate changes have positive jacobians).

We will distinguish two cases depending on K being contained in some coordinate neighborhood or not.

1.2.1 K is contained in some coordinate neighborhood

With $K \subset V_\alpha = f_\alpha(U_\alpha)$ (coordinate neigborhood), the local representation ω_α of ω in U_α is

$$\omega_\alpha = a_\alpha(x_1, \ldots, x_n)\, dx_1 \wedge \ldots \wedge dx_n$$

Similar to 1.1, we now define:

$$\int_M \omega = \int_{V_\alpha} \omega_\alpha = \int_{U_\alpha} a_\alpha dx_1 \ldots dx_n$$

Again, the right hand side is an integral in $\mathbb{R}^n$.

Given that otehr neighborhoods are possible, we still have to show that this definition is independent of the choice of coordinate neighborhood.

Let K be contained in another coordinate neighborhood $V_\beta = f_\beta(U_\beta)$. We can safely assume that $V_\alpha = V_\beta$. If that's not the case, we can contract both U_α and U_β until this applies.

With $(x_1, \ldots, x_n) \in U_\alpha$ and $(y_1, \ldots, y_n) \in U_\beta$, the change of coordinates

$$f = f_\alpha^{-1} \circ f_\beta : U_\beta \longrightarrow U_\alpha$$

is given by $x_i = f_i(y_1, \ldots, y_n), \;\; i = 1, \ldots, n.$

Because $\omega_\beta = f^*(\omega_\alpha)$, we have

$$\omega_\beta = \det(df) a_\beta dy_1 \wedge \ldots \wedge dy_n$$

where a_β is as follows:

$$a_\beta = a_\alpha(f_1(y_1, \ldots, y_n), \ldots, f_n(y_1, \ldots, y_n))$$

By applying the change of variables for multiple integrals in $\mathbb{R}^n$, we obtain:

$$\int_{V_\alpha} \omega_\alpha = \int_{U_\alpha} a_\alpha dx_1 \ldots dx_n = \int\limits_{U\beta} \det(df) a_\beta dy_1 \ldots dy_n \overset{1}{=} \int_{V_\beta} \omega_\beta$$

Note that the last equality (1) holds because $\det(df) > 0$, which explains the reason for the assumption that M is oriented: without the orientation, the sign of the integral wouldn't be well defined. $\square$

1.2.2 K is contained in no coordinate neighborhood

In the preceding section we saw that the integral of a differential form with a domain contained in a coordinate system was just a multiple integral. Now, to be able to integrate in more complex domains, we need to reduce that complexity in some way, which can be done in two different ways: by dividing the domain in simpler domains and adding up the results, or by decomposing the form into forms which are zero outside some simpler domains.

The second alternative is the one we will be using with the help of the *partition of unity*, which will enable us to give a simple definition of the form's integral :

$$\int_M \omega = \sum_{i=1}^{m} \int_M \varphi_i \omega$$

Note: the $\sum\limits_{i=1}^{m} \varphi_i$ contained in the definition is the aforementioned partition of unity

The Partition of Unity

Definition 1. *Given a covering $\{V_\alpha\}$ of a compact differentiable manifold M, a differentiable partition of the unity subordinate to the covering $\{V_\alpha\}$ is a family of differentiable functions $\varphi_1, \ldots, \varphi_m$ such that:*

1. $\sum\limits_{i=1}^{m} \varphi_i = 1$
2. $0 \leq \varphi_i \leq 1 \;\; i \in \{1, \ldots, m\}$ *and the support of φ_i is contained in some $V_{\alpha_i} = V_i$.*

Note: if M is orientable, $\{V_\alpha\}$ is chosen compatible with the said orientation.

We now want to prove the existence of a differentiable partition of the unity subordinate to a given covering by coordinate neighborhoods of a compact differentiable manifold Mn (not necessarily oriented).

We will first need two technical lemma that will later help us to prove the existence of the partition of unity.

We define $B_r(0) = \{ p \in \mathbb{R}^n ; |p| < r \}$.

Lemma 1. *There exists a differentiable function $\varphi : B_3(0) \longrightarrow \mathbb{R}$ such that:*

1. $\varphi(p) = 1$, *if* $p \in B_1(0)$
2. $0 \leq \varphi(p) \leq 1$, *if* $p \in B_2(0)$
3. $\varphi(p) = 0$, *if* $p \in B_3(0) - B_2(0)$

Proof. Consider the C^∞ function $\alpha : \mathbb{R} \longrightarrow \mathbb{R}$ given by

$$\begin{aligned} \alpha(t) &= e^{-\frac{1}{(t+1)(t+2)}}, \qquad && t \in (-2,-1) \\ \alpha(t) &= 0, && t \notin (-2,-1) \end{aligned}$$

and the integral

$$\gamma(t) = \int_{-\infty}^{t} \alpha(s) ds$$

The maximum value of the differentiable function γ is given by $\int_{-2}^{-1} \alpha(s)ds = A$ and, by setting $\beta(t) = \gamma(t)/A$ we obtain a differentiable function with the following properties:

$$\begin{aligned} \beta(t) &= 0, \qquad && \text{if } t \leq -2 \\ 0 \leq \beta(t) &\leq 1, && \text{if } t \in (-2,-1) \\ \beta(t) &= 1, && \text{if } t \geq -1. \end{aligned}$$

We obtain the required function $\varphi : B_3(0) \longrightarrow \mathbb{R}$ by defining $\varphi(p) = \beta(-|p|)$, $p \in B_3(0)$. □

Lemma 2. *Let M^n be a differentiable manifold, let $p \in M$ and let $g : U \subset \mathbb{R}^n \longrightarrow M$ be a parametrization around p. Then, it is possible to obtain a parametrization $f : B_3(0) \longrightarrow M$ around p in such a way that $f(B_3(0)) \subset g(U)$ and that $f^{-1}(p) = (0, \dots, 0)$.*

Proof. Let $(x_1^0, \dots, x_n^0) \in U$ be such that $g(x_1^0, \dots, x_n^0) = p$. U is open so there exists an $r > 0$ such that $B_r(x_1^0, \dots, x_n^0) \subset U$. Let T be translation in $\mathbb{R}^n$ that takes $(x_1^0, \dots, x_n^0)$ to $(0, \dots, 0)$ and let $H : \mathbb{R} \longrightarrow \mathbb{R}$ be defined by $H(p) = \frac{3}{r}p$.

Using $H \circ T : B_r(x_1^0, \dots, x_n^0) \longrightarrow B_0(0, \dots, 0)$ we define the parametrization $f : B_3(0) \longrightarrow M$ by

$$f = g \circ T^{-1} \circ H^{-1}$$

which satisfies the required conditions. □

Proposition 1. *(Existence of a differentiable partition of unity). Let M be a compact manifold and let $\{V_\alpha\}$ be a covering of M by coordinate neighborhoods. Then there exists differentiable functions $\varphi_1, \dots, \varphi_m$ conforming a differentiable partition of the unity.*

Proof. For each $p \in M$ we consider the parametrization f_p given by Lemma 2 with $f_p(B_3(0)) = V_p \subset V_\alpha$ for some $V_\alpha \in \{V_\alpha\}$. Set $W_p = f_p(B_1(0)) \subset V_p$.

The family $\{W_\alpha\}$ is an open covering of M and we can select from it a (compactness) finite covering $W_1, \dots, W_m$ and the corresponding covering $V_1, \dots, V_m$.

We define functions $\theta_i : M \longrightarrow \mathbb{R},\ \ i = 1, \dots, m$, by

$$\theta_i = \varphi \circ f_i^{-1} \text{ in } V_i; \quad \theta_i = 0 \text{ in } M - V_i \quad (\varphi \text{ as given by Lemma 1})$$

These functions are differentiable and the support of θ_i is contained in V_i.

To finish up, we define the family φ_i that satisfies the required conditions:

$$\varphi_i(p) = \frac{\theta_i(p)}{\sum_{j=1}^{m} \theta_j(p)}\ ,\ \ p \in M$$

□

Armed with the partition of unity we can now define the integral of an n-form ω on M^n, a compact oriented differential manifold, as follows:

Let $\{\varphi_i\}$ a differentiable partition of unity subordinate to the covering $\{V_\alpha\}$. The support of the form $\varphi_i\omega$ is contained in V_i. We now set:

$$\int_M \omega = \sum_{i=1}^{m} \int_M \varphi_i \omega$$

We only have to demonstrate the independence from the chosen covering (and its partition of unity).

Consider hence another covering $\{W_\beta\}$ of M with the same orientation as $\{V_\alpha\}$, and let $\{\psi_j\}\ ,\ j = 1, \dots, s$ be its subordinate partition of unity. Then, $\{V_\alpha \cap W_\beta\}$ will be a covering of M with a subordinate partition of unity $\{\varphi_i\psi_j\}$. Because for each i, respectively j, the functions are defined in V_i, respectively in W_j, we obtain the following equalities:

$$\sum_{i=1}^{m} \int_M \varphi_i \omega = \sum_{i=1}^{m} \int_M \varphi_i \left(\sum_{j=1}^{s} \psi_j \right) \omega = \sum_{i,j} \int_M \varphi_i \psi_j \omega$$
$$\sum_{j=1}^{s} \int_M \psi_j \omega = \sum_{j=1}^{s} \int_M \left(\sum_{i=1}^{m} \varphi_i \right) \psi_j \omega = \sum_{i,j} \int_M \varphi_i \psi_j \omega$$

□

2 The Stokes Theorem

In this section, we want to prove Stokes Theorem, which states that, given an n-form ω on a bounded manifold M, the integral of $d\omega$ in M equals the one of $i^*\omega$ in ∂M (i being the inclusion of the boundary in the manifold).

2.1 Preliminaries

First we will need some new definitions!

Definition 2. *A half-space of* $\mathbb{R}^n$ *is the set*

$$H^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n;\ x_1 \leq 0\}$$

An open set in H^n is the intersection with H^n of an open set U in $\mathbb{R}^n$

A function $f : V \longrightarrow \mathbb{R}$ defined in an open set $V \in H^n$ is differentiable if there exists an open set $U \in V$ and a differentiable function $\overline{f}$ in U such that $\overline{f}_{|V} = f$. The differential of f at $p \in V$ is defined as $df_p = d\overline{f}_p$.

Definition 3. *An n-dimensional differentiable manifold with (regular) boundary is a set M and a family of injective maps $f_\alpha : U_\alpha \subset H^n \longrightarrow M$ of open sets of $U_\alpha \subset H^n$ into M such that:*

1. $\bigcup\limits_\alpha f_\alpha(U_\alpha) = M$
2. *For all pairs α, β with $f_\alpha(U_\alpha) \cap f_\beta(U_\beta) = W \neq \varnothing$ the sets $f_\alpha^{-1}(W)$ and $f_\beta^{-1}(W)$ are open sets in H^n and the maps $f_\beta^{-1} \circ f_\alpha$, $f_\alpha^{-1} \circ f_\beta$ are differentiable.*
3. *The family is maximal relative to 1 and 2.*

A point $p \in M$ is said to be a point in the boundary of M if for some parametrization around p we have that $f(0, x_2, \ldots, x_n) = p$.

Note that the definition is almost identical to the definition of differential Manifold, only adding the boundary and replacing $\mathbb{R}^n$ by H^n. All other definitions are introduced the same way just replacing $\mathbb{R}^n$ by H^n.

The set of points in the boundary of M is called the boundary of M and denoted by ∂M. If $\partial M = \varnothing$ the above definition agrees with the one of a differentiable manifold given in the previous lecture.

Lemma 3. *The definition of point in the boundary does not depend on the parametrization*

Proof. Let $f_1 : U_1 \longrightarrow M$ be a parametrization around p with $f_1(q) = p$, $q = (0, x_2, \ldots, x_n)$ and assume the existence of a parametrization $f_2 : U_2 \longrightarrow M$ around p with $f_2(q_2) = p$, $q_2 = (x_1, x_2, \ldots, x_n)$ and $x_1 \neq 0$. Furthermore, let $W = f_1(U_1) \cap f_2(U_2)$.

The map

$$f_1^{-1} \circ f_2 : \, f_2^{-1}(W) \longrightarrow f_1^{-1}(W)$$

is a diffeomorphism. Because $x_1 \neq 0$, there is a neighborhood U of q_2, $U \subset f_2^{-1}(W)$, that does not intersect the x_1-axis.

The restriction $f_1^{-1} \circ f_2$ to U is differentiable with non-zero jacobian and, because of the inverse function theorem, it will take a neighborhood $V \subset U$ of q_2 diffeomorphically onto $f_1^{-1} \circ f_2(V)$. But this would mean that $f_1^{-1} \circ f_2(V)$ contains points $(x_1, x_2, \ldots, x_n)$ with $x_1 > 0$ and thus not in H^n, which is a contradiction to our assumption. $\square$

Proposition 2. *The boundary ∂M of an n-dimensional differentiable manifold M with boundary is an (n-1)-differentiable manifold. Furthermore, if M is orientable, an orientation for M induces an orientation for ∂M.*

Proof. Let $p \in M$ be a point in the boundary of M and let $f_a : U_\alpha \longrightarrow M^n$ be a parametrization around p.

Given $f_\alpha^{-1}(p) = (0, x_2, \ldots, x_n) \in U_\alpha$, let $\overline{U}_\alpha = U_\alpha \cap \{(x_1, \ldots, x_n) \in \mathbb{R}^n ;\ x_1 = 0\}$, which is an open set in $\mathbb{R}^{n-1}$, and $\overline{f}_\alpha = f_{\alpha|\overline{U}_\alpha}$.

According to Lemma 3, $\overline{f}_\alpha(\overline{U}_\alpha) \subset \partial M$ and by letting p run the points of ∂M, we see that $\{(\overline{U}_\alpha, \overline{f}_\alpha)\}$ is a differentiable structure of ∂M and ∂M a (n-1)-differentiable manifold.

Now assume that M is orientable and set an orientation $\{(U_\alpha, f_\alpha)\}$ for it. $\{(\overline{U}_\alpha, \overline{f}_\alpha)\}$ is, as per above, a differentiable structure for ∂M. We want to show that $\det(d(\overline{f}_\alpha^{-1} \circ \overline{f}_\beta)_q) > 0$; $q \in \partial M$ (the jacobian of the change of coordinates is positive).

For a point q whose image is on the boundary, because the change of coordinates $f_\alpha \circ f_\beta^{-1}$ takes a point $(0, x_2^\beta, \ldots, x_n^\beta)$ into a point $(0, x_2^\alpha, \ldots, x_n^\alpha)$, we get:

$$\det(d(\overline{f}_\alpha^{-1} \circ f_\beta)) = \frac{\partial x_1^\alpha}{\partial x_1^\beta} \det(d(\overline{f}_\alpha^{-1} \circ \overline{f}_\beta))$$

$\frac{\partial x_1^\alpha}{\partial x_1^\beta} > 0$ and, since $\det(d(f_\alpha^{-1} \circ f_\beta))$ is positive by hypothesis, it follows that $\det(d(\overline{f}_\alpha^{-1} \circ \overline{f}_\beta)_q) > 0$; $q \in \partial M$. ☐

2.2 Stokes Theorem

Theorem 1. *(Stokes Theorem) Let M^n be a differentiable manifold with boundary, compact and oriented. Let ω be a differential (n-1)-form on M, and let $i : \partial M \longrightarrow M$ be the inclusion map of the boundary ∂M into M. Then*

$$\int_{\partial M} i^*\omega = \int_M d\omega$$

Proof. Let K be the support of ω. We will consider two cases: K is contained and K is not contained in a coordinate neighborhood.

A K is contained in some coordinate neighborhood $V = f(U)$ of a parametrization $f : U \subset H^n \longrightarrow M$. In U,

$$\omega = \sum_{j=1}^{n} a_j dx_1 \wedge \ldots \wedge dx_{j-1} \wedge dx_{j+1} \wedge \ldots \wedge dx_n$$

Where $a_j = a_j(x_1, \ldots, x_n)$ is a differentiable function on U. $d\omega$ is given by:

$$d\omega = \left(\sum_{j=1}^{n} (-1)^{j-1} \frac{\partial a_j}{\partial x_j} \right) dx_1 \ldots dx_n$$

A1 We will first examine the case where the border isn't contained in the coordinate neighborhood.

In such a case ω is zero in ∂M and $i^*\omega = 0$ which gives us $\int_{\partial M} i^* M = 0$ and we only need to show that $\int_M d\omega = 0$ to end the proof.

Let us first extend a_j to H^n:

$$\begin{aligned} &a_j(x_1, \ldots, x_n) = a_j(x_1, \ldots, x_n), && \text{if} \quad (x_1, \ldots, x_n) \in U \\ &a_j(x_1, \ldots, x_n) = 0, && \text{if} \quad (x_1, \ldots, x_n) \in H^n - U \end{aligned}$$

The functions are differentiable in H^n ($f^{-1}(K) \subset U$). We now construct a parallelepiped $Q \subset H^n$ with boundaries $x_j^1 \leq x_j \leq x_j^0$ containing $f^{-1}(K)$. We then proceed to explicitly calculate the integral:

$$\begin{aligned} \int_M d\omega &= \int_U \left(\sum_{j=1}^{n} (-1)^{j-1} \frac{\partial a_j}{\partial x_j} \right) dx_1 \ldots dx_n = \sum_j (-1)^{j-1} \int_Q \frac{\partial a_j}{\partial x_j} dx_1 \ldots dx_n \\ &= \sum_j (-1)^{j-1} \int [a_j(x_1, \ldots, x_{j-1}, x_j^0, x_{j+1}, \ldots, x_n) \\ &\qquad - a_j(x_1, \ldots, x_{j-1}, x_j^1, x_{j+1}, \ldots, x_n)] \, dx_1 \ldots dx_{j-1} dx_{j+1} \ldots dx_n = 0 \end{aligned}$$

With the last equality being a result of $a_j = 0$ on the edges of Q.

A2 Lets now assume that $f(U) \cap \partial M \neq \varnothing$. We write the inclusion map i as $x_1 = 0,\ x_j = x_j$

We will extend the a_j functions as in A1 and construct a parallelepiped Q containing $f^{-1}(K)$, given by $x_1^1 \leq x_1 \leq 0,\ x_j^1 \leq x_j \leq x_j^0,\quad j \in \{2, \dots, n\}$. Again, we only need to calculate the integral:

$$\begin{aligned}\int_M d\omega &= \sum_j (-1)^{j-1} \int_Q \frac{\partial a_j}{\partial x_j} dx_1 \dots dx_n \\ &= \int\limits_Q [a_1(0, x_2, \dots, x_n) - a_1(x_1^1, x_2, \dots, x_n) dx_2 \dots dx_n \\ &\quad + \sum_{j=2}^{n} (-1)^{j-1} \int_Q [a_j(x_1, \dots, x_j^0, \dots, x_n) - a_j(x_1, \dots, x_j^1, \dots, x_n)]\, dx_1 \dots dx_{j-1} dx_{j+1} \dots dx_n\end{aligned}$$

Because $a_j(x_1, \dots, x_j^0, \dots, x_n) = a_j(x_1, \dots, x_j^1, \dots, x_n) = 0$ for $2 \leq j \leq n$ and $a_1(x_1^1, x_2, \dots, x_n) = 0$, we obtain the desired result:

$$\int_M \omega = \int\ a_1(0, x_2, \dots, x_n) dx_2 \dots dx_n = \int_{\partial M} i^* \omega$$

B We now want to consider the general case. Let $\{V_\alpha\}$ be a covering of M by coordinate neighborhoods compatible with orientation and $\varphi_1, \dots, \varphi_m$ be its subordinate differential partition of unity. The forms $\omega_j = \varphi_j \omega,\ j \in 1, \dots, m$ satisfy the conditions of case A and, since $\sum_j d\varphi_j = 0$, we have

$$\sum \omega_j = \omega, \quad \sum d\omega_j = d\omega$$

. Hence,

$$\int_M d\omega = \sum_{j=1}^{m} \int_M d\omega_j = \sum_{j=1}^{m} \int_{\partial M} i^* \omega_j = \int_{\partial M} i^* \sum_j \omega_j = \int_{\partial M} i^* \omega$$

□

3 Divergence Theorem

Definition 4. *Let υ the n-form in $\mathbb{R}^n$ defined by $\upsilon = dx_1 \wedge \dots \wedge dx_n$, i.e. the n-form such that $\upsilon(e_1, \dots, e_n) = 1$, where $\{e_i\}$, $i = 1, \dots, n$ is the canonical basis of $\mathbb{R}^n$. Then υ is called the* volume element *of $\mathbb{R}^n$.*

Definition 5. *(Hodge star operation)*
Given a k-form ω in $\mathbb{R}^n$, we will define an $(n-k)$-form $*w$ by setting

$$*(dx_{i_1} \wedge \dots \wedge dx_{i_k}) = (-1)^\sigma (dx_{j_1} \wedge \dots \wedge dx_{j_{n-k}})$$

and extending it linearly, where $i_1, \dots, i_k$, $j_1, \dots, j_{n-k}$, $(i_1 \dots i_k, j_1 \dots j_{n-k})$ is a permutation of $(1, \dots, n)$ and σ is 0 or 1 according to the permutation is even or odd, respectively.

Example 1. If we have a differentiable vector field in $\mathbb{R}^n$, that can be considered as a differentiable map $v : \mathbb{R}^n \to \mathbb{R}^n$, we can obtain a 1-form ω by the canonical isomorphism induced by the inner product: $\omega(-) = \langle v, - \rangle$, or, in coordinates $\omega = v_1 dx_1 \wedge \dots \wedge v_3 dx_3$. Then we have $*\omega$ is an $(n-1)$-form given by

$$*\omega = v_1 dx_2 \wedge \dots \wedge dx_n - v_2 dx_1 \wedge (dx_2) \dots \wedge dx_n + \dots + (-1)^{n-1} v_n dx_2 \wedge \dots \wedge dx_n$$

Now we can compute $d(*\omega)$ that is an n-form and it is equal to:

$$d(*\omega) = \sum_1^n \frac{\partial v_i}{\partial x_i} dx_1 \wedge \cdots \wedge dx_n$$

So, we have that

$$d(*\omega) = (divv)\upsilon$$

Theorem 2. *(Divergence theorem) Let M be a bounded region of $\mathbb{R}^3$ such that the boundary ∂M of M is a regular hyper-surface of $\mathbb{R}^3$; M is then a compact 3-dimensional manifold with boundary ∂M. Let v be a differentiable vector field in $\mathbb{R}^3$, and let w be the 1-form in $\mathbb{R}^3$ dual to v in the natural inner product of $\mathbb{R}^3$. Then,*

$$\int_M div(v)\ \upsilon = \int_{\partial M} \langle v, N\rangle \sigma$$

Proof. Let v be a differentiable vector field in $\mathbb{R}^3$, and let w be the 1-form in $\mathbb{R}^3$ dual to v in the natural inner product of $\mathbb{R}^3$. Then, $d(*\omega) = (divv)\ \upsilon$, where υ is the volume element of $\mathbb{R}^3$. Now choose an orientation for $\mathbb{R}^3$ and let N be the unit normal vector of ∂M in the induced orientation. Finally, let σ be the area element of ∂M.
Consider, in a neighbourhood $U \subset \mathbb{R}^3$ of $p \in M$, differentiable orthonormal fields e_1, e_2, N such that, in the points of ∂M, e_1, e_2 are tangent to ∂M.
Then

$$i^* * \omega(e_1, e_2) = \omega(N) = \langle v, N\rangle$$

i.e. $i^*(*\omega) = \langle v, N\rangle\sigma$.
In fact, if we change basis from the standard one to $\{e_1, e_2, N\}$, we see that $\omega(-) = \omega(e_1)\langle e_1, -\rangle + \omega(e_2)\langle e_2, -\rangle + \omega(N)\langle N, -\rangle$, in coordinates $\omega = \omega(e_1)de_1 + \omega(e_2)de_2 + \omega(N)dN$.
So, $(*\omega) = \omega(e_1)de_2 \wedge dN - \omega(e_2)de_1 \wedge dN + \omega(N)de_1 \wedge de_2$.

$$i^* * \omega(N)\ (e_1, e_2) = \omega(N) = \langle v, N\rangle$$

Thus, in this case

$$\int_M d(*\omega) = \int_{\partial M} i^*(*\omega)$$

can be written as

$$\int_M div(v)\ \upsilon = \int_{\partial M} \langle v, N\rangle \sigma$$

4 Poincaré's Lemma

Definition 6. Let M^n an n-dimensional differentiable manifold. A differential k-form ω is said to be *exact* if there exist a $(k-1)$-form β such that $d\beta = \omega$. It is said to be *locally exact* if for each $p \in M^n$ there exist a neighbourhood V of p in which it is exact.
ω is said to be *closed* if $d\omega = 0$. It is said to be *locally closed* if for each $p \in M^n$ there exist a neighbourhood V of p in which it is exact.

Remark 1. Since $d^2 = 0$, an exact form is closed, but the converse of the above fact does not hold in general.

Example 2. Let's consider the 1-form $\omega = \frac{xdy - ydx}{x^2+y^2}$, defined in $\mathbb{R}^2 - \{(0,0)\} = U$.
We see that ω is closed. In fact,

$$\begin{aligned} d\omega =& d\left(\frac{x}{x^2+y^2}\right) \wedge dy + \frac{x}{x^2+y^2} d^2 y - d\left(\frac{y}{x^2+y^2}\right) \wedge dx + \frac{y}{x^2+y^2} d^2 x \\ =& d\left(\frac{x}{x^2+y^2}\right) \wedge dy - d\left(\frac{y}{x^2+y^2}\right) \wedge dx = 0 \end{aligned}$$

On the other hand, there exist no 1-form, i.e. differentiable function g in U such that $dg = \omega$, otherwise, by Stokes theorem, if $C = (x, y) \in \mathbb{R}^2, \quad x^2 + y^2 = 1$,

$$\int_C \omega = \int_C dg = \int_{\partial C} g = 0$$

But it is easy to see that this is impossible, because it is easy to prove that $\int_C \omega = 2\pi$.
It is possible, however, to find a neighbourhood $V \subset U$ of each $p \in U$ and a differentiable function g_V such that $dg_V = \omega$.

In this section we will show that the situation of this example is completely general, in other words, we will show that the condition $dw = 0$, i.e . w is a closed form, is a sufficient condition for ω to be locally exact.

Definition 7. An n-dimensional differentiable manifold M^n is said to be *contractible* to some point $p_0 \in M$ if there exist a differentiable map $H : M^n \times R \to M^n, \;\; H(p, t) \in M, \;\; p \in M, \;\; t \in R$ such that

$$H(p, 1) = p, \quad H(p, 0) = p_0, \quad \forall p \in M$$

Example 3. $\mathbb{R}^n$ is contractible to an arbitrary point. In fact, we can define

$$H(p, t) = p_0 + (p - p_0)t$$

With the same argument, we can show that the ball $B_r(0) = \{p \in \mathbb{R}^n ; |p| < r\}$ is contractible to the origin 0.

Now we are going to prove the Poincaré's Lemma, in other words, we are going to show that, if M is smoothly contractible to a point, then every closed form $d\omega$ is exact.
At first, we need to define a differential form on the product $M^n \times R$. To do that, we need to remember the projection map $\pi : M \times R \to M, \pi(p, t) = p$.
In addition, if we have a k-form ω in M^n, we can consider the k-form in $M \times R$, given by

$$\omega_1 = H^* \omega$$

where H is the map of the definition of contractibility.

To prove the theorem, we start with a Lemma.

Lemma 4. *Every k-form w_1 in $M \times R$ can be written uniquely as*

$$w_1 = w_0 + dt \wedge \eta$$

where w_0 is a k-form on $M \times R$ with the property that $w_0(v_1, \dots, v_k) = 0$, if some $v_i, i = 1, .., k$, belongs to the kernel of $d\pi$ and η is a $(k-1)$-form with a similar property.

Proof. Let $p \in M$ and $f : U \to M$ be a parametrization around p. Then $f(U) \times R$ is a coordinate neighbourhood of $M \times R$, with coordinates, say, $(x_1 \circ \pi, \dots, x_n \circ \pi, t)$, that we will call $(\bar{x_1}, \dots, \bar{x_n}, t)$.
In $f(U) \times R$, ω_1 can be written as

$$\omega_1 = \sum_{i_1, \dots, i_k} a_{i_1, \dots, i_k} dx_{i_1} \wedge .. \wedge dx_{i_k} + dt \wedge \sum b_{i_1, \dots, i_{k-1}} dx_{i_1} \wedge \dots \wedge dx_{i_{k-1}} = \omega_0 + dt \wedge \eta.$$

It is very easy to see that ω_0 and η have the required properties.

Furthermore, if the decomposition

$$w_1 = w_0 + dt \wedge \eta$$

holds in all of M, it has to be locally equal of the form

$$\omega_1 = \sum_{i_1,\dots,i_k} a_{i_1,\dots,i_k} dx_{i_1} \wedge \dots \wedge dx_{i_k} + dt \wedge \sum b_{i_1,\dots,i_{k-1}} dx_{i_1} \wedge \dots \wedge dx_{i_{k-1}} = \omega_0 + dt \wedge \eta.$$

To prove the existence, it is sufficient to define ω_0 and η on each coordinate neighbourhood using 2. In the intersection of two of such neighbourhood, the definition agree by uniqueness, thus ω_0 and η can be extended to the whole M satisfying the decomposition of the Lemma.
This proves the claim. □

Now let's consider the map $i_t : M \to M \times R$, given by $i_t(p) = (p, t)$.
It is the inclusion of M into $M \times R$ at the "level" t.
We want now to define a map I that takes k-forms of $M \times R$ into $(k-1)$-forms of M. If $p \in M$ and $v_1 \dots v_k \in T_pM$, then, the required map, at p is the following:

$$(I\omega_1)(v_1..v_{k-1}) = \int_0^1 \{\eta(p,t)(di_t(v_1), \dots, di_t(v_{k-1})\} dt$$

where η is the form found in Lemma.

The crucial point of the theorem is contained in the following Lemma.

Lemma 5.

$$i_1^*\omega_1 - i_0^*\omega_1 = d(I\omega_1) + I(d\omega_1).$$

Proof. Let $p \in M$. We can use the same coordinate system $(x_1, \dots, x_n, t)$ of $M \times R$ introduced in Lemma 4. At first, we notice that the operation I is additive,

$$I(\alpha_1 + \alpha_2) = I(\alpha_1) + I(\alpha_2)$$

It follows that, since $\omega_1 = \omega_0 \wedge dt \wedge \eta$, it suffices to consider the following two cases:

1. $\omega_1 = f dx_{i_1} \wedge .. \wedge dx_{i_k}$ and

2. $\omega_1 = f dt \wedge dx_{i_1} \wedge \dots \wedge dx_{i_{k-1}}$.

1. We know that $\omega_1 = f dx_{i_1} \wedge .. \wedge dx_{i_k}$. Let's calculate $d\omega_1 = \frac{\partial f}{\partial t} dt \wedge dx_{i_1} \wedge \dots \wedge dx_{i_k}$ + terms without dt.
   Since we are in the coordinate system $(x_1, \dots, x_n, t)$, the operation I allow to integrate the local representation of ω_1 along the second factor t. Therefore,

   $$I(d\omega_1)(p) = \left(\int_0^1 \frac{\partial f}{\partial t} dt \right) dx_{i_1} \wedge \dots \wedge dx_{i_k} = (f(p,1) - f(p,0)) dx_{i_1}, \dots, dx_{i_k} = i_1^*\omega(p) - i_0^*\omega(p)$$

   , where we used the fact that $(i_t \circ \pi)^* = (id)^*$ in order to prove the first of the equations.
   Since $I\omega_1 = 0$, we conclude the lemma in case (1).

2. If $\omega_1 = f dt \wedge dx_{i_1} \wedge .. \wedge dx_{i_{k-1}}$, then $i_1^*\omega = 0 = i_0^*\omega$. It is easy to show that considering the fact that $i_t^*(dt) = d(i_t^*(t)) = d(t \circ i_t) = d(const) = 0$.
   On the other hand,
$$d\omega_1 = \sum_{\alpha=0}^{n} \frac{\partial f}{\partial x_\alpha} dx_\alpha \wedge dt \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_{k-1}}$$
   Therefore,
$$I(d\omega_1)(p) = -\sum_\alpha \Big(\int_0^1 \frac{\partial f}{\partial x_\alpha} dt \Big) dx_\alpha \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_{k-1}}$$
   and
$$d(I\omega_1)(p) = d\{ \Big(\int_0^1 f dt \Big) dx_{i_1} \wedge \cdots \wedge dx_{i_{k-1}} \} = \sum_\alpha \Big(\int_0^1 \frac{\partial f}{\partial x_\alpha} dt \Big) dx_\alpha \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_{k-1}}$$
   which complete the Case (2) and the proof of the Lemma.

□

Now we are ready to deal with the Poincaré's Lemma.

Theorem 3. *Let M be a contractible differentiable manifold, and let w be a differentiable k-form in M with $dw = 0$. Then w is exact, i.e., there exist a $(k-1)$-form α in M such that $d\alpha = \omega$.*

Proof. Let $\pi : M \times R \to M, \pi(p,t) = p$ the projection map and $\omega_1 = H^*\omega$ as defined above. Since M is contractible, we have
$$H \circ i_1 = identity$$
and
$$H \circ i_0 = const = p_0 \in M$$
Thus,
$$\omega = (H \circ i_1)^*\omega = i_1^*(H^*\omega) = i_1^*\omega_1$$
$$0 = (H \circ i_0)^*\omega = i_0^*(H^*\omega) = i_0^*\omega_1$$
Now, since $d\omega = 0$, we obtain that $d\omega_1 = H^* d\omega = 0$. It follows by Lemma 5 that
$$\omega = i_1^*\omega = d(I\omega_1) = d(\alpha)$$
where $\alpha = I\omega_1$. □

5 Exercises

Exercise 1. *Let $g : \mathbb{R}^3 \to \mathbb{R}, f : \mathbb{R}^3 \to \mathbb{R}$ be a differentiable function and let $M^3 \subset \mathbb{R}^3$ be a compact differentiable manifold with boundary ∂M^2.*
Prove that:

1. *(First Green's identity)*
$$\int_M \langle \mathrm{grad} f, \mathrm{grad} g \rangle \ \upsilon + \int_M f \nabla^2 g \ \upsilon = \int_{\partial M} f \langle \mathrm{grad} g, N \rangle \ \sigma$$
   where υ and sigma are, respectively, the volume element of M and the area element of ∂M, and N is the normal of ∂M

2. *(Second Green's identity)*

$$\int_M (f\nabla^2 g - g\nabla^2 f)\upsilon = \int_{\partial M} (f\langle \text{grad}g, N\rangle - g\langle \text{grad}g, N\rangle)\sigma$$

Proof. 1. The proof of the first Green's identity is a simple application of the divergence theorem proved above. We know that

$$\int_M div(v)\ \upsilon = \int_{\partial M} \langle v, N\rangle\sigma$$

If we choose our vector field $v = f\text{grad}g$, we obtain:

$$\int_M div(f\text{grad}g)\ \upsilon = \int_{\partial M} \langle f\text{grad}g, N\rangle\sigma$$

But it is very easy to check that

$$div(f\text{grad}g) = f\nabla^2 g + \langle \text{grad}f, \text{grad}g\rangle$$

and so, we have:

$$\int_M div(v)\ \upsilon = \int_M f\nabla^2 g\ \upsilon + \int_M \langle \text{grad}f, \text{grad}g\rangle\ \upsilon = \int_{\partial M} \langle f\text{grad}g, N\rangle\sigma$$

2. To prove the second Green's identity, we will use the first one. In fact, we know that

$$\int_{\partial M} \langle f\text{grad}g, N\rangle\sigma = \int_M f\nabla^2 g\ \upsilon + \int_M \langle \text{grad}f, \text{grad}g\rangle\ \upsilon$$

and

$$\int_{\partial M} \langle g\text{grad}f, N\rangle\sigma = \int_M g\nabla^2 f\ \upsilon + \int_M \langle \text{grad}g, \text{grad}f\rangle\ \upsilon$$

Now, we just need to subtract the two equations in order to prove our statement

$$\int_{\partial M} \langle f\text{grad}g, N\rangle - \langle g\text{grad}f, N\rangle\sigma = \int_M f\nabla^2 g\ \upsilon + \int_M \langle \text{grad}f, \text{grad}g\rangle\ \upsilon - \int_M g\nabla^2 f\ \upsilon + \int_M \langle \text{grad}g, \text{grad}f\rangle\ \upsilon$$

Since $\langle \text{grad}f, \text{grad}g\rangle = \langle \text{grad}g, \text{grad}f\rangle$, the statement is proved.

□

Definition 8. *Given a differentiable function $f : \mathbb{R}^n \to \mathbb{R}$, we will define the Laplacian $\nabla f : \mathbb{R}^n \to \mathbb{R}$ by*

$$\nabla f = div(\text{grad}f)$$

Is it easy to show that this definition in equivalent to

$$\nabla f = \sum \frac{\partial^2 f}{\partial x_i^2}$$

and, with this definition, we can see that the Laplacian is a linear operator, i.e.

$$\nabla(f + g) = \nabla f + \nabla g$$

.

Exercise 2. *(introduction to potential theory in $\mathbb{R}$) A differentiable function $g : \mathbb{R}^3 \to \mathbb{R}$ is said to be harmonic in a subset $B \subset \mathbb{R}^3$ if $\nabla^2 g = 0$ for all $p \in B$. Let $Min\mathbb{R}^3$ be a bounded region with regular boundary ∇M. Prove that:*

1. *If g_1 and g_2 are harmonic in M and $g_1 = g_2$ in ∇M, then $g_1 = g_2$ in M.*

2. *If g is harmonic in M and we define*

$$\frac{\partial g_1}{\partial N} = \langle gradg, N \rangle = 0$$

   in ∂M, where N is the unit normal vector of ∂M, then $g = const.$ in M.

3. *If g_1 and g_2 are harmonic in M and*

$$\frac{\partial g_1}{\partial N} = \frac{\partial g_2}{\partial N}$$

   in ∂M, then $g_1 = g_2 + const$ in M.

4. *If g is harmonic in M, then*

$$\int_{\partial M} \frac{\partial g}{\partial N} \sigma = 0$$

5. *The function $\frac{1}{(x^2+y^2+z^2)^{\frac{1}{2}}}$ is harmonic in $\mathbb{R}^3 - \{0\}$*

6. *(Mean value theorem). Let f be harmonic in the region*

$$B_r = \{p \in \mathbb{R}^3 \mid \ \|p - p_0\|^2 \leq r^2\}$$

   whose boundary is the sphere S_r with center in p_0. Then

$$f(p_0) = \frac{1}{4\pi r^2} \int_{S_r} f\sigma$$

Proof. 1. We can use the first Green's identity with $f = g = g_1 - g_2$ and we obtain:

$$\int_M \langle \mathrm{grad}(g_1 - g_2), \mathrm{grad}(g_1 - g_2) \rangle \ \upsilon + \int_M (g_1 - g_2) \nabla^2 (g_1 - g_2) \ \upsilon = \int_{\partial M} (g_1 - g_2) \langle \mathrm{grad}(g_1 - g_2), N \rangle \ \sigma$$

Since $g_1 = g_2$ in ∇M, $f = g = 0$ in ∇M and g_1, g_2, i.e. $\nabla^2(g_1 - g_2) = 0$ are harmonic functions:

$$\int_M \langle \mathrm{grad}(g_1 - g_2), \mathrm{grad}(g_1 - g_2) \rangle \ \upsilon = 0$$

The norm of the gradient is equal to 0 in M and so, the function $f = g = g_1 - g_2$ is constant in M and so, it must be null in M.
As a consequence, $g_1 = g_2$ in M.

2. We can use the first Green's identity with $f = g$ and we obtain:

$$\int_M \langle \mathrm{grad} g, \mathrm{grad} g \rangle \ \upsilon + \int_M g \nabla^2 g \ \upsilon = \int_{\partial M} g \langle \mathrm{grad} g, N \rangle \ \sigma$$

As in the previous case, we have:

$$\int_M \langle \mathrm{grad}(g), \mathrm{grad}(g) \rangle \ \upsilon = 0$$

and so, g must be constant in M.

3. Let's define the function $f = g_1 - g_2$. We know that $\nabla^2 g_1 = \nabla^2 g_2 = 0$ and
$$\frac{\partial g_1}{\partial N} = \frac{\partial g_2}{\partial N}$$
or, in other words,
$$\langle \mathrm{grad} g_1, N\rangle = \langle \mathrm{grad} g_2, N\rangle \Rightarrow \langle \mathrm{grad} g_1 - \mathrm{grad} g_2, Nrangle = \langle \mathrm{grad} f, N\rangle = 0$$
We also know that $\nabla^2(f) = 0$ because the two functions g_1 and g_2 are harmonic: the function f is harmonic. So, using 2), $f = const$, i.e. $g_1 = g_2 + const$.

4. Let's use the first Green's identity with $f = 1$. We obtain:
$$\int_M \langle \mathrm{grad}1, \mathrm{grad} g\rangle\ \upsilon + \int_M \nabla^2 g\ \upsilon = \int_{\partial M} \langle \mathrm{grad} g, N\rangle\ \sigma$$
Since $\mathrm{grad}1 = 0$ and $\nabla^2 g = 0$,
$$\int_{\partial M} \frac{\partial g}{\partial N} = \int_{\partial M} \langle \mathrm{grad} g, N\rangle\ \sigma = 0$$

5. It follows with easy calculations by the definition of Laplacian operator.

6. Let's use the second Green's identity in the region $D = B_r - B_\rho, \rho < r$, with $f = f$ and $g = \frac{1}{r}$. Since g and f are harmonic,
$$\int_{S_\rho} (f \frac{\partial}{\partial N}(\frac{1}{r}) - (\frac{1}{r})(\frac{\partial f}{\partial N}))\sigma = \int_{S_r} (f \frac{\partial}{\partial N}(\frac{1}{r} - \frac{1}{r}\frac{\partial f}{\partial N})\sigma$$
Furthermore, we know that $\frac{\partial}{\partial N}(\frac{1}{r}) = \frac{\partial}{\partial r}(\frac{1}{r}) = \frac{(-1)}{r^2}$ and we can also use the point (4):
$$\frac{1}{4\pi\rho^2}\int_{S_\rho} f\sigma = \frac{1}{4\pi r^2}\int_{S_r} f\sigma =$$
And if we just let $\rho \to 0$ we prove the claim because:
$$\lim_{\rho\to 0} \frac{1}{4\pi\rho^2}\int_{S_\rho} f\sigma = \lim_{\rho\to 0} \frac{1}{4\pi r^2}\int_{S_r} f\sigma = \frac{1}{4\pi r^2}\int_{S_r} f\sigma$$
$$\lim_{\rho\to 0} \frac{1}{4\pi\rho^2}\int_{S_\rho} f\sigma = f(P) = \frac{1}{4\pi r^2}\int_{S_r} f\sigma$$

□

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