The Frobenius Theorem

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Abstract

The main purpose of this talk is to present the *Frobenius Theorem.* A classical theorem of the Differential Geometry that connects distributions or families of vector fields with sub-manifolds of a smooth manifold M.

Motivation

Let M be a C^∞ manifold, X a vector field on M and $p \in M$. We know that there exists a unique maximal integral curve,

$$C_p(t) : (a, b) \longrightarrow M,$$

such that $C_p(0) = p$ and its tangent velocity vector field is given by $\dot{C}_p(t) = X_{c(t)}$, in particular $\dot{C}_p(0) = X_p$. We also use the notation $C_p(t) = \Phi(t, p)$ ans say that $C_p(t)$ is the integral curve of Y at p.

We can see (locally) that $Im(C_p) = N$ is a sub-manifold of M such that for any $q \in N$, it holds $T_qN =< X_q >_{\mathbb{R}}$.

We will say that $D =< X >_{C^\infty(M)}$ is a one-dimensional distribution on M, and, that N is an integral manifold of D.

Our goal is to generalize the last result for higher dimensional distributions.

Local Parameter Groups and Commutative Vector Fields

Before we state and prove the Frobenius Theorem, we are going to learn some properties about vector fields on a manifold M and the one parameter group generated by them.

Remark. *From now on we are going to assume that M is a C^∞ n-dimensional manifold.*

Remember that given a vector field X on a manifold M, it has an associated family of local diffeomorphisms $\{\varphi_t\}$, called the one parameter group of local transformations generated by X. The local diffeomorphims φ_t is given by,

$$\begin{aligned} \varphi_t : U_t &\longrightarrow M \\ p &\mapsto \Phi(t, p), \end{aligned}$$

where $U_t = \{p \in M \mid a < t < b\}$ and $(a, b) = dom(\Phi(t, p))$.

Proposition 1. *Let X, Y be vector fields on M and $\{\varphi_t\}$, $\{\psi_t\}$ their corresponding one parameter groups. The following statements are equivalent,*

i. X, Y are commutative, i.e, $[X, Y] = 0$.

ii. Y is invariant bu φ_t, i. e, for an arbitrary $t \in \mathbb{R}$, $(\varphi_t)_*Y = Y$.

iii. φ_t and ψ_t are mutually commutative, i.e, for any $t, s \in \mathbb{R}$, $\varphi_t \circ \psi_s = \psi_s \circ \varphi_t$.

Proof. *i.* $\Rightarrow$ *ii.*
First note that $\varphi_0 = id_M \quad \Rightarrow \quad (\varphi_0)_* = id_{T_pM}$. Therefore, for any $p \in M$ it holds $(\varphi_0)_* Y_p = Y_p$.

On the other hand, the derivative with respect the parameter t of $(\varphi_t)_* Y$ at $t = t_0$ is given by,

$$\frac{d}{dt}((\varphi_t)_* Y)|_{t=t_0} = lim_{t \to t_0} \frac{(\varphi_{t+t_0})_* Y - (\varphi_{t_0})_* Y}{t} = lim_{t \to t_0} (\varphi_{t_0})_* \frac{(\varphi_t)_* Y - Y}{t}$$

$$= (\varphi_{t_0})_* lim_{t \to t_0} \frac{(\varphi_t)_* Y - Y}{t} = (\varphi_{t_0})_* [-X, Y] = 0.$$

The latter tells us that $(\varphi_t)_* Y$ is a constant vector field, i.e, it doesn't depend on t. Therefore, for all t we have $(\varphi_t)_* Y = (\varphi_{t_0})_* Y = Y$.

ii. $\Rightarrow$ *iii.*
Let $\varphi : U \longrightarrow M$ be a local diffeomorphism. Let us take $p \in U$, Y a vector field on M and $c(t)$ and integral curve of Y.
Note the following, the curve $\gamma(t) = \varphi \circ c(t) : (a, b) \to M$ is a smooth curve on M, s.t,

$$\frac{d}{dt} \varphi \circ c(t) = (\varphi)_* \frac{c(t)}{dt} = (\varphi)_* \dot{C}(t) = (\varphi)_* Y_{c(t)} = [(\varphi)_* Y]_{\varphi \circ c(t)},$$

which implies that $\varphi \circ C(t)$ is an integral curve of $(\varphi)_* Y$.

Now, let us assume that $\varphi \circ c(0) = p$, since φ is a local diffeomorphism we have $c(0) = \varphi^{-1}(p)$.

The last remark tells us the following, if $c(t)$ is the integral curve of Y at $\varphi^{-1}(p)$ (i.e, $c(t) = \Phi(t, \varphi^{-1}(p))$), then the curve $\gamma(t) = \varphi \circ c(t)$ is the integral curve of $(\varphi)_* Y$ at p (i.e $\gamma(t) = \tilde{\Phi}(t, p)$)

Let us take an element $\tilde{\psi_t}$ of the one parameter family generated by $(\varphi)_* Y$. Then, for all $p \in U_t$ we have,

$$\tilde{\psi_t}(p) = \tilde{\Phi}(t, p) = \varphi \circ \Phi(t, \varphi^{-1}(p)) = \varphi \circ \psi_t(\varphi^{-1}(p)) = \varphi \circ \psi_t \circ \varphi^{-1}(p) \quad \Rightarrow \tilde{\psi_t} = \varphi \circ \psi_t \circ \varphi^{-1}. \qquad (1)$$

From now on, let φ_s be an element of the one parameter subgroup generated by X. It holds,

1. Since $(\varphi_s)_* Y = Y$, it's clear that the local diffeomorphisms $\tilde{\psi_t}$ and ψ_t are the same.
2. From the latter and equation (1) we get that $\psi_t = \varphi_s \circ \psi_t \circ \varphi_s^{-1} \quad \Rightarrow \quad \psi_t \circ \varphi_s = \varphi_s \circ \psi_t$.

iii. $\Rightarrow$ *i.*
By assumption we know that $\varphi_s \circ \psi_t \circ \varphi_s^{-1} = \psi_t$, let us take $p \in M$

a. By letting the parameter t vary, we get that $\psi_t(p) = \Phi(t, p)$ is the integral curve at p of Y. Therefore, we have that $\frac{d}{dt} \psi_t(p)|_{t=0} = Y_p$.

b. In the last proof we showed that $\frac{d}{dt} \varphi_s \circ \psi_t \circ \varphi_s^{-1}(p)|_{t=0} = [(\varphi_s)_* Y]_p$.

Since, $\varphi_s \circ \psi_t \circ \varphi_s^{-1}(p) = \psi_t(p)$, from a. and b. it follows that $(\varphi_s)_* Y = Y$, and so, we get that

$$[X, Y] = lim_{s \to 0} \frac{-(\varphi_{-s})_* Y - Y}{s} = 0$$

$\square$

Frobenius Theorem

We are going to study completely integrable distributions. In particular, we will state and prove the Frobenius Theorem, which gives us the conditions to generalize the result that was given in the motivation.

Definition. *An r-dimensional distribution D on M is an smooth assigment of an r-dimensional subspace D_p of T_pM at each point $p \in M$, such that D_p is C^∞ with respect to p.*
We also say that a vector field X on M belongs to D if $X_p \in D_p$ for any point $p \in M$.

Definition. *A submanifold N of M is called an integral manifold of D, if $T_pN = D_p$ for any point $p \in N$. Moreover, if an integral manifold of D exists through each point of M, D is said to be completely integrable.*

Example. *The distribution $D =< X >_{C^\infty(M)}$ is completely integrable, since for all $p \in M$ we can find the integrable curve $c(t) = \Phi(t, p)$ at p.*

Theorem (Frobenius Theorem)**.** *Let D be a distribution on a C^∞ manifold M. Then, D is completely integrable if and only if for any two vector fields X, Y belonging to D, the lie-bracket $[X, Y]$ also belongs to D (a distribution with this property is said to be involutive).*

Proof. "$\Rightarrow$" We want to see that for any vector fields X, Y belonging to D, and, an arbitrary $p \in M$, it holds $[X, Y]_p \in D_p$.
Since D is completely integrable there exist an integrable manifold N of D through p. Let $r = dim(N)$, we can choose a coordinate neighborhood $(U, x_1, ..., x_n)$ of p, such that, $x_1(p) = x_2(p) = ... = x_n(p) = 0$ and $(N \cap U, x_1, x_2, ..., x_r, x_{r+1} = 0, ..., x_n = 0)$ is a local chart of N. Due to the fact that for any $q \in N D_q = T_qN$, we have

$$D_q =< \frac{\partial}{\partial x_1}|_q, ..., \frac{\partial}{\partial x_r}|_q >_{\mathbb{R}}.$$

On the other hand, the following holds:

1. The local expressions of X and Y are given by $X = \sum_{i=1}^n a_i \frac{\partial}{\partial x_i}$ and $Y = \sum_{i=1}^n b_i \frac{\partial}{\partial x_i}$.

2. Since X, Y belong to D, we get that $a_j(x_1, ..., x_r, 0, ..., 0) = b_j(x_1, ..., x_r, 0, ..., 0) = 0$ for $j > r$. Consequently,

   $\frac{\partial a_j}{\partial x_i}(0) = 0$ for $i \leq r$ and $j > r$,

   $\frac{\partial b_j}{\partial x_i}(0) = 0$ for $i \leq r$ and $j > r$.

3. Finally, we know that $[X, Y] = \sum_{i=1}^n c_j \frac{\partial}{\partial x_j}$, where $c_j = \sum_{i=1}^n a_i \frac{\partial b_j}{\partial x_i} - b_i \frac{\partial a_j}{\partial x_i}$.

Let us take $j > r$, since $a_i = b_i = 0$ for $i > r \quad \Rightarrow c_j = \sum_{i=1}^r a_i \frac{\partial b_j}{\partial x_i} - b_i \frac{\partial a_j}{\partial x_i}$. Moreover, we know that $\frac{\partial a_j}{\partial x_i}(0) = \frac{\partial b_j}{\partial x_i}(0) = 0$ for $i \leq r \quad \Rightarrow \quad c_j(0) = 0$.

From the latter, we get $[X, Y]_p = \sum_j^r c_j(p) \frac{\partial}{\partial x_j}|_p \quad \Rightarrow \quad [X, Y]_p \in D_p$.

"$\Leftarrow$" Let us assume that D is involutive, and, let us take an arbitrary point $p \in M$. We want to find an integral manifold N of D through p.
We can choose a small coordinate neighborhood $(U, x_1, ..., x_n)$ of p, and, vector fields $Y_1, ..., Y_r$ in D, such that, $\{Y_1, ..., Y_n\}$ is a linearly independent set for all $q \in U$. With respect to $(U, x_1, ..., x_n)$ we can write

$$Y_i = \sum_{j=1}^{n} a_{ij} \frac{\partial}{\partial x_j} \text{ for } i = 1, ..., r.$$

Since $\{Y_1, ..., Y_n\}$ is linear independent set, we can assume, w.l.o.g (change the order if necessary), that the matrix $A = [a_{ij}]_{i,j=1,...,r}$ is invertible. Let $B = [b_{ij}]_{i,j=1,...,r} = A^{-1}$ and let us take the vector fields

$$X_i = \sum_{k=1}^{r} b_{ik} Y_k.$$

By definition $[AB]_k ij = \sum_{k=1}^{n} b_{ik} a_{kj} = \delta_{ij}$ for $i, j = 1, ..., r$, and so, X_i can be given in local coordinates as follows

$$\begin{aligned} X_i &= \sum_{j=1}^{r} \sum_{k=1}^{n} b_{ik} a_{kj} \frac{\partial}{\partial x_j} + \sum_{j=r+1}^{n} \sum_{k=1}^{r} b_{ik} a_{kj} \frac{\partial}{\partial x_j} \\ &= \sum_{j=1}^{r} \delta_{ij} \frac{\partial}{\partial x_i} + \sum_{j=r+1}^{n} c_{ij} \frac{\partial}{\partial x_j} \\ &= \frac{\partial}{\partial x_i} + \sum_{j=r+1}^{n} c_{ij} \frac{\partial}{\partial x_j}. \end{aligned}$$

It's clear that X_i belongs to D for all $i = 1, ..., r$, and,$\{X_1, ..., X_n\}$ is a linearly independent set. Consequently, $D_q =< \{X_1(q), ..., X_n(q)\} >$ for all $q \in D$.

Also note that

$$[X_i, X_j] = [\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}] + \sum_{k=r+1}^{n} [\frac{\partial}{\partial x_i}, c_{ik} \frac{\partial}{\partial x_k}] + \sum_{k=r+1}^{n} [c_{jk} \frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_j}] + \sum_{k,k'=r+1}^{n} [c_{ik} \frac{\partial}{\partial x_k}, c_{jk'} \frac{\partial}{\partial x_{k'}}],$$

and, since

$[a \frac{\partial}{\partial x_k}, b \frac{\partial}{\partial x_{k'}}] = a \frac{\partial b}{\partial x_k} \frac{\partial}{\partial x_{k'}} - b \frac{\partial a}{\partial x_{k'}} \frac{\partial}{\partial x_k}$, it tells us that $[X_i, X_j] = \sum_{k=r+1}^{n} \tilde{C}_k \frac{\partial}{\partial x_k}$.

Moreover, since D is involutive we know that $[X_i, X_j]$ belongs to D, and, for that reason

$$[X_i, X_j] = \sum_{k=1}^{r} f_k X_k, \text{ where } f_k \in C^\infty(U).$$

From the last computations and considerations we have that:

$$\begin{aligned} &[X_i, X_j] = \sum_{k=i}^{r} f_k \frac{\partial}{\partial x_k} + \sum_{k=r+1}^{n} \hat{C}_k \frac{\partial}{\partial x_k} \quad \Rightarrow \sum_{k=i}^{r} f_k \frac{\partial}{\partial x_k} = 0 \\ &\quad \Rightarrow f_k = 0 \text{ for all } k = 1, ..., r \quad \Rightarrow [X_i, X_j] = 0. \end{aligned}$$

The latter tells us that, the set $\{X_1, ..., X_n\}$ is a linear independent set of mutually commutative vector fields, which generate and belong to D.

Now, let $\{\varphi^i_{t_i}\}$ be the one parameter group generated by X_i, and, let us take a small neighborhood V of the origin in $\mathbb{R}^r$, we can define a map

$$\varphi : V \longrightarrow U$$

$$\text{by} \quad \varphi(t_1, ..., t_r) = \varphi^1_{t_1} \circ \varphi^2_{t_2} \circ ... \circ \varphi^r_{t_r}(p).$$

Since the vector fields X_i are mutually commutative, we have that

$$\varphi(t_1, ..., t_r) = \varphi^i_{t_i} \circ \varphi^1_{t_1} \circ ... \circ \varphi^{i-1}_{t_{i-1}} \circ \varphi^{i+1}_{t_{i+1}} \circ ... \circ \varphi^r_{t_r}(p)$$

and so, at $t = 0$, and, for $f \in C^\infty(U)$ we have that

$$(\varphi)_*(\frac{\partial}{\partial t_i})f = \frac{\partial f \circ \varphi^i_{t_i}(p)}{\partial t_i}|_{t_i=0} = X_i(p)f \quad \Rightarrow \; (\varphi)_*(\frac{\partial}{\partial t_i}) = X_i(p).$$

As a consequence, we get that $(\varphi)_* : T_0\mathbb{R}^r \longrightarrow T_pM$ is an injective map, and, by tacking V as small as necessary, φ is an embedding.

Let $N = Im\varphi$, by the last result, N is a submanifold of M with the property that $D_p = T_pN$.

We want to see now that for all $q \in N$, $D_q = T_qN$. Since φ is an embedding there is a $t = (t_1, ..., t_r)$,

such that, $q = \varphi(t_1, ..., t_r) = \varphi^1_{t_1} \circ ... \circ \varphi^r_{t_r}(p)$,

in the same way as we did before, as the vector fields X_i are mutually commutative, we have that

$$q = \varphi^i_{t_i} \circ \varphi^1_{t_1} \circ ...\varphi^{i-1}_{t_{i-1}} \circ \varphi^{i+1}_{t_{i+1}} \circ ... \circ \varphi^r_{t_r}(p).$$

Let us take

$$\tilde{p} = \varphi^1_{t_1} \circ ...\varphi^{i-1}_{t_{i-1}} \circ \varphi^{i+1}_{t_{i+1}} \circ ... \circ \varphi^r_{t_r}(p)$$

, and, a small interval $(a, b) \subset \mathbb{R}$, such that, $t_i \in (a, b)$ and the mapping $\gamma(t) = \varphi^i_t(\tilde{p})$ defines a curve on N.

It is clear that $\gamma(t_i) = q$. Moreover, since $\varphi^i_t(\tilde{p})$ is by definition an integral curve of $X_i \quad \Rightarrow$

$$\dot{\gamma}(t_i) = X_i(q) \quad \Rightarrow \quad X_i(q) \in T_qN.$$

We also know that $\{X_1(q), ..., X_r(q)\}$ is a linear independent set and $dim(T_qN) = r \quad \Rightarrow$

$$T_qN = D_q$$

□

Reference:

Shigeyuki Morita, Geometry of Differential Forms, American Mathematical Society, 2001