Seifert Surfaces and Genus

Andrea Rincon

July 25, 2014

The aim of this paper is to introduce the concept of the genus for a given knot K. For this purpose, I'm going to give a brief introduction to orientable surfaces and to introduce the Seifert's Algorithm, which gives us an explicit way to find an orientable surface with one boundary component, such that, this boundary component is an already known knot. The latter will lead us to define Seifert Surfaces and the genus for a knot.

1 Introduction to Surfaces

Definition A *Surface S* or a *Two-Manifold* is any object, such that, for any point $p \in S$, there is a small region U on the surface, which surronds and contains the point p.

Some examples of surfaces are the sphere S^2 and the glaze of a doughnut, which we will call a *Torus*.

On the other hand, we can think that surfaces are made of rubber, so that we can deform the rubber to go from an initial surface to a new one. This kind of rubber deformations are called *Isotopies* and we say that two surfaces are *Isotopic* if we can find an *Isotopy* between the two of them.

To understand better the properties of surfaces it is better to cut them into triangles, such that, the triangles fit together nicely along their edges (as in Figure 1) and they cover the entire surface.

Definition Such a division of a surface into triangles is called a *Triangulation*.

Now given a surface with a triangulation, we can cut it into the individual triangles keeping track of the original surface by:

- Labeling the edges that should be glued back together.
- Placing matching arrows on the pair of the edges that are to be glued together.

Remark: Note that each triangle can be deform into a disk with boundary. Moreover, We can apply rubber deformations to the triangles without changing the labeling on the edges and the matching arrows.

Definition Two surfaces are *Homeomorphic* if one them can be triangulated, then cut along a subset of the edges into pieces, and then glue back together according to the instructions given by the orientation and labels on the edges, in order to obtain the second surface.

Example

- Two isotopic surfaces are also homeomorphic surfaces.
- The sphere S^2 and the torus T are not homeomorphic.

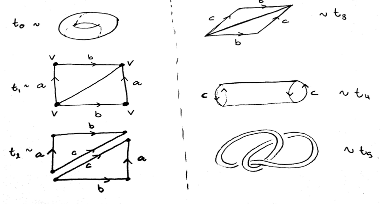


Figure 1: Triangulation of the Torus and Homeomorphism.

Definition Let S be a surface and T triangulation of S. Let us assume that V is the number of vertices in T, E is the number of edges in T and F is the number of faces (triangles or polygons) in T. The *Euler Characteristic* of the triangulation is

$$\chi(T) = V - E + F.$$

Example For the triangulation of the torus given in Figure 1 we have $V = 1$, $E = 3$ and $F = 2$ and so we get $\chi(T) = 1 - 3 + 2 = 0$

Facts:

- Given a surface S and two triangulations T_1 and T_2 of it, the euler characteristic of the two triangulations is the same. In other words, the euler characteristic does not depend on the triangulation. Therefore, instead of talking about the euler characteristic of a triangulation, we talk about the euler characteristic of a surface S.
- From the last example we have $\chi(T) = 0$, where T is the Torus, and it is well know that $\chi(S^2) = 2$.

Definition A surface is *compact* if it has a triangulation with finite number of triangles.

Definition A surface is *orientable* it it has two sides that can be painted in different colors, say black and withe, such that, the black paint never meets the white paint, except along the boundary (in the case that the surface has a boundary).

Example

- From Figure 1, it is clear that the Torus T is a compact surface. It is also easy to see that the sphere S^2 is a compact surface.
- We can imagine that the interior of the Torus is black and its exterior is white, which would imply that the torus is orientable. We can use the same reasoning to see that the sphere S^2 is orientable.
- An example of a non-orientable surface is the Möbius band.

Connected sum of Tori: Given two tori T_1 and T_2 we can perform the following operation:

1. Remove a disk from each tori.
2. Glue the tori together along the resulting the resulting circle boundaries.

The result of this operation will be a torus with two holes, which we will denote by $T^{(2)}$.

We can generalize the latter operation for any two surfaces S and S', we call it the *connected sum of S and S'* and is denoted by $S \sharp S'$.

In terms of triangulations of surfaces, the connected sum is equivalent to remove the interior of a triangle from each triangulation, and then, glue the boundaries of the missing triangles.

Note:

- For any surface S, $\quad S^2 \sharp S \cong S$.
- For any two surfaces S_1 and S_2, $\quad \chi(S_1 \sharp S_2) = \chi(S_1) + \chi(S_2) - 2$.
- We can perform the connected sum of g Tori, $\underbrace{T \sharp T \sharp \ldots \sharp T}$, in that case we obtain a Torus with g holes that we will called $T^{(g)}$.
- If we have a torus $T^{(g)}$, we call the number of holes *genus* and we can see that $\chi(T^{(g)}) = 2 - 2g$.

Theorem 1.1. *(Classification Theorem) Any oriented, connected and closed (compact) surface is homeomorphic to the sphere or the connected sum of g tori.*

We can assume that the sphere has genus zero (no holes), note also that $\chi(S^2) = 2 - 2 \bullet 0$. Now let S be a compact oriented surface, the euler characteristic of S must be of the form

$$\chi(S) = 2 - 2g,$$

we call g the *genus* of S.

Finally note that the connected sum of two compact and oriented surfaces is also a surface with those properties.

Theorem 1.2. *Let S_1 and S_2 be two compact oriented surfaces, then $g(S_1 \sharp S_2) = g(S_1) + g(S_2)$.*

Proof.

$$\chi(S_1 \sharp S_2) = 2 - 2g(S_1) + 2 - 2g(S_2) - 2 = 2 - 2(g(S_1) + g(S_2)) \Rightarrow g(S_1 \sharp S_2) = g(S_1) + g(S_2)$$

□

Surfaces with boundary

To obtain a surface with boundary, we just have to remove the interior of a disk from a surface that we already know. We can do the last process as long as we want, the final result will be surface whose boundaries are circles. We call those circles *Boundary Components.*

However, a surfaces with boundary and its boundary components may look pretty different. We can perform isotopies (rubber deformations) and homeomorphism (cut and paste) on the surface to obtain new surface with boundary. Note that we still have the same number of boundary components but the boundary components may now look like a link (knot).

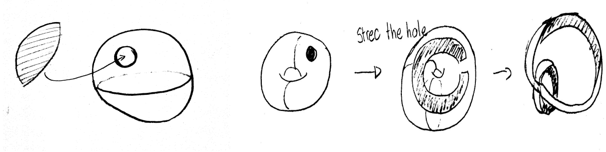


Figure 2: Torus with one boundary

A good way to think about surfaces with boundary is to picture them as follows:

1. *Boudaries* are wire frames. (They are knots).

2. *Surfaces* are soap films spanning the wires.

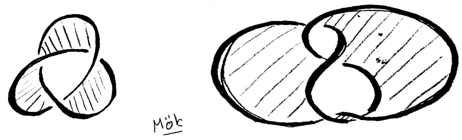


Figure 3: Surfaces whose boundary component is the Trefoilknot.

Now we would like to study some properties of the surfaces with boundary, in particular the euler characteristic. We would like to know how to compute it and if it can tells which kind of surface do we have, in the case that we have a complicated surface.

To compute the *Euler Characteristic* of a surface with boundary S, we have two possibilities:

1. Find a triangulation of S and compute $\chi(S)$. In this case we have to take into account that the resulting pieces of the boundary count as edges in our calculation.

2. If $\bar{S}$ is the corresponding surface without boundary , the euler characteristic $\chi(S)$ is

   $$\chi(\bar{S})\text{-(number of boundary components).}$$

   This must be true, since removing the interior of a disk from a surface $\bar{S}$ is equivalent to remove the interior of a triangle (polygon) from a triangulation of $\bar{S}$.

It is possible to find two non-homeomorphic surfaces with boundary, whose euler characteristic is the same. Nevertheless, the following fact tell us how to distinguish non-homeomorphic surfaces with boundary.

Fact: To classify a surface S with boundary we have to answer the following questions:

1. It is orientable?

2. Which is the number of boundary components?

3. What is its Euler Characteristic?.

To understand it better let S be an orientable surface, with n boundary components and euler characteristic $\chi(S)$. The corresponding surface without boundary $\bar{S}$ has euler characteristic

$$\chi(\bar{S}) = \chi(S) + n$$

and is homeomorphic to a torus $T^{(g)}$, whose genus is $g = \frac{2-\chi(\bar{S})}{2}$.(see Figure 4).

Now, it makes sense to make the following definition,

Definition Let S be an orientable surface with boundary, the genus $g(S)$ of the surface is the genus of the corresponding surface without boundary.

Example
- Let D be a disk, $g(D) = 0$.
- .

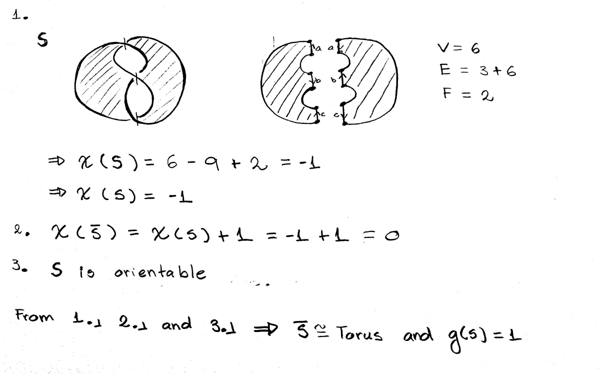


Figure 4: Surface with one boundary component and genus.

2 Seifert Surfaces

In the last section we saw that a knot can be seen as boundary component of a surface with boundary. Now we are going to see that given a knot K we can always find an orientable surface with one boundary component such that the boundary component of the surface is the knot K. Tacking this into account we are going to use the notion of genus of surfaces to define the genus of a knot.

Given a knot K the following algorithm, discovered by the german mathematician Herbert Seifert, return us and orientable surface S_K with one boundary component, such that,the boundary component is K.

Sifert's Algorithm(1934, Herbert Seifert)

- **Input** := K (a knot).
- **Output** := S_k (an orientable surface with boundary component K).

Algorithm:(Given K, see Figure 5)

1. Start whit a projection of K.
2. Give it an orientation.
3. Eliminate the crossings as follows: First note that at each crossing two strands come in and two come out. Then connect each of the strands coming into the crossing to the adjacent strand leaving the crossing.

   Remarks:

- The result of this step will be a set of oriented circles on the plane. These circles are called *Seifert Circles.*
- By changing the orientation we get the same Seifert Circles but with opposite orientation. Therefore the result is independent of the orientation.

4. Fill in the circles. (Each circle will bound a disk).
5. Connect the disks to one another, at the crossings of the knots, by twisted bands.

To see that the surface S_K is orientable we just have to paint the seifert circles with two different colors depending on their orientation. For example, if the seifert circle is clockwise oriented paint the upward pointing face white and the downward pointing face black. Do the same for a counterclockwise oriented circle but the other way around.

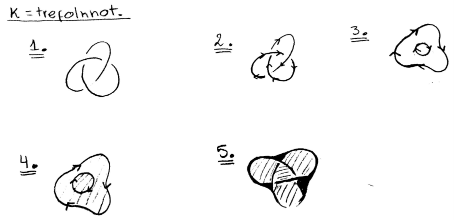


Figure 5: Seifert's Algorithm using the Trefoilknot

Definition Given a knot K, a *Seifert Surface for K* is an orientable surface with one boundary component such that the boundary component of the surface is K.

Note:

1 We can always find a Seifert Surface for a given knot K.(Use the Seifert's Algorithm).

2 There might be more than one Seifert Surface for the same knot. (We can use find more than one projection for the same knot).

Definition The *Genus* of a knot K is the least genus of any Seifert Surface for that knot.

Example

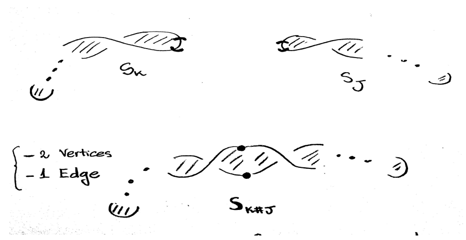
- $g(O) = 0$ since the unknot is the boundary of a disk.
- From Figure 4 we have: $g(\text{trefoilknot}) = 1$.

Now, a nice property of the genus.

Theorem 2.1. *Given two knots K and J $g(K\sharp J) = g(K) + g(J)$.*

Proof. First, let us see that $g(K\sharp J) \leq g(K) + g(J)$.

Let S_K be a seifert surface for K with minimal genus, and, let S_J be a seifert surface for J with minimal genus. Now, let us compose S_K and S_J along the boundary, i.e, remove a little piece along their boundaries and sew them together. We get a surface $S_{K\sharp J}$ whose boundary is $K\sharp J$.



To compute the genus of $S_{K\sharp J}$, first note that $g(K) = g(S_K)$ and $g(J) = g(S_J)$ and let $\bar{S}_K$ and $\bar{S}_J$ be the corresponding surfaces without boundary.
We have:

$\chi(\bar{S}_J) = \chi(S_J) + 1$, and, $\chi(\bar{S}_J) = \chi(S_J) + 1$.

$\chi(\bar{S}_K) = 2 - 2g(K) = \chi(S_K) + 1 \Rightarrow \chi(S_K) = 1 - 2g(K)$,

in the same way, $\chi(\bar{S}_J) = 1 - 2g(J)$.

Now let T_1 be a triangulation of S_K with V_1 vertices, E_1 edges and F_1 faces, and, let T_2 be a triangulation of S_J with V_2 vertices, E_2 edges and F_2 faces.
We get w.l.o.g

$$\chi(S_{K\sharp J}) = (V_1 + V_2 - 2) - (E_1 + E_2 - 1) + (F_1 + F_2) = \chi(S_k) + \chi(S_J) - 1$$

$$=1 - 2g(K) + 1 - 2g(J) - 1 = 1 - 2(g(K) + g(J)).$$

.
Let $\bar{S}_{K\sharp J}$ be the corresponding surface without boundary, then

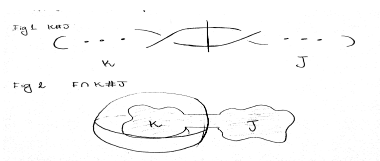
$$\chi(\bar{S}_{K\sharp J}) = \chi(S_{K\sharp J}) + 1 = 1 - 2(g(K) + g(J)) + 1 = 2 - 2(g(K) + g(J))$$

$\Rightarrow g(S_{K\sharp J}) = g(K) + g(J)$.

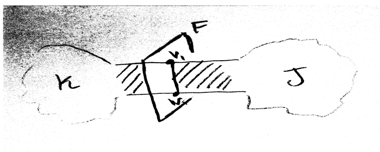
Now, since $S_{K\sharp J}$ is a seifert surface with boundary $K\sharp J$ we have $g(K\sharp J) \leq g(K) + g(J)$.

To prove $g(K) + g(J) \leq g(K\sharp J)$ is more difficult, so I will just give a sketch of the proof.

Let $S_{K\sharp J}$ be a seifert surface for $K\sharp J$ with minimal genus. Since $K\sharp J$ is a composite knot there is a sphere F such that <u>K is inside F</u>, <u>J is outside F</u> and it only intersects $K\sharp J$ in two points.



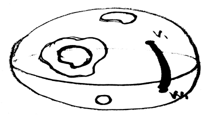
Now let us think about $F \cap S_{K\sharp J}$. First note that $F \cap S_{K\sharp J} \neq \emptyset$, the points V_1 and W_1 where F intersects $K\sharp J$ also lie on $F \cap S_{K\sharp J}$. Moreover, there is a path between V_1 and W_1 that lies on $F \cap S_{K\sharp J}$.



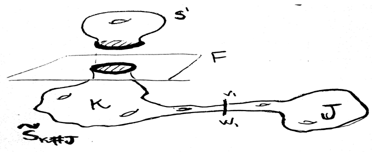
F may intersect $S_{K\sharp J}$ in a different way. Other ways of intersection might be a point, a loop, an arc etc... In general we can move (deform) the surface in such a way that $F \cap S_{K\sharp J}$ only consist of loops and arcs; this is called to put the surface in *General Position.*

We can also assume,w.l.o.g, that the path between V_1 and W_1 is the only arc in the intersection, no loop intersects this path, and both, V_1 and W_1, are on the same side of any loop on $F \cap S_{K\sharp J}$.

Over F we get a situation as the following:



We now take the loop that is the innermost on F, i.e, a loop that bounds a disk on F containing no other intersection curves. Let us call this loop C.
After that, we cut $S_{K\sharp J}$ open along C, we obtain two copies of C in the open cut of $S_{K\sharp J}$ and we glue a disk to each of the copies of C.



As a result of the last operation we get two new surfaces, a seifert surface $\tilde{S}_{K\sharp J}$ for $K\sharp J$ and a surface $S^{'}$ that we trow away. Note that by construction

$$g(S_{K\sharp J}) = g(S^{'}) + g(\tilde{S}_{K\sharp J}),$$

we must have that $g(S^{'}) = 0$ otherwise we would have that $g(S_{K\sharp J}) > g(\tilde{S}_{K\sharp J})$, which contradicts the fact that $S_{K\sharp J}$ has minimal genus.

We repeat this surgery operation until we get a seifert surface $\hat{S}_{K\sharp J}$ for $K\sharp J$ with minimal genus, such that, $\hat{S}_{K\sharp J} \cap F$ do not have any intersection circles.

F divides $\hat{S}_{K\sharp J}$ into two seifert surfaces S_K and S_J for K and J respectively.

We have:

1. $g(K\sharp J) = g(S_{K\sharp J}) = g(S_K) + g(S_J)$ (As in the last proof).
2. $g(S_K) \geq g(K)$.
3. $g(S_J) \geq g(J)$.

and so, from 1., 2. and 3. we have $g(K\sharp J) \geq g(K) + g(J)$

□

Some applications of the genus:

- We can see that the unknot is not a composite knot.
  We know that $g(O) = 0$ and $g(K) > 0$ if $K \neq O$.
  Now, If $O = K\sharp J$ for $K, J \neq O$, $g(O) = g(K\sharp J) = g(K) + g(J) > 0$ " $\Rightarrow\Leftarrow$ ".
- David Gabai from Caltech managed to show that the Kinoshita-Terasaka mutants do not have the same genus, which implies that they are distinct knots.

Reference:

Robin C. Adams, The Knot Book, American Mathematical Society, 2004.