Knot Polynomials

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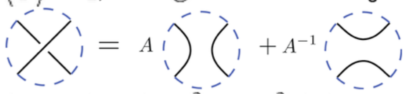
1 Why Polynomials?

First introduced by James Wadell Alexander II in 1923, knot polynomials have proved themselves by being one of the most efficient ways of classifying knots. In this spirit, one expects two different projections of a knot to have the same knot polynomial; one therefore demands that a good knot polynomial be invariant under the three Reidemeister moves (although this is not always case, as we shall find out). In this report, we present 5 selected knot polynomials: the Bracket, Kauffman X, Jones, Alexander and HOMFLY polynomials.

2 The Bracket Polynomial

2.1 Calculating the Bracket Polynomial

The Bracket polynomial makes for a great starting point in constructing knot polynomials. We start with three simple rules, which are then iteratively applied to all crossings in the knot:

- $\langle O \rangle = 1$, where O is the standard diagram of the unknot
- 

- $\langle O \cup L \rangle = (-A^2 - A^{-2})\langle L \rangle$

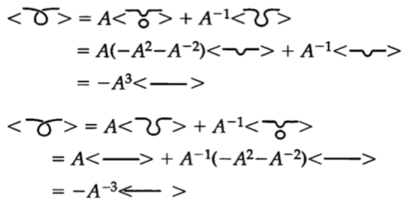
A direct application of the third rule leads to the following relation for (untangled) unknots:

$$\langle \underbrace{O \cup O \ldots O}_{n\ -\ times} \rangle = (-A^2 - A^{-2})^{n-1} \underbrace{\langle O \rangle}_{=1} = (-A^2 - A^{-2})^{n-1}$$

The process of obtaining the Bracket polynomial can be streamlined by evaluating the contribution of a particular sequence of actions in undoing the knot (*states*) and summing over all such contributions to obtain the net polynomial.

2.2 The Problem with Bracket Polynomials

The bracket polynomials can be shown to be invariant under types 2 and 3 Reidemeister moves. However by considering type 1 moves, its one major drawback becomes apparent. From:



we conclude that that the Bracket polynomial does not remain invariant under type 1 moves. This can be fixed by introducing the writhe of a knot, as we shall see in the next section. But despite its shortcomings, Bracket polynomials can be used to prove conjectures which are notoriously difficult to prove otherwise. One such conjecture is the statement that any two given reduced alternating projections of the same knot have the same number of crossings.

2.3 The Kauffmann X and Jones Polynomials

A type 1 invariant 'version' of the Bracket polynomial can be introduced by defining the writhe of a knot. To do so, we classify crossings as either L_- or L_+. A L_+ (L_-) crossing (say) is where the overstrand has a positive (negative) slope (with both over- and understrands 'pointed upwards'). To every L_+ (L_-) crossing we assign the number $+1$ (-1). The writhe is then given by the sum of these numbers.

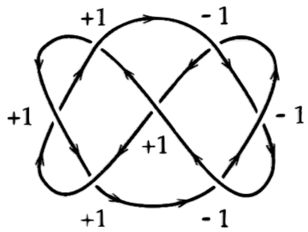


Figure 1: The writhe of this knot is given by $4 - 3 = 1$.

The Kauffmann X polynomial is then easily obtained from the Bracket polynomial, simply by multiplying in prefactor:

$$X(L) = (-A^3)^{-w(L)} \langle L \rangle$$

The Kauffmann X polynomial can be shown to be invariant also under type 1 Reidemeister (note that the writhe is invariant under type 2 and 3 moves). Furthermore, the Jones polynomial can be obtained by substituting $\frac{1}{t^4}$ for A, thereby *normalizing* the Kauffmann X polynomial.

3 The Alexander Polynomial

It was the first polynomial to be described in 1928 by the topologist James Alexander. It can be shown that the polynomial is invariant under the three Reidemeister moves, so we can change the projection of a knot while computing the polynomial.
We use two simplified rules discovered by John Conway in 1969 to compute the polynomial. But first let us introduce the so called skein relations.
Think of a knot with an orientation. Pick a specific crossing. According to the skein relation we can say it is either a L_+-knot or a L_--knot. By splitting the two strings of a crossing and gluing them together in a way that the orientation is preserved we get an L_0-knot.

Using these relations and the following rules we can now compute the polynomial of an oriented knot.

$$\Delta(\circ) = 1, \text{ for every projection of the unknot}$$
$$\Delta(L_+) - \Delta(L_-) + (t^{1/2} - t^{-1/2})\Delta(L_0) = 0$$

Pick a crossing an decide wether the knot is a L_+-knot or a L_--knot. Rearrange the equations to either $\Delta(L_+)$ or $\Delta(L_-)$ and compute the two leftover knots. One can show that after a finite number of steps one is left with just a finite number of trivial knots of which we know the polynomial.

Now let us arrange this process in a diagram, called the resolving tree to get an impression of the procedure.

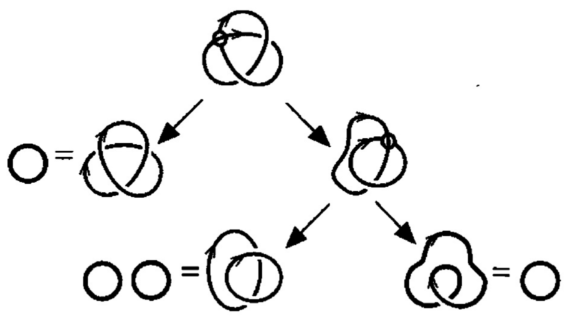


Figure 2: A resolving tree for the trefoil knot

4 HOMFLY Polynomial

The HOMFLY Polynomial is a generalisation of both the Jones and Alexander Polynomial and a Laurent Polynomial in 2 variables: m, l. Again we're using a skein relation to define it.

$$P(\circ) = 1, \text{ for every projection of the unknot}$$

$$lP(L_+) + l^{-1}(L_-) + mP(L_0) = 0$$

As with the Jones and Alexander Polynomial the HOMFLY Polynomial is invariant under the three Reidemeister moves and hence an invariant for knots.
Now, what does it mean that the HOMFLY Polynomial is a generalisation? By replacing $l = \sqrt{-1}$ and $m = \sqrt{-1}(t^{1/2} - t^{-1/2})$ we're left with the Alexander Polynomial.

$$\Delta(L_+) - \Delta(L_-) + (t^{1/2} - t^{-1/2})\Delta(L_0) = 0$$

By substituting with the right variables the same holds true for the Jones Polynomial.

An amazing property which is worth mentioning is that the polynomial of a composition knot is just the product of the polynomials of the factor knots.

$$P(L_1\#L_2) = P(L_1)P(L_2)$$

5 Amphichirality

A knot K ist amphichiral if it is ambient isotopic to its mirror image K*. It is obtained by changing every crossing of K to its opposite, while the orientation is preserved.

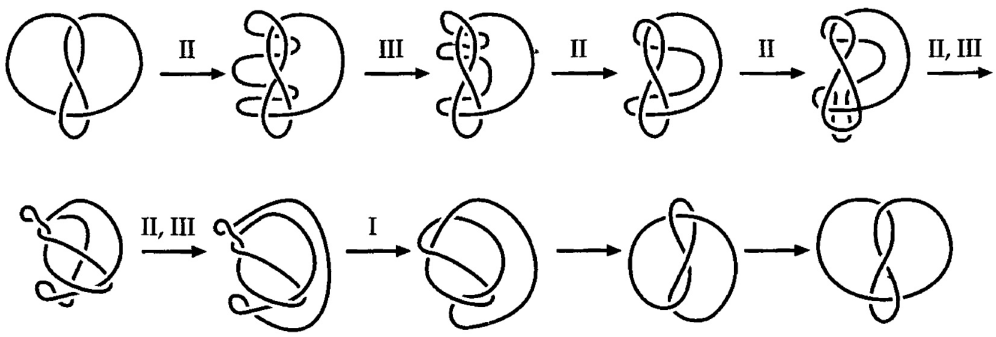


Figure 3: The figure-eight knot is amphichiral.

Now consider the Kaufmann Polynomial. One can show that the following holds true for an amphichiral knot K:

$$X_K(A) = X_K(A^{-1})$$

So the polynomial of an amphichiral knot K is palindromic, that is to say, the coefficients must be the same backwards and forwards.
Let's look at the Polynomial of the figure-eight knot, which is palindromic.

$$A^8 - A^4 + 1 - A^{-4} + A^8$$

In comparison, the polynomial of the trefoil knot is not palindromic and indeed there are two distinct projections of the trefoil knot, the left-handed and right-handed trefoil knot.

$$-A^{16} + A^{12} + A^4$$