# Arithmetic intersection numbers on Hilbert modular surfaces

Ulf Kühn, Humboldt Universität Berlin
E-mail: kuehn@mathematik.hu-berlin.de

Part of joint projects with:

J. Burgos, J. Bruinier, J. Kramer

# Arithmetic self intersection numbers

Let $\mathcal{X}$ be an arithmetic variety, i.e. a regular projective scheme, flat over $\mathrm{Spec}(\mathbb{Z})$, of relative dimension $n$.

Let $\overline{\mathcal{L}} = (\mathcal{L}, \|\cdot\|)$ be a good, hermitian line bundle on $\mathcal{X}$, i.e. a line bundle $\mathcal{L}$ on $\mathcal{X}$ together with a hermitian metric $\|\cdot\|$, which may have logarithmic singularities of certain kind along some divisor with normal crossings, on the induced complex line bundle on $\mathcal{X}(\mathbb{C})$.

Then the *arithmetic self intersection number of* $\overline{\mathcal{L}}$ is given by

$$\begin{aligned}\overline{\mathcal{L}}^{n+1} &= \widehat{\deg}\left(\pi_* \,\widehat{\mathrm{c}}_1(\overline{L})^{n+1}\right)\\ &= (s_1, \ldots, s_{n+1})_{\mathcal{X},\mathrm{fin}}\\ &\quad - \frac{1}{(2\pi i)^n} \int\limits_{\mathcal{X}(\mathbb{C})} \log\|s_1\| * \ldots * \log\|s_{n+1}\|;\end{aligned}$$

where $s_1, \ldots, s_{n+1}$ are rational sections intersecting properly on $\mathcal{X}(\mathbb{C})$.

# General Picture

Important examples of arithmetic varieties are given by regular models of Shimura varieties.

Construct an arithmetic intersection theory, where the Green's objects are allowed to have "automorphic singularities" (Burgos, Kramer, K.).

Determine the arithmetic self intersection number of the line bundle of modular forms $\overline{\mathcal{M}}_k$ equipped with the Petersson metric on a smooth compactification of a regular model.

Expectation: Such numbers should be given by logarithmic derivatives of certain $L$-functions (Maillot-Roessler, Köhler, Kramer, Kudla,...).

Examples (Bost, K.): If $\mathcal{X}(\Gamma)$ is a regular model of a modular curve $X(\Gamma)$, then

$$\overline{\mathcal{M}}_k(\Gamma)^2 = k^2[\Gamma(1):\Gamma]\zeta(-1)\left(\frac{\zeta'(-1)}{\zeta(-1)} + \frac{1}{2}\right).$$

# Hilbert modular surfaces

$K = \mathbb{Q}(\sqrt{\Delta})$: a real quadratic field of discriminant $\Delta > 0$ with $\Delta$ a prime $\equiv 1 \pmod 4$.

$\mathcal{O}_K$: the ring of integers in $K$.

$\Gamma_K = \mathrm{SL}_2(\mathcal{O}_K)$ the Hilbert modular group.

$Y_K = \Gamma_K \backslash \mathbb{H}^2$ is quasi-projective algebraic variety.

It is the coarse moduli space for Abelian surfaces with real multiplication by $\mathcal{O}_K$ and a certain class of polarizations.

$X_K = \overline{\Gamma_K \backslash \mathbb{H}^2}$, compactified by adding $h_K$ cusps.

# Desingularization

Toroidal resolution of the singularities at the cusps (Hirzebruch, Mumford et. al. ,...):

$$\widetilde{\pi} : \widetilde{X_K} \longrightarrow X_K,$$

such that the inverse image of each cusp is a DNC in $\widetilde{X_K}$.

$D := \widetilde{X_K} \setminus X_K$ divisor at infinity, $D$ is a DNC.

# Hilbert modular forms

A holomorphic function $F : \mathbb{H}^2 \longrightarrow \mathbb{C}$ is called a *Hilbert modular form of weight* $k$, if $\forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_K$

$$F\left(\frac{az_1+b}{cz_1+d}, \frac{a'z_2+b'}{c'z_2+d'}\right)$$
$$= \Big((cz_1+d)(c'z_2+d')\Big)^k F(z_1, z_2).$$

$M_k(\Gamma_K)$ vector space of Hilbert modular forms.

A Hilbert modular form $F$ has a Fourier expansion at the cusp $\infty$ of the form

$$F(z_1, z_2) = a_0 + \sum_{\substack{\nu \in \mathfrak{d}^{-1} \\ \nu \gg 0}} a_\nu \, e^{2\pi i\left(\nu z_1 + \nu' z_2\right)},$$

and analogous expansions at the other cusps.

Hilbert modular forms are global sections of the line bundle of modular forms $\mathcal{M}_k$ on $\widetilde{X_K}$.

Notice on $\widetilde{X_K}$ the divisor of the section $\widetilde{F}$ of $\mathcal{M}_k$ associated with a Hilbert modular form $F$ decomposes as

$$\mathrm{div}(\widetilde{F}) = \mathrm{div}(F) + \widetilde{D}_F,$$

with $\mathrm{supp}\,\widetilde{D}_F \subseteq D$.

The *Petersson metric* on $\mathcal{M}_k$ is on $\mathbb{H}^2$ defined by

$$\|F\|^2_{Pet} := |F(z_1, z_2)|^2 (16\pi^2 y_1 y_2)^k,$$

where $F \in M_k(\Gamma_K)$.

**Observation:** $\overline{\mathcal{M}}_k = (\mathcal{M}_k, \|\quad\|_{Pet})$ is a good hermitian line bundle on $\widetilde{X_K}$.

# Formula for the star product

**Theorem.** *Let* $F_1$, $F_2$, $F_3 \in M_k(\Gamma_K)$*. If* $\mathrm{div}(F_1) \cap \mathrm{div}(F_2) \cap \mathrm{div}(F_3) = \emptyset$ *and if* $\mathrm{div}(F_1) \cap \mathrm{div}(F_2)$ *is disjoint to the cusps, then*

$$\frac{1}{(2\pi i)^2} \int\limits_{\widetilde{X_K}} \log \|F_1\|_{Pet} * \log \|F_2\|_{Pet} * \log \|F_3\|_{Pet}$$

$$= k^2 \int\limits_{\Gamma_K \backslash \mathbb{H}^2} \log \|F_1\|_{Pet}\, \omega^2$$

$$+ k \int\limits_{\mathrm{div}(F_1)} \log \|F_2\|_{Pet}\, \omega$$

$$+ \int\limits_{\mathrm{div}(F_1) \cap \mathrm{div}(F_2)} \log \|F_3\|_{Pet},$$

*where*

$$\omega = \frac{dx_1 \wedge dy_1}{4\pi y_1^2} + \frac{dx_2 \wedge dy_2}{4\pi y_2^2}.$$

Proof uses:

• Stoke's theorem applied to a more general formula of the star product which involves certain partitions of unity.

• good knowledge of the desingularisation and of the local behavior of the Petersson norm of a modular form.

Remarks:

• formula is independent of $D = \widetilde{X_K} \setminus X_K$, this is special for Hilbert modular surfaces.

• formula coincides with the usual formula for the star product.

• need to find sections for which we can calculate all the integrals.

# Hirzebruch-Zagier divisors

The Hirzebruch-Zagier divisor of discriminant $m$, $m \in \mathbb{N}$, on $\mathbb{H}^2$ is given by the equations

$$\bigcup_{\substack{(a,b,\lambda)\in\mathbb{Z}\times\mathbb{Z}\times\mathfrak{d}^{-1} \\ ab-\mathsf{N}(\lambda)=m/\Delta}} \left\{ az_1z_2 + \lambda z_1 + \lambda' z_2 + b = 0 \right\}.$$

It descends to an algebraic divisor $T(m)$ on the quotient $Y_K$.

If $m$ square free and $\chi_\Delta(m) = 1$, then

$$\begin{aligned} T(m) &\cong Y_0(m) = \Gamma_0(m) \setminus \mathbb{H}, && m = \lambda\lambda', \\ T(m) &\cong Sh(m), && m \neq \lambda\lambda'. \end{aligned}$$

Intersection points of Hirzebruch-Zagier divisors are special points, e.g. Heegner points.

# Borcherds products

Borcherds products $F$ are meromorphic Hilbert modular forms, whose divisors are finite linear combinations of Hirzebruch-Zagier divisors.

A Borcherds product is called *integral* if the Fourier coefficients of $F$ are integers. Given a Borcherds product $F$, then $F^m$ is integral for some $m \in \mathbb{N}$.

**Theorem.** *Let $F_3$ be any integral Borcherds product. Then there exist infinitely many integral Borcherds products $F_2$ and $F_1$ of non-zero weights such that*

*1.)* $\operatorname{div}(F_1) \cap \operatorname{div}(F_2) \cap \operatorname{div}(F_3) = \emptyset$*,*

*2.)* $F_2(\kappa) = 1$ *for all cusps* $\kappa$ *of* $X_K$*,*

*3.)* $\operatorname{div}(F_1)$ *is a linear combination of Hirzebruch-Zagier divisors of prime discriminant* $m$ *such that* $m$ *is a norm of* $\mathcal{O}_K$ *coprime to* $\Delta$*.*

Proof uses:

- Borcherds Theorem,
- properties of the obstruction space,
- representation numbers of quadratic forms,
- properties of $\ell$-adic Galois representations.

**Proposition**: *Let $T(N)$ be any Hirzebruch-Zagier divisor. Then there exist infinitely many integral Borcherds products $F$ of non-zero weight such that*

$$\mathrm{div}(F) = c(N)T(N) + \sum_{\substack{p \text{ prime} \\ \chi_\Delta(p)=1}} c(p)T(p)$$

*with suitable coefficients $c(.)$ and $c(N) \neq 0$.*

# Example for $K = \mathbb{Q}(\sqrt{5})$

Gundlach constructed Hilbert cusp form $\Theta(z_1, z_2)$ of weight 5 by means of theta series and showed:

$$\operatorname{div}(\Theta) = T(1).$$

In fact $2^{-6}\Theta(z_1, z_2)$ is an integral Borcherds product and equals (cf. Bruinier-Bundschuh)

$$e\left(\frac{\varepsilon_0 z_1 - \varepsilon_0' z_2}{\sqrt{5}}\right) \prod_{\substack{\nu \in \mathfrak{d}^{-1} \\ \varepsilon_0 \nu' - \varepsilon_0' \nu > 0}} \Big(1 - e(\nu z_1 + \nu' z_2)\Big)^{\tilde{c}(5\nu\nu')}.$$

Here $\varepsilon_0 = \frac{1}{2}(1 + \sqrt{5})$ is the fundamental unit and $\tilde{c}(.)$ are essentially the Fourier coefficients of a certain elliptic modular function of level 5. The integral of the Gundlach theta function equals

$$\int_{\widetilde{X_K}} \log \|2^{-6}\Theta(z_1, z_2)\|_{Pet}\, \omega^2$$

$$= 5\zeta_K(-1)\left(\frac{L'(\chi_5, -1)}{L(\chi_5, -1)} + \frac{1}{2} + \frac{1}{2}\log(5)\right).$$

**Theorem.** *Let $F_1$, $F_2$ and $F_3$ be three integral Borcherds products of weights $k_1$, $k_2$ and $k_3$ as before:*

*If* $\mathrm{div}(F_1) = \sum a(m)T(m)$, $\mathrm{div}(F_2) = \sum b(n)T(n)$ *and* $\mathrm{div}(F_3) = \sum c(n')T(n')$, *then*

$$\frac{1}{(2\pi i)^2} \int\limits_{\widetilde{X_K}} \log \|F_1\|_{Pet} * \log \|F_2\|_{Pet} * \log \|F_3\|_{Pet}$$

$$= k_1 k_2 k_3 \zeta_K(-1)$$

$$\times \left( \frac{\zeta_K'(-1)}{\zeta_K(-1)} + \frac{\zeta'(-1)}{\zeta(-1)} + \frac{3}{2} + \frac{1}{2} \log \Delta \right)$$

$$+ \sum_{m,n,n'>0} a(m)b(n)c(n') \cdot \Big( P_m(n), P_m(n') \Big)_{\mathcal{X}_0(m),\mathrm{fin}},$$

*here $P_m(n)$, $P_m(n')$ are certain linear combination of Heegner points on the minimal regular model $\mathcal{X}_0(m)$ of $X_0(m)$.*

Main ingredients of proof:

• Fourier expansions of automorphic Green's functions (Bruinier) or the Siegel-Weil formula (Kudla) determine

$$\int_{\widetilde{X_K}} \log \|F_1\|_{Pet}\, \omega^2.$$

in terms of $L'/L$, sum of $\log(m)$'s, ...

• Arithmetic intersection numbers associated to $X_0(m) \cong T(m)$ and the action of the Atkin-Lehner involution on $\mathcal{X}_0(m)$ are used to show

$$\begin{aligned}
&\int_{X_0(m)} \log \|F_2\|_{Pet}\, k_3\, \omega + \int_{X_0(m)\cap \mathrm{div}(F_2)} \log \|F_3\|_{Pet} \\
&= \sum_{n,n'<0} b(n)c(n') \cdot \Big(P_m(-n), P_m(-n')\Big)_{\mathcal{X}_0(m),\mathrm{fin}} \\
&\quad + \mathrm{vol}(X_0(m)) \cdot k_2 k_3 \frac{1-m}{2(m+1)} \log(m) \\
&\quad + \mathrm{vol}(X_0(m)) \cdot 4 k_2 k_3 \left( \frac{\zeta'(-1)}{\zeta(-1)} + \frac{1}{2} \right).
\end{aligned}$$

# A regular model

We introduce a level structure.

$A \in \mathbb{N}$, square-free, prime factors inert in $K$.

$\mathcal{A} = (A) \subset \mathcal{O}_K$.

$\Gamma_{00}(\mathcal{A}) = \left\{ \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \Gamma_K; \quad c,\, a-1,\, d-1 \in \mathcal{A} \right\}$.

$\Gamma_{00}(\mathcal{A}) \backslash \mathbb{H}^2$ parameterizes isomorphism classes of Abelian varieties with real multiplication by $\mathcal{O}_K$ and additional structure.

**Theorem** (Pappas). *There exists an arithmetic variety $\mathcal{X}_{00}(\mathcal{A})$ such that $\mathcal{X}_{00}(\mathcal{A})(\mathbb{C})$ is a smooth compactification of $\Gamma_{00}(\mathcal{A}) \backslash \mathbb{H}^2$.*

Let $\overline{\mathcal{M}}_k$ be the line bundle of modular forms on $\mathcal{X}_{00}(\mathcal{A})$ equipped with the Petersson metric.

*$q$-expansion principle*: global sections of $\mathcal{M}_k$ correspond to modular forms of weight $k$ for $\Gamma_{00}(\mathcal{A})$ with integral Fourier coefficients.

**Theorem.**(Bruinier, Burgos, K.) *The arithmetic self intersection number* $\overline{\mathcal{M}}_k^3$ *of the line bundle of modular forms* $\overline{\mathcal{M}}_k$ *on* $\mathcal{X}(\Gamma_{00}(\mathcal{A}))$ *is given by*

$$\overline{\mathcal{M}}_k^3 = -k^3[\Gamma_K : \Gamma_{00}(\mathcal{A})]\,\zeta_K(-1) \times \left(\frac{\zeta_K'(-1)}{\zeta_K(-1)} + \frac{\zeta'(-1)}{\zeta(-1)} + \frac{3}{2} + \frac{1}{2}\log(\Delta)\right).$$

Remarks:

- observe $\mathrm{vol}(X_K) = \zeta_K(-1)$.
- conjectured 1999 in thesis of K.
- compatible to conjectures of Maillot-Roessler.
- related to derivatives of Eisenstein series.

Proof uses:

- previous explicit formula for the star-product.

- integrality of Borcherds products.

- irreducibility of the coarse moduli space of abelian surfaces with real multiplication (Rapoport, Deligne-Pappas).

- Deligne-Rapoport description of $\mathcal{X}_0(m)$.

Same techniques can be used to calculate Faltings-heights of Hirzebruch-Zagier divisors.

**Theorem.**(Bruinier, Burgos, K.) *If* $T(n)$ *is any Hirzebruch-Zagier divisor, then the Faltings height of its natural model* $\widetilde{\mathcal{T}(n)} \in \mathrm{Z}^1(\mathcal{X}_{00}(\mathcal{A}))_{\mathbb{Q}}$ *is independent of the choice made in the toroidal compactification. If* $(n, \triangle) = 1$ *, then it is given by the equality*

$$\mathrm{ht}_{\overline{\mathcal{M}}(\Gamma_{00}(\mathcal{A}))_k}(\widetilde{\mathcal{T}(n)}) = k^2 \,\mathrm{vol}(T(n)) \times \Big( \frac{\zeta'(-1)}{\zeta(-1)} + \frac{1}{2}$$

$$- \frac{\sum_{d|n} d(\chi_\Delta(d) + \chi_\Delta(n/d)) \log(d)}{2 \sum_{d|n} d(\chi_\Delta(d) + \chi_\Delta(n/d))} + \frac{1}{4} \log(n) \Big).$$

Remarks:

• These Faltings-heights turn out to be related to the coefficients of the derivative of a certain Eisenstein series for $\Gamma_0(D)$ of weight 2, confirming a general conjecture of Kudla in this case.

• One may deduce the self intersection number of the Hodge bundle on certain Shimura curves. (Kudla-K.)

• If $m$ is square free norm of $\mathcal{O}_K$, then $\mathcal{T}(m)$ is the image of a regular model for the modular curve $X(m, \mathsf{N}(\mathcal{A}))$ and

$$\mathrm{ht}_{\overline{\mathcal{M}}(\Gamma_{00}(\mathcal{A}))_k}(\mathcal{T}(m)) =$$
$$\mathrm{ht}_{\overline{\mathcal{M}}(\Gamma_0(m)\cap\Gamma_1(\mathsf{N}(\mathcal{A})))_{2k}}(\mathcal{X}(m, \mathsf{N}(\mathcal{A}))$$
$$- k^2 \,\mathrm{vol}(X(m, \mathsf{N}(\mathcal{A}))) \left( \frac{1}{2} \sum_{p|m} \frac{p}{p+1} \log(p) - \frac{1}{4} \log m \right).$$

It seems to be a challenge to compute this difference of Faltings heights by means of algebraic geometry, i.e. using the change of moduli problems. A positive answer would give a generalization of Raynauds theorem on the change of Faltings heights under isogenies.