Arakelov Chow groups of non compact Shimura Varieties

Joint work with J. Bruinier, J. Kramer, U. Kühn.

Heights and log-singular hermitian line bundles

From the proof of the Mordell Conjecture by Faltings.

Let $M = M_{g,l}/\mathbb{C}$ the moduli space of genus g abelian varieties with some level structure. Let $A \longrightarrow M$ be the universal family and $s : M \longrightarrow A$ the zero section.
Write

$$\omega_{A/M} = s^*(\Omega^g_{A/M}).$$

It is an ample line bundle with a natural metric $\| \bullet \|$ given by

$$\langle \alpha, \beta \rangle = \left(\frac{i}{2}\right)^g \int_A \alpha \wedge \overline{\beta}.$$

Let r such that $\mathscr{M} = (\omega_{A/M})^r$ is very ample. $\overline{M}$ the corresponding compactification of M.
Lemma. There is a unique extension of $\mathscr{M}$ over $\overline{M}$, denoted $\overline{\mathscr{M}}$ such that the induced metric $\|\bullet\|$ on $\overline{\mathscr{M}}$ has logarithmic singularities along $\overline{M} \setminus M$.

Lemma.
$X \subset \mathbb{P}^n_{\mathbb{Z}}$ Zariski closed subset. $Y \subset X$ closed. $\|\bullet\|$ hermitian metric on $\mathcal{O}(1)|_{X(\mathbb{C})\setminus Y(\mathbb{C})}$, with logarithmic singularities along Y.
K number field. h the height associated to $\| \bullet \|$. $c > 0$. Then

$$\{x \in X(K) \setminus Y(K) \mid h(x) \leq c\}$$

is finite.

<u>Conclusion:</u> An hermitian metric with logarithmic singularities can be used (in some cases) as if it were smooth.

Good metrics.

From the proof of Hirzebruch's proportionality theorem in the non compact case by Mumford

$\overline{X}$ complex manifold of dimension n.
$X = \overline{X} \setminus D$, D a normal crossings divisor.
U open coordinate set.
$U \setminus D \cong (\Delta_\epsilon^*)^k \times \Delta_\epsilon^{n-k}$, ϵ small.
Poincaré metric on Δ_ϵ^*: $ds^2 = \frac{|dz|^2}{|z|^2(\log|z|)^2}$
Standard metric in Δ_ϵ: $ds^2 = |dz|^2$.
ω_U the product metric in $U \setminus D$.

Definition. A complex valued p-form, η, has *Poincaré growth* if, for any open coordinate set U as above, one has the estimate

$$|\eta(t_1, \ldots, t_p)|^2 \leq C\omega_U(t_1, t_1) \ldots \omega_U(t_p, t_p).$$

for all tangent vectors $t_1, \ldots, t_p$ in $U \setminus D$.

Definition. A p-form η is *good* if η and $d\eta$ have Poincaré growth.

Proposition. i) A good form is locally L^1.
ii) Let $[\eta]$ be the associated current. Then $d[\eta] = [d\eta]$.

Definition. E vector bundle on $\overline{X}$. h hermitian metric in $E|_X$. We say h is *good* if, for all open coordinate sets as above, and local frames of E
i) $|h_{ij}|, (\det h)^{-1} \leq C(\sum_{i=1}^{k} \log |z_i|)^N$, $C > 0$.
ii) The 1-form $(\partial h . h^{-1})_{ij}$ are good.

Theorem. If h is a good metric of E, then, for all k, the k-th Chern form $c_k(E, h)$ is a good form and the current $[c_k(E, h)]$ represents the k-th Chern class of E.

Automorphic bundles

$D = K\backslash G$ a hermitian symmetric domain.
Γ a neat arithmetic group acting on D.
$X = D/\Gamma$, $\overline{X}$ a smooth toroidal compactification of X.
E_0 a rank r equivariant analytic vector bundle on D.
h_0 a G-equivariant hermitian metric on E_0.
$E = E_0/\Gamma$ the induced vector bundle on X.
h the induced metric.
We call E an automorphic vector bundle and h the Petersson metric.

Theorem. E admits a unique extension to a vector bundle $\overline{E}$ on $\overline{X}$ such that h is a singular good hermitian metric on $\overline{X}$.

Goal of the project

To develop a theory of arithmetic Chow rings that has all the usual properties:

1. Exact sequences.
2. Associative and commutative products.
3. Functoriality.
4. Arithmetic degrees.

But that receives characteristic classes from automorphic vector bundles with Petersson metric.

The case of modular curves

J.B. Bost in his proof of Lefschetz theorems for arithmetic surfaces, has introduced hermitian metrics of type L_1^2 and has defined arithmetic Chow groups with L_1^2-Green functions: $\widehat{\mathrm{CH}}^1(X, L_1^2)$.
Moreover he has shown that there is a well defined intersection pairing and arithmetic degrees.

Example: If X is a modular curve, $\mathcal{L}$ a modular line bundle and h is the Petersson metric then h is of L_1^2 type.

Simultaneously, U. Kühn has studied the case of modular curves, introducing the arithmetic Picard group of line bundles with logarithmically singular hermitian metrics. He has given a formula for the arithmetic intersection pairing that agrees with the L_1^2 intersection pairing.

Log-log forms

Let X be a smooth complex variety of dimension n, $D \subset X$ a normal crossings divisor.
$V = X \setminus D$, $j : V \longrightarrow X$ the inclusion.
U open coordinate subset with coordinates $(z_1, \ldots, z_n)$. $z_1 \ldots z_k = 0$ a local equation for D. We say that U is adapted to D.
Write $r_i = \|z_i\|$.

Definition. A function f on V is said to be log-log along D if, for any coordinate subset as above, the following estimate holds near D

$$|f(z_1, \ldots, z_n)| \prec \left| \prod_{i=1}^{k} \log(\log(1/r_i)) \right|^{M},$$

$$\left| \frac{\partial^{|\alpha|}}{\partial z^{\alpha}} \frac{\partial^{|\beta|}}{\partial \overline{z}^{\beta}} f(z_1, \ldots, z_n) \right| \prec \frac{\left| \prod_{i=1}^{k} \log(\log(1/r_i)) \right|^{M_{\alpha,\beta}}}{\prod_{i=1}^{k} r_i^{\alpha_i+\beta_i} \log(1/r_i)^{\min(\alpha_i,1)+\min(\beta_i,1)}}.$$

Definition. The sheaf of differential forms on X log-log along D, $\mathscr{E}_X^*\langle\langle D\rangle\rangle$,
is the subalgebra of $j_*\mathscr{E}_V^*$ generated locally by the log-log functions and the forms

$$\frac{dz_i}{z_i \log(1/r_i)}, \ \frac{d\overline{z}_i}{\overline{z}_i \log(1/r_i)}, \text{ for } i = 1, \dots, k,$$
$$dz_i, \ d\overline{z}_i, \qquad \text{for } i = k+1, \dots, n.$$

Theorem. The natural inclusion

$$\Omega_X^* \longrightarrow \mathscr{E}_X^*\langle\langle D\rangle\rangle$$

is a filtered quasi-isomorphism with respect to the Hodge filtration.

Functorial properties of log-log forms

Proposition. Let X, Y be smooth complex varieties. $D \subset X$, $E \subset Y$ be normal crossing divisors. $f : X \longrightarrow Y$ a morphism such that $f^{-1}(E) \subset D$. Then

$$f^* \mathscr{E}_Y^* \langle\langle E \rangle\rangle \subset \mathscr{E}_X^* \langle\langle D \rangle\rangle .$$

Proposition. The forms of log-log growth are locally integrable with zero residue. In particular, if X is proper of dimension n, there is a degree map

$$E_X^{2n} \langle\langle D \rangle\rangle \longrightarrow \mathbb{C}.$$

Proposition. X, D as above. C a curve. $f : X \longrightarrow C$ a proper dominant morphism. We can write $D = D_h \cup D_v$. D_h horizontal components, D_v vertical components. Put

$$S = f(D_v) \cup \{\text{singular values}\}.$$

Then

$$f_* E_X^* \langle\langle D \rangle\rangle \subset E_C^* \langle\langle S \rangle\rangle .$$

Log forms

Definition. A function f on V is said to be log along D if, for any coordinate subset as above, the following estimate holds near D

$$|f(z_1,\ldots,z_n)| \prec \left|\prod_{i=1}^{k} \log(1/r_i)\right|^{M},$$

$$\left|\frac{\partial^{|\alpha|}}{\partial z^{\alpha}}\frac{\partial^{|\beta|}}{\partial \overline{z}^{\beta}} f(z_1,\ldots,z_n)\right| \prec \frac{\left|\prod_{i=1}^{k} \log(1/r_i)\right|^{M_{\alpha,\beta}}}{\prod_{i=1}^{k} r_i^{\alpha_i+\beta_i}}.$$

Definition. The sheaf of differential forms on X log along D, $\mathscr{E}^*_X\langle D\rangle$, is the subalgebra of $j_*\mathscr{E}^*_V$ generated locally by the log functions and the forms

$$\frac{dz_i}{z_i},\ \frac{d\overline{z}_i}{\overline{z}_i},\ \text{for } i = 1,\ldots,k,$$
$$dz_i,\ d\overline{z}_i,\ \text{for } i = k+1,\ldots,n.$$

Log and log-log forms

Let D_1, D_2 be two normal crossing divisors such that $D_1 \cup D_2$ is also a normal crossing divisor.
We can define the sheaf $\mathscr{E}_X^* \langle D_1 \langle D_2 \rangle\rangle$ of forms that are log along D_1 and log-log along D_2.

Theorem. The natural inclusion

$$\Omega_X^*(\log D_1) \longrightarrow \mathscr{E}_X^* \langle D_1 \langle D_2 \rangle\rangle$$

is a filtered quasi-isomorphism.

U a non proper smooth complex variety.
$D \subset U$ a normal crossing divisor.

$$\mathscr{E}_l^* \langle\langle D \rangle\rangle (U) = \varinjlim_{(X,D')} \mathscr{E}_X^* \left\langle D' \left\langle \overline{D} \right\rangle \right\rangle,$$

where the limit is taken over all compactifications (X, D') of U, such that $D' \cup \overline{D}$ is a normal crossing divisor.

Deligne-Beilinson cohomology

The space of global sections $E_l^*\langle\langle D\rangle\rangle(U)$ is a Dolbeault algebra.

Let $\mathcal{D}^*(E_l\langle\langle D\rangle\rangle(U),*)$ be the associated Deligne algebra.

Proposition. The complex $\mathcal{D}^*(E_l\langle\langle D\rangle\rangle(U),*)$ computes the real Deligne-Beilinson cohomology of U.

If $U_{\mathbb{R}}$ is real and $D_{\mathbb{R}}$ is defined over $\mathbb{R}$, then

Proposition. The complex

$$\mathcal{D}^*(E_l\langle\langle D_{\mathbb{R}}\rangle\rangle(U_{\mathbb{R}}),p)=\mathcal{D}^*(E_l\langle\langle D_{\mathbb{C}}\rangle\rangle(U_{\mathbb{C}}),p)^{F_\infty}$$

computes the real Deligne-Beilinson cohomology of the real variety $U_{\mathbb{R}}$, where F_∞ is the involution that acts as complex conjugation in the space and the coefficients.

Arithmetic Chow rings with log-log forms

Let X be an arithmetic variety over the arithmetic ring A, of relative dimension n.
Let D be a fixed normal crossing divisor of $X_{\mathbb{R}}$.
We denote by $\underline{X}$ the pair (X, D). The assignment

$$U_{\mathbb{R}} \longmapsto \mathcal{D}^n(E_l \left\langle\left\langle D \right\rangle\right\rangle (U_{\mathbb{R}}), p)$$

is a presheaf in the Zariski topology of $X_{\mathbb{R}}$, denoted $\mathcal{D}^n_{l,\underline{X}}(p)$. This presheaf satisfies the Mayer-Vietoris principle, hence it is an acyclic sheaf.
There is a well defined map

$$\mathcal{D}_{\mathrm{log},X_{\mathbb{R}}}(p) \longrightarrow \mathcal{D}^*_{l,\underline{X}}(p).$$

That turns $\mathcal{D}^*_{l,\underline{X}}(*)$ into a $\mathcal{D}_{\mathrm{log}}$-algebra over X.
Since $\mathcal{D}_{\mathrm{log}}$ is a Gillet complex, we can define the arithmetic Chow groups with log-log forms

$$\widehat{\mathrm{CH}}^*(\underline{X}) = \widehat{\mathrm{CH}}^*(X, \mathcal{D}^n_{l,\underline{X}}(p)).$$

Exact sequences

Theorem. The following sequences are exact

$$\mathrm{CH}^{p-1,p}(X) \xrightarrow{\rho} \widetilde{\mathcal{D}^{2p-1}_{l,\underline{X}}}(X_{\mathbb{R}},p) \xrightarrow{\mathsf{a}} \widehat{\mathrm{CH}}^{p}(X,\mathcal{D}_{l,\underline{X}}) \xrightarrow{\zeta} \mathrm{CH}^{p}(X) \longrightarrow 0,$$

$$\mathrm{CH}^{p-1,p}(X) \xrightarrow{\rho} H^{2p-1}_{\mathcal{D}}(X_{\mathbb{R}},p) \xrightarrow{\mathsf{a}} \widehat{\mathrm{CH}}^{p}(X,\mathcal{D}_{l,\underline{X}}) \xrightarrow{(\zeta,-\omega)}$$
$$\mathrm{CH}^{p}(X) \oplus Z\mathcal{D}^{2p}_{l,\underline{X}}(X_{\mathbb{R}},p) \xrightarrow{\mathrm{cl}+h} H^{2p}_{\mathcal{D}_{l,\underline{X}}}(X_{\mathbb{R}},p) \longrightarrow 0,$$

$$\mathrm{CH}^{p-1,p}(X) \xrightarrow{\rho} H^{2p-1}_{\mathcal{D}}(X_{\mathbb{R}},p) \xrightarrow{\mathsf{a}} \widehat{\mathrm{CH}}^{p}(X,\mathcal{D}_{l,\underline{X}})_0 \xrightarrow{\zeta} \mathrm{CH}^{p}(X)_0 \longrightarrow 0.$$

Properties

Theorem.

1. $\widehat{\mathrm{CH}}^p(X, \mathcal{D}_{l,\underline{X}}) \otimes \mathbb{Q}$ is a commutative and associative ring.

2. Let $f : X \longrightarrow Y$ be a morphism of arithmetic varieties, D and E normal crossing divisors of $X_{\mathbb{R}}$ and $Y_{\mathbb{R}}$ respectively. If $f^{-1}(E) \subset D$, then there is a morphism
$$f^* : \widehat{\mathrm{CH}}^*(Y, \mathcal{D}_{l,\underline{Y}}) \longrightarrow \widehat{\mathrm{CH}}^*(X, \mathcal{D}_{l,\underline{X}}).$$

3. If X is proper over A, of relative dimension n, there is a well defined map
$$\widehat{\deg} : \widehat{\mathrm{CH}}^{n+1}(X, \mathcal{D}_{l,\underline{X}}) \longrightarrow \widehat{\mathrm{CH}}^1(A).$$

Comparison with the groups of Gillet and Soulé

X an arithmetic variety over A. D normal crossing divisor of $X_{\mathbb{R}}$. y a codimension p algebraic cycle of X,

Proposition. A pair (ω_y, g_y) is a Green object for the cycle y if and only if

$$-2\partial\bar{\partial}[g_y] = [\omega_y] - \delta_y.$$

Proposition. Let y and z be algebraic cycles of complementary codimension, such that $y_{\mathbb{R}}$ and $z_{\mathbb{R}}$ intersect properly and

$$y_{\mathbb{R}} \cap z_{\mathbb{R}} \cap D = \emptyset.$$

Let $\mathfrak{g}_y = (\omega_y, g_y)$ and $\mathfrak{g}_z = (\omega_z, g_z)$ be Green objects for the cycles y and z. Then

$$\int_X \mathfrak{g}_y * \mathfrak{g}_z = \int_X g_y \wedge \omega_z + (2\pi i)^p \int_Y g_z.$$

Proposition. If $X_{\mathbb{R}}$ is projective and D is empty, there is a natural isomorphism

$$\widehat{\mathrm{CH}}^*(X, \mathcal{D}_{l,\underline{X}}) \longrightarrow \widehat{\mathrm{CH}}^*(X).$$

Good metrics II

X complex variety, D a normal crossing divisor, $U = X \setminus D$.
E vector bundle on X. h hermitian metric on $E|_U$.
Definition. The metric h is *good* if, for all open coordinate sets adapted to D, and local frames of E
i) $|h_{ij}|, (\det h)^{-1} \leq C(\sum_{i=1}^{k} \log |z_i|)^N$, $C > 0$.
ii) The 1-forms $(\partial h.h^{-1})_{ij}$ are log-log along D.

Theorem. (Mumford + ε) $X = D/\Gamma$, $\overline{X}$ a toroidal compactification. E an automorphic vector bundle on X with the Petersson metric. Then E can be extended uniquely to a vector bundle on $\overline{X}$ with a good metric.

Bott-Chern forms of good metrics

X a smooth complex algebraic variety.
D a normal crossing divisor. $U = X \setminus D$.
E a vector bundle over X.
h a smooth hermitian metric on $E|_U$, which is good as a metric on E.
h' a smooth hermitian metric on E.

Theorem. The Bott-Chern forms for the change of metrics h, h' belongs to the space

$$\bigoplus_p \mathcal{D}^{2p-1}(E_X \langle\langle D \rangle\rangle, p).$$

h'' a second smooth metric on E.

Theorem. The iterated Bott-Chern form for the change of metrics h, h', h'' belongs to the space

$$\bigoplus_p \mathcal{D}^{2p-2}(E_X \langle\langle D \rangle\rangle, p).$$

Characteristic classes

X arithmetic variety with $X_{\mathbb{R}}$ projective. D a normal crossing divisor. E a vector bundle on X with h a hermitian metric on $E_{\mathbb{R}}$, good along D. Let φ be a power series in the Chern classes.

Definition. The characteristic class $\widehat{\varphi}(E,h)$ is defined by choosing a smooth hermitian metric h' and writing

$$\widehat{\varphi}(E,h) = \widehat{\varphi}(E,h') + \mathrm{a}(\widetilde{\varphi}(h',h)) \in \widehat{\mathrm{CH}}^{*}(X, \mathcal{D}_{l,\underline{X}}).$$

These characteristic classes satisfy the usual properties.