Automorphic Green's functions

Jan Hendrik Bruinier
Universität Heidelberg
E-mail: bruinier@mathi.uni-heidelberg.de

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Motivation

Important examples of arithmetic varieties are given by regular models of Shimura varieties.

Determine arithmetic self intersection numbers of "natural" hermitian line bundles on a smooth compactification of a regular model.

More generally: Faltings heights of sub-varieties with respect to such line bundles.

Here: We focus on the line bundle of modular forms $\overline{\mathcal{M}}_k$ equipped with the (logarithmically singular) Petersson metric.

The appropriate intersection theory for line bundles with logarithmically singular metric has been provided by Burgos-Kramer-Kühn.

Expectation: Such numbers should be given by logarithmic derivatives of certain L-functions (Maillot-Roessler, Köhler, Kramer, Kudla,...).

Analytic side

Compute star-products on Shimura varieties.

This essentially boils down to computing

$$\int_{\Gamma\backslash D} \log \|F\|_{Pet}\, d\mu, \tag{1}$$

for a modular form F; and more generally to computing

$$\int_{\Gamma\backslash D} G_T\, d\mu, \tag{2}$$

where G_T is some "nice" Green's function associated with a "modular" divisor T.

Both integrals cannot be computed in general (at present). But:

(1) can be computed for *Borcherds products* on orthogonal groups $\mathrm{O}(2,p)$ (Kudla, Bruinier-Kühn).

(2) can be computed for *Heegner divisors* on $\mathrm{O}(2,p)$ and certain automorphic Green's functions.

Today:

We would like to indicate

- how this works,
- give some background,
- explain the relation of (1) and (2).

For simplicity we concentrate on a certain $\mathrm{O}(2,2)$ case: Hilbert modular surfaces.

Recall: $\mathrm{O}(2,2) \cong \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$.

Most of the arguments also work in general for $\mathrm{O}(2,p)$.

Hilbert modular surfaces

$K = \mathbb{Q}(\sqrt{D})$: the real quadratic field of discriminant $D > 0$.

For simplicity: D a prime $\equiv 1 \pmod 4$.

$\mathcal{O}_K$: the ring of integers in K.

$\Gamma_K = \mathrm{SL}_2(\mathcal{O}_K)$ the Hilbert modular group. It acts on $\mathbb{H}^2$, where $\mathbb{H} = \{z \in \mathbb{C};\ \Im(z) > 0\}$.

$Y_K = \Gamma_K \backslash \mathbb{H}^2$ is quasi-projective algebraic variety.

It is the coarse moduli space for Abelian surfaces with real multiplication by $\mathcal{O}_K$ and a certain class of polarizations.

It can be compactified by adding h_K cusps.

Hirzebruch-Zagier divisors

Let $m \in \mathbb{N}$. The Hirzebruch-Zagier divisor of discriminant m on $\mathbb{H}^2$ is given by the equations

$$az_1z_2 + \lambda z_1 + \lambda' z_2 + b = 0,$$

where $a, b \in \mathbb{Z}$, $\lambda \in \mathfrak{d}^{-1}$, and $ab - \lambda\lambda' = m/D$.

It can be viewed as an algebraic divisor $T(m)$ on the quotient Y_K.

$T(m) = 0 \Leftrightarrow \chi_D(m) = -1$.

If m square free and $\chi_D(m) = 1$, then

$$\begin{aligned} T(m) &\cong Y_0(m), \qquad & m &= \lambda\lambda', \\ T(m) &\cong Sh(m), \qquad & m &\neq \lambda\lambda'. \end{aligned}$$

$T(m)$ has a moduli interpretation.

Hilbert modular forms

A meromorphic Hilbert modular form F of weight k is a meromorphic function on $\mathbb{H}^2$ satisfying

$$F\left(\frac{az_1+b}{cz_1+d}, \frac{a'z_2+b'}{c'z_2+d'}\right) = (cz_1+d)^k(c'z_2+d')^k F(z_1,z_2)$$

for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_K$.

Borcherds products on Y_K are certain meromorphic Hilbert modular forms, which have product expansions and arise as liftings of elliptic modular forms of weight 0. Their zeros and poles lie on Hirzebruch-Zagier divisors.

Input data for Borcherds' lifting

$A_k(D, \chi_D)$ = nearly holomorphic modular forms of weight k for $\Gamma_0(D)$ with character χ_D.

$f \in A_k(D, \chi_D)$ has a Fourier expansion

$$f(\tau) = \sum_{n \gg -\infty} c(n) q^n.$$

Define subspaces:

$A_k^+ = \{f \in A_k(D, \chi_D); \quad f = \sum_{\chi_D(n)=0,1} c(n) q^n\}$.

M_k^+ = holomorphic modular forms in A_k^+.

Notation: For $f = \sum c(n) q^n \in \mathbb{C}[[q]]$ we put

$$\tilde{c}(n) = \begin{cases} c(n), & n \not\equiv 0 \pmod{D}, \\ 2c(n); & n \equiv 0 \pmod{D}. \end{cases}$$

Borcherds' lifting

Theorem (Borcherds).
Let $f = \sum_{n \gg -\infty} c(n) q^n \in A_0^+$, *such that* $\tilde{c}(n) \in \mathbb{Z}$ *for* $n < 0$. *Then there exists a Hilbert modular form* F *for* Γ_K *with*

i) weight$(F) = c(0)$,

ii) $\mathrm{div}(F) = \sum_{n<0} \tilde{c}(n) T(-n)$,

iii) Product expansion (put $e(z) = e^{2\pi i z}$*):*

$$F = e(\rho z_1 + \rho' z_2) \prod_{\substack{\nu \in \mathfrak{d}^{-1} \\ \nu > W}} \Big(1 - e(\nu z_1 + \nu' z_2)\Big)^{\tilde{c}(D\nu\nu')}.$$

If $\mathrm{div}(F)$ is compact in Y_K, then $\rho = 0$ and $\nu > W$ becomes $\nu \gg 0$.

Example:

Consider $K = \mathbb{Q}(\sqrt{5})$.

Fundamental unit: $\varepsilon_0 = \frac{1}{2}(1+\sqrt{5})$.

Gundlach constructed Hilbert cusp form $\Theta(z_1, z_2)$ of weight 5 by means of theta series.

$\mathrm{div}(\Theta) = T(1)$.

The Borcherds lifting of

$$f_1 = q^{-1} + 5 + 11\,q - 54\,q^4 + 55\,q^5 + 44\,q^6 + \ldots$$

in $A_0^+(5, \chi_5)$ has the same properties.

We find that $2^{-6}\Theta(z_1, z_2)$ is equal to

$$e\left(\frac{\varepsilon_0 z_1 - \varepsilon_0' z_2}{\sqrt{5}}\right) \prod_{\substack{\nu \in \mathfrak{d}^{-1} \\ \varepsilon_0\nu' - \varepsilon_0'\nu > 0}} \left(1 - e(\nu z_1 + \nu' z_2)\right)^{\tilde{c}(5\nu\nu')}.$$

Remarks

Every $f = \sum_{n \gg -\infty} c(n) q^n \in A_0^+$ is uniquely determined by its *principal part*

$$\sum_{n<0} c(n) q^n \in \mathbb{C}[q^{-1}].$$

The Galois group $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ acts on A_0^+.

Hence: If f has principal part in $\mathbb{Q}[q^{-1}]$, then all coefficients of f are rational (with bounded denominators).

Some power of any Borcherds product for Γ_K has Fourier coefficients in $\mathbb{Z}$.

It defines a section of the line bundle of modular forms over $\mathbb{Z}$, by the *q-expansion principle*.

Generalized Borcherds products

Idea: construct "log |Borcherds product|" directly regarding the properties:

- logarithmic singularity on $T(m)$,
- Γ_K-invariance.

More precisely: construct "Green's function" for *every* $T(m)$.

Ansatz:

$$\phi_m(z_1, z_2) = \sum_{\substack{a,b,\lambda \\ ab-\mathsf{N}(\lambda)=m/D}} \log \left| \frac{az_1z_2 + \lambda z_1 + \lambda' z_2 + b}{az_1\bar{z}_2 + \lambda z_1 + \lambda' \bar{z}_2 + b} \right|.$$

Problem: Series diverges!

Regularization

If we put $Q_0(z) = \frac{1}{2}\log\frac{z-1}{z+1}$, we have

$$\phi_m(z_1, z_2)$$
$$= \sum_{\substack{a,b,\lambda \\ ab-\mathsf{N}(\lambda)=m/D}} Q_0\left(1 + \frac{|az_1z_2 + \lambda z_1 + \lambda' z_2 + b|^2}{2y_1y_2m/D}\right).$$

Replace Q_0 by the 1-parameter family Q_{s-1} of *Legendre functions* of the second kind:

$$Q_{s-1}(z) = \int_0^\infty (z + \sqrt{z^2-1}\cosh u)^{-s} du.$$

We get a series $\Phi_m(z_1, z_2, s)$, which depends of an additional complex parameter s.

It converges for $\Re(s) > 1$. Can be meromorphically continued (Fourier expansion).

It is an eigenfunction of the Laplacian Δ, because of the differential equation satisfied by $Q_{s-1}(z)$.

Formally, we have

$$\phi_m(z_1, z_2) = \Phi_m(z_1, z_2, 1).$$

We define the *regularized automorphic Green's function* for $T(m)$ by:

$G_m(z_1, z_2) =$ constant term in the Laurent expansion of $\Phi_m(z_1, z_2, s)$ at $s = 1$ (plus some explicit normalizing constant L_m).

Remark: $\Phi_m(z_1, z_2, s)$ can also be understood in the framework of the regularized theta-lifting.

Properties

- Logarithmic singularity on $T(m)$.
- Known behavior at the boundary.
- $G_m(z_1, z_2) = \log|\Psi_m(z_1, z_2)| + \xi_m(z_1, z_2)$.
- $dd^c\xi_m(z_1, z_2)$ is the first Chern form of $T(m)$.
- $\Delta G_m(z_1, z_2) = B(m) =$ constant.

Theorem (Bruinier).
i) Let F be a Borcherds product of weight k with divisor $\sum_{n<0}\tilde{c}(n)T(-n)$. Then

$$\begin{aligned}\log\|F(z_1, z_2)\|_{Pet} &= \log\left|F(z_1, z_2)(16\pi^2 y_1 y_2)^{k/2}\right| \\ &= \sum_{n<0}\tilde{c}(n)G_{-n}(z_1, z_2).\end{aligned}$$

ii) Every modular form for Γ_K with divisor as above is a constant multiple of a Borcherds product.

Integrals of Borcherds products

Compute $\int_{Y_K} \log \|F\|_{Pet}\, d\mu$.

• Kudla: Use the regularized integral representation of $\log \|F\|_{Pet}$, Siegel-Weil formula, Stoke's type argument.

• Bruinier-Kühn: By the latter theorem it suffices to compute $\int_{Y_K} G_m\, d\mu$. Use a Kronecker limit formula approach to compute the integral.

Integrals of Green's functions

By construction:

$$G_m(z_1, z_2) = \lim_{s \to 1} \left(\Phi_m(z_1, z_2, s) - \frac{B(m)}{s-1} \right) + L_m.$$

Consequently

$$\int_{Y_K} G_m(z_1, z_2)\, d\mu = L_m \,\mathrm{vol}(Y_K) + \lim_{s \to 1} \left(\int_{Y_K} \Phi_m(z_1, z_2, s)\, d\mu - \frac{B(m)\,\mathrm{vol}(Y_K)}{s-1} \right).$$

Proposition. *Let $h : Y_K \to \mathbb{C}$ be any bounded eigenfunction of Δ with eigenvalue λ. Then*

$$\int_{Y_K} \Phi_m(z_1, z_2, s) h(z_1, z_2)\, \Omega^2 = \frac{1}{s(s-1) - \lambda} \cdot \int_{T(m)} h(z_1, z_2)\, \Omega,$$

where $\Omega = \frac{dx_1 dy_1}{4\pi y_1^2} + \frac{dx_2 dy_2}{4\pi y_2^2}$ (and $\Omega^2 = d\mu$).

• Hence $\int_{Y_K} G_m(z_1,z_2)h(z_1,z_2)\,\Omega^2$ can computed for bounded eigenfunctions h of Δ.

• Example $h=1$: One gets (Bruinier-Kühn)

$$\frac{1}{\deg(T(m))}\int\limits_{\Gamma_K\backslash\mathbb{H}^2} G_m(z_1,z_2)\,\Omega^2$$
$$=2\frac{L'(\chi_D,-1)}{L(\chi_D,-1)}+1+2\frac{\sigma_m'(2)}{\sigma_m(2)}+\log(Dm).$$

Example: $K=\mathbb{Q}(\sqrt{5})$. The integral of the Gundlach theta function equals

$$\int_{Y_K}\log\|2^{-6}\Theta(z_1,z_2)\|_{Pet}^2\,\Omega^2$$
$$=2\zeta(-1)\left(2\frac{L'(\chi_5,-1)}{L(\chi_5,-1)}+1+\log(5)\right).$$

Generalization to $\mathrm{O}(2,p)$.

Replace $\mathrm{O}(2,2)$ by $\mathrm{O}(2,p)$,

Γ_K by the discriminant kernel $\Gamma(L)$ of an even lattice L of signature $(2,p)$,

Y_K by the quotient $X_L = \Gamma(L)\backslash\mathbb{H}_p$,

$T(m)$ by the Heegner divisor $H(\beta,m)$ $(m<0)$,

$G_m(z_1,z_2)$ by $G_{\beta,m}(Z)$.

• Kudla-Yang: $\int_{X_L} \log\|F\|_{Pet}\,\Omega^p$ by Siegel-Weil formula, Stoke's type argument, ...

• Bruinier-Kühn: $\int_{Y_K} G_{\beta,m}\,\Omega^p$ by Kronecker limit formula approach, results of Shintani, Siegel, ...

Theorem (Bruinier-Kühn). *Let $r = 2 + p$ be the rank of L and $\kappa = r/2$. If r is even, then the integral*

$$\frac{2}{\deg H(\beta, m)} \int_{X_L} G_{\beta,m}(Z)\Omega^p$$

is equal to

$$2\frac{L'(\chi_{D_0}, 1-\kappa)}{L(\chi_{D_0}, 1-\kappa)} + 2\frac{\sigma'_{\beta,-m}(\kappa)}{\sigma_{\beta,-m}(\kappa)} + \log|mD_0^2| + \sum_{j=1}^{\kappa-1} \frac{1}{j},$$

where D_0 denotes the discriminant of $\mathbb{Q}(\sqrt{d})$ with $d = (-1)^{r/2}\det(L)$.

If r is odd, then it is equal to

$$4\frac{\zeta'(2-2\kappa)}{\zeta(2-2\kappa)} - 2\frac{L'(\chi_{D_0}, 3/2-\kappa)}{L(\chi_{D_0}, 3/2-\kappa)} + 2\frac{\sigma'_{\beta,-m}(\kappa)}{\sigma_{\beta,-m}(\kappa)}$$
$$+ \log|4m/D_0^2| + \sum_{j=1}^{\kappa-1/2} \frac{2}{2j-1},$$

where D_0 is the discriminant of $\mathbb{Q}(\sqrt{md})$ with $d = 2(-1)^{(r-1)/2}\det(L)$.