# Non-vanishing of scalar products of Fourier-Jacobi coefficients of Siegel cusp forms

by

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Juni 1997

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## Introduction

Let $F$ be a holomorphic cusp form of integral weight $k$ w.r.t. the Siegel modular group $\Gamma_n$ of degree $n \geq 2$ and write $f_m$ for its Fourier-Jacobi coefficients, where $m \in \mathbf{N}$. Denote by $\langle\, , \,\rangle$ the usual Petersson scalar product on the space of Jacobi cusp forms of weight $k$, index $m$ and degree $n-1$.

In sect. 1 of the present paper we shall show that if $F \neq 0$ and $N \in \mathbf{N}$, then for any given $a \in \mathbf{Z}$ with $(a, N) = 1$ there exist infinitely many $m \in \mathbf{N}$ such that $f_m \neq 0$ and $m \equiv a \pmod{N}$ (Thm. 1). The main ingredient in the proof (apart from some auxiliary Lemmas) is that the Rankin-Dirichlet series whose general coefficient is $\langle f_m, f_m \rangle$, and its character twists have meromorphic continuations to $\mathbf{C}$ [KS,Kr,K,KKS], together with a classical result of Landau on Dirichlet series with non-negative coefficients.

Suppose that $N \geq 5$ is odd and squarefree and let $\epsilon \in \{\pm 1\}$. Let $G$ be another holomorphic cusp form of weight $k$ w.r.t. $\Gamma_n$ with Fourier-Jacobi coefficients $g_m \quad (m \in \mathbf{N})$. In sect. 2 we shall prove that if $\langle f_{m_0}, g_{m_0} \rangle \neq 0$ for at least one $m_0$ with $(m_0, N) = 1$, then there exist infinitely many $m \in \mathbf{N}$ such that $\langle f_m, g_m \rangle \neq 0$ and $(\frac{m}{N}) = \epsilon$ (Thm. 2). The proof again uses the analytic properties given in [KS,Kr,K,KKS] of the Rankin-Dirichlet series $D_{F,G}(s)$ whose general coefficient essentially is $\langle f_m, g_m \rangle$, and its character twists. However, one cannot proceed as in the proof of Thm. 1 since $\langle f_m, g_m \rangle$ in general will not be real and non-negative. Instead —assuming that $\langle f_m, g_m \rangle \neq 0$ and $(\frac{m}{N}) = \epsilon$ for only finitely many $m$— we shall use functional equations and compare root numbers in certain of the above series, and from this obtain an identity between fourth powers of certain Gauss sums. Taking into account well-known properties of the latter (in particular relating them to Jacobi sums) and using some elementary Galois theory, one arrives at a contradiction.

In sect. 3 we shall specialize to the case $n = 2$, $k$ even. If $f_1 \neq 0$, observing the relation between $D_{F,G}(s)$ and the spinor zeta function $Z_F(s)$ for $F$ a Hecke eigenform and $G$ in the Maass space [KS], from Thm. 2 one immediately deduces the existence of infinitely many $m$ with $(\frac{m}{N}) = \epsilon$ and such that the eigenvalue $\lambda_m$ of $F$ under the usual Hecke operator $T_m$ is non-zero (Thm. 3). In particular, there always is a prime $p$ with $(\frac{p}{N}) = -1$ and $\lambda_p \neq 0$ (Cor.). If one exploits the fact that $Z_F(s)$ has an Euler product, the latter result can be sharpened: if $f_1 \neq 0$, one can always find infinitely many primes $p$ with $(\frac{p}{N}) = -1$ and $\lambda_p \neq 0$ (Thm. 4).

We note that the above two non-vanishing results for $\lambda_p$ also easily follow from the existence of the $\ell$-adic Galois representation attached to the holomorphic form $F$, cf. [L]. The existence of the latter is a deeper fact, and the proof uses the trace formula in the context of Shimura varieties.

Our method here is based only on analytic properties of the corresponding objects. At least in principle it could also be applied to non-holomorphic Siegel modular forms.

Finally note that the results and proofs of sects. 1 and 2 remain valid also in the case $n = 1$, if one interpretes $f_m$ resp. $g_m$ as usual Fourier coefficients and so the Dirichlet series involved are character twists of usual Rankin-Dirichlet series. The analytic properties of the latter were studied in [MP]. However, the root numbers given there, as it seems, cannot so easily be related to fourth powers of Gauss sums, a fact needed for the proof of Thm. 2. (The easiest way to get this relation, indeed, seems to take the roundabout via the theory for $n = 2$ and the Maass space, cf. [K].)

Nevertheless, in any case we think that Thms. 1 and 2 for $n = 1$ come down to well-known results about elliptic modular forms, although in general we do not know any explicit references (in the case of a Hecke eigenform one can use the existence of the $\ell$-adic Galois representation; cf. [S]).

## Notation

For $n \in \mathbf{N}$ we denote by $\mathcal{H}_n$ the Siegel upper space of degree $n$. Let $k \in \mathbf{N}$. If $F$ is a complex valued function on $\mathcal{H}_n$ and $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GSp_n^+(\mathbf{R})$, we set $(F|_k\gamma)(Z) = \det(CZ + D)^{-k} F((AZ + B)(CZ + D)^{-1})$. We usually write $| = |_k$ if the number $k$ is clear from the context.

For $A$ and $B$ matrices of appropriate sizes we set $A[B] = B^t AB$.

For $n$ and $k$ natural numbers we let $S_k(\Gamma_n)$ be the space of cusp forms of weight $k$ w.r.t. the Siegel modular group $\Gamma_n := Sp_n(\mathbf{Z})$ of degree $n$. In the following we will always suppose that $n \geq 2$.

## 1. Non-vanishing of Fourier-Jacobi coefficients

If $Z \in \mathcal{H}_n$ and we write $Z = \begin{pmatrix} \tau & z^t \\ z & \tau' \end{pmatrix}$ with $\tau \in \mathcal{H}_{n-1}$, $\tau' \in \mathcal{H}_1$ and $z \in \mathbf{C}^{n-1}$, then any $F \in S_k(\Gamma_n)$ has a Fourier-Jacobi expansion

$$F(Z) = \sum_{m \geq 1} f_m(\tau, z) e^{2\pi i m \tau'}.$$

The functions $f_m$ are in the space $J_{k,m}^{cusp}$ of cusp forms of weight $k$ and index $m$ w.r.t. the Jacobi group $\Gamma_{n-1}^J := \Gamma_{n-1} \propto (\mathbf{Z}^{n-1} \times \mathbf{Z}^{n-1})$. For $f, g \in J_{k,m}^{cusp}$ we let

$$\langle f, g \rangle = \int_{\Gamma_{n-1}^J \backslash \mathcal{H}_{n-1} \times \mathbf{C}^{n-1}} f(\tau, z) \overline{g(\tau, z)} (\det v)^k e^{-4\pi m v^{-1}[y^t]} dV_{n-1}^J$$

be the usual scalar product of $f$ and $g$; here $\tau = u + iv, z = x + iy$ and $dV_{n-1}^J = (\det v)^{-n-1} du dv dx dy$ is the invariant measure on $\mathcal{H}_{n-1} \times \mathbf{C}^{n-1}$.

**Theorem 1.** *Let $F$ be a non-zero function in $S_k(\Gamma_n)$ with Fourier-Jacobi coefficients $f_m \quad (m \in \mathbf{N})$. Let $N \in \mathbf{N}$, $N > 1$ and $a \in \mathbf{Z}$ with $(a, N) = 1$. Then there exist infinitely many $m \in \mathbf{N}$ with $f_m \neq 0$ and $m \equiv a \pmod N$.*

Observe that the corresponding assertion of Thm. 1 for $N = 1$ (i.e. any non-zero $F$ has infinitely many non-zero coefficients $f_m$) is trivially true; this is immediately seen by acting on the Fourier coefficients of $F$ with appropriate unimodular matrices.

The rest of this section is devoted to the *proof* of Theorem 1.

For $M \in \mathbf{N}$ we denote by $C_{n,1}(M, M^2)$ the set of integral symplectic matrices $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ ($A, B, C, D$ of size $n$) with the following property: if $c$ and $d$ denote the last rows of $C$ and $D$, respectively, then the first $n-1$ entries of both $c$ and $d$ are divisible by $M$, and the last entry of $c$ is divisible by $M^2$; cf. [K,KKS]. It is easy to check that $C_{n,1}(M, M^2)$ is a subgroup of $\Gamma_n$.

Observe that $\begin{pmatrix} E & S \\ 0 & E \end{pmatrix}$ is in $C_{n,1}(M, M^2)$ for $S \in \mathbf{Z}^{n,n}, S = S^t$. Therefore any modular form $H$ of weight $k$ w.r.t. $C_{n,1}(M, M^2)$ has a Fourier-Jacobi expansion

$$H(Z) = \sum_{m \geq 0} h_m(\tau, z) e^{2\pi i m \tau'}, \tag{1}$$

where $Z = \begin{pmatrix} \tau & z^t \\ z & \tau' \end{pmatrix}$ as before. If $H$ is a cusp form, then $m$ in the expansion (1) runs over positive integers only.

**Lemma 1.** *Let $H$ be a modular (resp. cusp) form of weight $k$ w.r.t. $C_{n,1}(M, M^2)$ with Fourier-Jacobi expansion (1). Let $p$ be a prime not dividing $M$. Then*

$$(B_p H)(Z) := \sum_{m \geq 0} h_{pm}(\tau, z) e^{2\pi i p m \tau'}$$

*is a modular (resp. cusp) form of weight $k$ w.r.t. $C_{n,1}(pM,(pM)^2)$.*

**Proof.** We only indicate the proof, since it is completely analogous to the proof of the Lemma in sect. 2 of [K] or that of Propos. 1 in [KKS]. For $\nu \in \mathbf{Z}$ we let $\gamma_\nu = \begin{pmatrix} 1_n & S_\nu \\ 0 & 1_n \end{pmatrix}$ with $S_\nu$ the $(n,n)$-matrix having $(n,n)$-component equal to $\nu/n$ and all other components equal to zero. Then

$$B_pH = \frac{1}{p}\sum_{\nu(p)} H|\gamma_\nu.$$

If $\gamma$ is in $C_{n,1}(pM,(pM)^2)$ and $d_n$ is the last entry of $d$, one easily checks that $\gamma_\nu\gamma\gamma^{-1}_{\nu d_n^2}$ is in $C_{n,1}(M,M^2)$ (in fact, is in $C_{n,1}(pM,(pM)^2)$). Hence since $p$ does not divide $d_n$, it follows that $B_pH$ is invariant under $C_{n,1}(pM,(pM)^2)$. Since $H$ is a modular (resp. cusp) form, each

$H|\gamma_\nu$ is a modular (resp. cusp) form, and so is $B_pH$.

**Lemma 2.** *Let $H$ be a modular form of weight $k$ w.r.t. $C_{n,1}(M,M^2)$ and let $p$ be a prime not dividing $M$. Suppose that in the Fourier-Jacobi expansion (1) of $H$ one has $h_m = 0$ for $p$ not dividing $m$. Then $H = 0$.*

**Proof.** We consider the embedding

$$SL_2(\mathbf{R}) \hookrightarrow Sp_n(\mathbf{R}), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{\downarrow} := \begin{pmatrix} 1_{n-1} & 0 & 0_{n-1} & 0 \\ 0 & a & 0 & b \\ & & & \\ 0_{n-1} & 0 & 1_{n-1} & 0 \\ 0 & c & 0 & d \end{pmatrix}.$$

Then from the hypothesis we see that

$$H|\gamma^{\downarrow} = H$$

for all $\gamma$ in the subgroup of $SL_2(\mathbf{R})$ generated by $\Gamma_0(M^2)$ and $\begin{pmatrix} 1 & \frac{1}{p} \\ 0 & 1 \end{pmatrix}$. We easily see that this subgroup also contains the element $\delta_p := \begin{pmatrix} p & 0 \\ 0 & \frac{1}{p} \end{pmatrix}$ (in fact, $\begin{pmatrix} 1 & \frac{1}{p} \\ 0 & 1 \end{pmatrix}$ and $\delta_p$ generate the same double coset w.r.t. $\Gamma_0(M^2)$ as $p$ does not divide $M$); hence since $\delta_p^{-1}\begin{pmatrix} 1 & \frac{1}{p^\nu} \\ 0 & 1 \end{pmatrix}\delta_p = \begin{pmatrix} 1 & \frac{1}{p^{\nu+2}} \\ 0 & 1 \end{pmatrix}$ $(\nu \in \mathbf{N})$, this subgroup contains all translations of the form $\begin{pmatrix} 1 & \frac{1}{p^\nu} \\ 0 & 1 \end{pmatrix}$ $(\nu \in \mathbf{N})$.

Thus

$$H\begin{pmatrix} \tau & z^t \\ z & \tau' + \frac{1}{p^\nu} \end{pmatrix} = H\begin{pmatrix} \tau & z^t \\ z & \tau' \end{pmatrix} \quad (\nu \in \mathbf{N}),$$

and so $H$ does not depend on $\tau'$ at all.

If $H$ is cuspidal, this already finishes the proof that $H = 0$. In the general case, writing

$$H(Z) = \sum_{T \geq 0} a(T) e^{2\pi i tr(TZ)}$$

one sees that $a(T) = 0$ unless $T$ is of type $T = \begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix}$ with $T_1$ of size $n-1$. If $T = \begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix}$ is not the zero matrix, we can find an integral column vector $x$ such that $T_1[x] \neq 0$.

On the other hand, $\begin{pmatrix} U^t & 0 \\ 0 & U^{-1} \end{pmatrix}$ is in $C_{n,1}(M, M^2)$ if $U$ has the form $U = \begin{pmatrix} 1_{n-1} & 0 \\ Mx^t & 1 \end{pmatrix}$. For such $U$ we have

$$a(T) = a(T[U^t]),$$

therefore $a(T)$ must be zero, because the element in the right lower corner of $T[U^t]$ equals $M^2 T_1[x]$ which is different from zero by our choice of $x$. Therefore $H$ is zero.

**Lemma 3.** *Let $F$ be in $S_k(\Gamma_n)$, $F \neq 0$. Then there is at least one $m_0$ with $(m_0, N) = 1$ and such that the Fourier-Jacobi coefficient $f_{m_0}$ is non-zero.*

**Proof.** Suppose that

$$F(Z) = \sum_{m \geq 1} f_m(\tau, z) e^{2\pi i m \tau'}$$

and $f_m = 0$ whenever $(m, N) = 1$.

If $N$ is a prime, Lemma 2 shows $F = 0$. In the general case, suppose that $p_1, ..., p_r$ ($r \geq 2$) are the different prime divisors of $N$ and define

$$H := (\prod_{\nu=1}^{r-1} (1 - B_{p_\nu}))F.$$

Note that $1 - B_{p_\nu}$ is the projection operator onto Fourier-Jacobi series with coefficients $f_m$ such that $p_\nu$ does not divide $m$. Hence by the hypothesis on the coefficients of $F$, we see that the coefficients $h_m$ of $H$ are zero unless $p_r$ divides $m$. Since by Lemma 1 the function $H$ is modular w.r.t. the group $C_{n,1}(p_1...p_{r-1}, (p_1...p_{r-1})^2)$, we conclude from Lemma 2 that $H = 0$, i.e. $f_m = 0$ for $(m, p_1 p_2 ... p_{r-1}) = 1$. By induction therefore $F = 0$. This concludes the proof of Lemma 3.

We can now give the proof of Theorem 1. It suffices to show that the series

$$\sum_{m \geq 1, m \equiv a \pmod{N}} \frac{\langle f_m, f_m \rangle}{m^s} \quad (\mathrm{Re}(s) \gg 0) \tag{2}$$

is convergent for $\mathrm{Re}(s) > k$ and has a pole at $s = k$.

By [KS,Kr] the series

$$\zeta(2s - 2k + 2n) \sum_{m \geq 1} \frac{\langle f_m, f_m \rangle}{m^s} \qquad (\mathrm{Re}(s) \gg 0)$$

has a meromorphic continuation to $\mathbf{C}$ which is holomorphic except for a simple pole at $s = k$, hence by a Theorem of Landau converges for $\mathrm{Re}(s) > k$. Hence (2) converges for $\mathrm{Re}(s) > k$.

By the orthogonality relations for characters one has

$$(3) \qquad \sum_{m \geq 1, m \equiv a \pmod N} \frac{\langle f_m, f_m \rangle}{m^s} = \frac{1}{\phi(N)} \sum_{\chi \pmod N} \overline{\chi}(a) \Big( \sum_{m \geq 1} \frac{\chi(m) \langle f_m, f_m \rangle}{m^s} \Big) \qquad (\mathrm{Re}(s) > k),$$

where $\phi$ is Euler's phi-function.

If $\chi$ is any Dirichlet character modulo $N$, we put

$$D_F(s, \chi) := L(2s - 2k + 2n, \chi^2) \sum_{m \geq 1} \frac{\chi(m) \langle f_m, f_m \rangle}{m^s} \qquad (\mathrm{Re}(s) > k).$$

By [KKS, sect. 5], if $\chi$ is different from the principal character $\chi_0$ modulo $N$, then $D_F(s, \chi)$ extends to an entire function, while $D_F(s, \chi_0)$ has a holomorphic continuation to $\mathbf{C}$ except for a *possible* simple pole at $s = k$. Since the series

$$D_F(s, \chi_0) := L(2s - 2k + 2n, \chi_0) \sum_{m \geq 1, (m,N)=1} \frac{\langle f_m, f_m \rangle}{m^s} \qquad (\mathrm{Re}(s) > k)$$

has non-negative coefficients, by the Theorem of Landau it either must have a pole at its abscissa of convergence or must converge for all $s \in \mathbf{C}$. In the second case, since $L(2s - 2k + 2n, \chi_0)$ converges for $\mathrm{Re}(s) > k - n + \frac{1}{2}$ and has a pole at $s = k - n + \frac{1}{2}$, it would follow that $\sum_{m \geq 1, (m,N)=1} \frac{\langle f_m, f_m \rangle}{m^s}$ converge for $\mathrm{Re}(s) > k - n + \frac{1}{2}$ and

$$\lim_{\sigma \to k - n + \frac{1}{2}+} \sum_{m \geq 1, (m,N)=1} \frac{\langle f_m, f_m \rangle}{m^\sigma} = 0,$$

hence $\langle f_m, f_m \rangle = 0$ for all $m$ with $(m, N) = 1$. This contradicts Lemma 3.

Therefore $D_F(s, \chi_0)$ must have a pole at $s = k$ and so the same is true for

$$\sum_{m \geq 1, (m,N)=1} \frac{\langle f_m, f_m \rangle}{m^s}.$$

From (3) it now follows that (2) has a pole at $s = k$. This concludes the proof of Theorem 1.

**Remark.** In [KKS, Thm. 1] it is shown that the residue of $D_F(s,\chi_0)$ at $s=k$ up to some trivial non-zero factor is the Petersson scalar product $\langle Tr(F_{\chi_0}), F\rangle$, where $F_{\chi_0}$ is the "$\chi_0$-twist of $F$" and $Tr$ is a trace operator. Thus the above considerations imply $\langle Tr(F_{\chi_0}), F\rangle \neq 0$ if $F \neq 0$. We are not aware of any simple proof of this in a more direct way.

## 2. Non-vanishing of scalar products

**Theorem 2.** *Let $F$ and $G$ be in $S_k(\Gamma_n)$ and denote by $f_m$ resp. $g_m$ their Fourier-Jacobi coefficients. Let $N \geq 5$ be an odd squarefree natural number and $\epsilon \in \{\pm 1\}$. Suppose that there exists at least one $m_0$ such that $(m_0, N) = 1$ and $\langle f_{m_0}, g_{m_0}\rangle \neq 0$. Then there exist infinitely many $m \in \mathbf{N}$ with $\langle f_m, g_m\rangle \neq 0$ and $(\frac{m}{N}) = \epsilon$.*

**Proof.** We write $\psi = (\frac{}{N})$ and put $r = 3N$ or $r = N$ according as 3 divides $N$ or not. According to our assumption on $N$, we can find a primitive Dirichlet character $\chi$ modulo $r$ such that $\chi^2$ and $\chi\psi$ are primitive modulo $r$. In fact, it is sufficient to give such a character for each prime power component of $r$, so we may suppose $r = p$ a prime $\geq 5$ or $r = 9$. In the first case, any $\chi$ different from 1 and $(\frac{}{p})$ will satisfy our requirements, and in the second case (since $(\mathbf{Z}/9\mathbf{Z})^* \approx \mathbf{Z}/6\mathbf{Z}$) any $\chi$ with with $\chi \neq 1, (\frac{}{3})$ will do. For $s \in \mathbf{C}$ with $\mathrm{Re}(s) \gg 0$ we set

$$D_{F,G}(s,\chi) = L(2s-2k+2n,\chi^2)\sum_{m\geq 1}\chi(m)\langle f_m, g_m\rangle m^{-s}$$

and

$$D^*_{F,G}(s,\chi) = (\frac{2\pi}{r})^{-2s}\Gamma(s)\Gamma(s-k+n)D_{F,G}(s,\chi).$$

As was shown in [K,KKS], under the assumption that $r > 1$ and $\chi, \chi^2$ are both primitive modulo $r$, the function $D^*_{F,G}(s,\chi)$ has a holomorphic continuation to $\mathbf{C}$ and satisfies the functional equation

$$D^*_{F,G}(2k-n-s,\chi) = (\frac{\mathcal{G}(\chi)}{\sqrt{r}})^4 D^*_{F,G}(s,\overline{\chi}), \tag{4}$$

where $\mathcal{G}(\chi) = \sum_{\nu(r)}\chi(\nu)e^{2\pi i\nu/r}$ is the Gauss sum attached to $\chi$. (We would expect that (4) is also true if $\chi^2 = 1$, but as far as we know this has not yet been proved.)

Clearly for $\mathrm{Re}(s) \gg 0$ we have

$$D_{F,G}(s,\chi) + \epsilon D_{F,G}(s,\chi\psi) = 2L(2s-2k+2n,\chi^2)\sum_{m\geq 1,\psi(m)=\epsilon}\chi(m)\langle f_m, g_m\rangle m^{-s}. \tag{5}$$

Now assume that there are only finitely many $m$ with $\psi(m) = \epsilon$ and $\langle f_m, g_m\rangle \neq 0$. Then the series over $m$ on the right-hand side of (5) is finite, and by analytic continuation for all $s \in \mathbf{C}$ we have

$$D^*_{F,G}(s,\chi) + \epsilon D^*_{F,G}(s,\chi\psi) = 2(\frac{2\pi}{r})^{-2s}\Gamma(s)\Gamma(s-k+n)L(2s-2k+2n,\chi^2)\sum_{m=1}^{M}c_m m^{-s}, \tag{6}$$

where $M$ is an appropriate positive integer and $c_m$ is equal to $\chi(m)\langle f_m, g_m\rangle$ if $\psi(m) = \epsilon$ and 0 otherwise.

The $\Gamma$-factor on the right of (6) has double poles at $s = l, l-1, l-2, \ldots$ where $l := \min\{0, k-n\}$; on the other hand, since $\chi^2$ is even, $L(2s-2k+2n, \chi^2)$ has first order zeroes at these points. Since the left-hand side of (5) is holomorphic, one infers that

$$\sum_{m=1}^{M} c_m m^{-l} m^{-s} = 0 \qquad (s = 0, 1, 2, \ldots),$$

hence $c_1 = c_2 = \ldots = c_M = 0$ (Vandermonde determinant), and therefore

$$D^*_{F,G}(s,\chi) = -\epsilon D^*_{F,G}(s,\chi\psi). \tag{7}$$

Replacing $s$ by $2k-n-s$ and using the functional equation (4), we obtain from (7)

$$\mathcal{G}(\chi)^4 D^*_{F,G}(s,\overline{\chi}) = -\epsilon \mathcal{G}(\chi\psi)^4 D^*_{F,G}(s,\overline{\chi}\psi).$$

If one repeats the above argument with $\overline{\chi}$ instead of $\chi$, we see that (7) also holds true with $\chi$ replaced by $\overline{\chi}$. As by hypothesis $\langle f_{m_0}, g_{m_0}\rangle \neq 0$ for at least one $m_0$ with $(m_0, N) = 1$ and so $D_{F,G}(s,\chi)$ is not identically zero, we thus deduce that

$$\mathcal{G}(\chi)^4 = \mathcal{G}(\chi\psi)^4.$$

To complete the proof it is therefore sufficient to show

**Lemma 4.** *Let $M \in \mathbf{N}$ and let $p$ be an odd prime which is greater than every prime divisor of $M$. Let $t$ be the product of all the primes dividing $M$ and put $N = tp$. Let $\psi = (\frac{\cdot}{N})$. Let $\chi$ be a primitive Dirichlet character modulo $Mp$ such that also $\chi\psi$ is primitive modulo $Mp$. Then $\mathcal{G}(\chi)^4 \neq \mathcal{G}(\chi\psi)^4$.*

**Proof.** Suppose on the contrary that

$$\mathcal{G}(\chi)^4 = \mathcal{G}(\chi\psi)^4. \tag{8}$$

Since $p$ does not divide $M$, we can write $\chi = \chi_1\chi_2$ with $\chi_1$ resp. $\chi_2$ primitive Dirichlet characters modulo $M$ resp. modulo $p$. Also $\psi = \psi_1\psi_2$ with $\psi_1 = (\frac{\cdot}{t})$ and $\psi_2 = (\frac{\cdot}{p})$.

Since $(M,p) = 1$ we have

$$\mathcal{G}(\chi) = \chi_1(p)\chi_2(M)\mathcal{G}(\chi_1)\mathcal{G}(\chi_2)$$

and

$$\mathcal{G}(\chi\psi) = \mathcal{G}(\chi_1\psi_1\chi_2\psi_2) = \chi_1(p)(\frac{p}{t})\chi_2(M)(\frac{M}{p})\mathcal{G}(\chi_1\psi_1)\mathcal{G}(\chi_2\psi_2),$$

so (8) implies

$$\Big(\frac{\mathcal{G}(\chi_1)}{\mathcal{G}(\chi_1\psi_1)} \cdot \frac{\mathcal{G}(\chi_2)\mathcal{G}(\psi_2)}{\mathcal{G}(\chi_2\psi_2)}\Big)^4 = \mathcal{G}(\psi_2)^4.$$

Therefore

$$\eta \frac{\mathcal{G}(\chi_1)}{\mathcal{G}(\chi_1\psi_1)} \cdot \frac{\mathcal{G}(\chi_2)\mathcal{G}(\psi_2)}{\mathcal{G}(\chi_2\psi_2)} = \mathcal{G}(\psi_2), \tag{9}$$

where $\eta$ is a fourth root of unity.

By hypothesis $\chi_2, \psi_2$ and $\chi_2\psi_2$ are not equal to the unit character modulo $p$, hence

$$\frac{\mathcal{G}(\chi_2)\mathcal{G}(\psi_2)}{\mathcal{G}(\chi_2\psi_2)} = J(\chi_2, \psi_2),$$

where

$$J(\chi_2, \psi_2) = \sum_{x,y(p), x+y\equiv 1(p)} \chi_2(x)\psi_2(y)$$

is the Jacobi sum attached to $\chi_2$ and $\psi_2$ [H, sect.20, 4.]. Clearly $J(\chi_2, \psi_2) \in \mathbf{Q}(\zeta_{p-1})$ (as usual, $\zeta_d$ for $d \in \mathbf{N}$ denotes a primitive $d$-th root of unity). Also $\mathcal{G}(\chi_1), \mathcal{G}(\chi_1\psi_1) \in \mathbf{Q}(\zeta_{\phi(M)}, \zeta_M)$. Hence the left-hand side of (9) is contained in $\mathbf{Q}(\zeta_{4(p-1)M\phi(M)})$.

On the other hand, since $\psi_2$ is quadratic, $\mathcal{G}(\psi_2) \in \mathbf{Q}(\zeta_p)$; in fact, one has $\mathcal{G}(\psi_2) = \epsilon_p\sqrt{p}$ where $\epsilon_p = 1$ or $i$ according as $p \equiv 1 \pmod 4$ or not. Since $p$ is odd and greater than every prime divisor of $M$, we have $\big(p, 4(p-1)M\phi(M)\big) = 1$, hence by Galois theory

$$\mathbf{Q}(\zeta_{4(p-1)M\phi(M)}) \cap \mathbf{Q}(\zeta_p) = \mathbf{Q}.$$

Thus (9) implies $\epsilon_p\sqrt{p} \in \mathbf{Q}$, a contradiction.

**Remarks.** i) By the same reasoning as above, one can show that $\mathcal{G}(\chi)^a \neq \mathcal{G}(\chi\psi)^a$ for any integer $a$ prime to $p$.

ii) Let $p$ be an odd prime and $\chi$ a primitive character modulo $p^a$ with $a \geq 2$ such that $\chi\psi$ is primitive modulo $p^a$. Then $\mathcal{G}(\chi)^2 = \mathcal{G}(\chi(\frac{\cdot}{p}))^2$. The easy proof is left to the reader.

## 3. Non-vanishing of Hecke eigenvalues

We now specialize to the case $n = 2$ and $k$ even. For $m \in \mathbf{N}$ we denote by $T_m$ the usual Hecke operator acting on $S_k(\Gamma_2)$ by

$$T_m F = m^{2k-3} \sum_{\gamma \in \Gamma_2 \backslash \mathcal{O}_{2,m}} F|\gamma,$$

where

$$\mathcal{O}_{2,m} = \{\gamma \in \mathbf{Z}^{4,4} \mid \gamma[J] = J\} \quad (J = \begin{pmatrix} 0 & 1_2 \\ -1_2 & 0 \end{pmatrix})$$

is the set of symplectic similitudes with scale $m$. Recall that the (local) Hecke algebra of the pair $(\Gamma_2, GSp_2^+(\mathbf{Z}[p^{-1}]))$ is generated by $p1_4, T_p$ and $T_{p^2}$.

Let $F \in S_k(\Gamma_2)$ be a Hecke eigenform with $T_m F = \lambda_m F$ for all $m$, and as before let $f_m$ $(m \in \mathbf{N})$ be the Fourier-Jacobi coefficients of $F$. Then by the main result of [KS], for any $G \in S_k(\Gamma_2)$ contained in the Maass space with Fourier-Jacobi coefficients $g_m$ $(m \in \mathbf{N})$ one has

$$\langle f_1, g_1 \rangle \lambda_m = \langle f_m, g_m \rangle.$$

Taking for $G$ the Maass lift of $f_1$ (and so $f_1 = g_1$), we derive from Thm. 2

**Theorem 3.** *Let $k$ be even and $F \in S_k(\Gamma_2)$ be a Hecke eigenform with $T_m F = \lambda_m F$ for all $m \in \mathbf{N}$. Suppose that $f_1 \neq 0$. Let $N \geq 5$ be an odd squarefree integer and $\epsilon \in \{\pm 1\}$. Then there are infinitely many eigenvalues $\lambda_m$ with $(\frac{m}{N}) = \epsilon$ and $\lambda_m \neq 0$.*

**Corollary.** *Under the same assumptions as in Thm. 2, there is at least one prime $p$ with $(\frac{p}{N}) = -1$ and $\lambda_p \neq 0$.*

**Proof.** Applying Thm. 3 with $\epsilon = -1$ and taking into account that $m \mapsto \lambda_m$ is a multiplicative function, we find that there is a prime $p$ and $\ell \in \mathbf{N}$, $\ell$ odd such that $(\frac{p}{N}) = -1$ and $\lambda_{p^\ell} \neq 0$. Since

$$\frac{1 - p^{2k-4}X^2}{1 - \lambda_p X + (\lambda_p^2 - \lambda_{p^2} - p^{2k-4})X^2 - \lambda_p p^{2k-3} X^3 + p^{4k-6} X^4} = \sum_{\ell \geq 0} \lambda_{p^\ell} X^\ell \tag{10}$$

[A], it follows that $\lambda_p \neq 0$.

We will now sharpen the Corollary and prove

**Theorem 4.** *Let $F \in S_k(\Gamma_2)$ be a Hecke eigenform with $T_m F = \lambda_m F$ for all $m \in \mathbf{N}$. Suppose that the first Fourier-Jacobi coefficient $f_1$ of $F$ is non-zero. Let $N \geq 5$ be an odd squarefree integer and $\epsilon \in \{\pm 1\}$. Then there exist infinitely many primes $p$ with $(\frac{p}{N}) = -1$ and $\lambda_p \neq 0$.*

**Proof.** We use the same notations as in the proof of Thm. 2; in particular $r = 3N$ or $r = N$ according as $3|N$ or not, and $\chi$ is a primitive Dirichlet character modulo $r$ such that $\chi^2$ and $\chi\psi$ is primitive modulo $r$. We denote by

$$Z_F(s) = \prod_p Z_{F,p}(p^{-s})^{-1} \qquad (\mathrm{Re}(s) \gg 0)$$

the spinor zeta function of $F$, where $Z_{F,p}(X)$ is given by the left-hand side of (10). According to [A] one has

$$Z_F(s) = \zeta(2s - 2k + 4) \sum_{m \geq 1} \lambda_m m^{-s} \qquad (\mathrm{Re}(s) \gg 0).$$

We also let

$$Z_F(s, \chi) = \prod_p Z_{F,p}(\chi(p) p^{-s})^{-1} \qquad (\mathrm{Re}(s) \gg 0)$$

be the twist of $Z_F(s)$ by $\chi$.

Let

$$\mathcal{P} := \{p \text{ prime} \mid (\frac{p}{N}) = -1,\ \lambda_p \neq 0\}.$$

Then

$$Z_F(s,\chi) \cdot \prod_{p\in\mathcal{P}} Z_{F,p}(\chi(p)p^{-s}) = Z_F(s,\chi\psi) \cdot \prod_{p\in\mathcal{P}} Z_{F,p}((\chi\psi)(p)p^{-s}).$$

Since $f_1 \neq 0$, the functions $Z_F(s,\chi)$ resp. $Z_F(s,\chi\psi)$ are proportional to $D_{F,G}(s,\chi)$ resp. $D_{F,G}(s,\chi\psi)$ where $G$ is the Maass lift of $f_1$. Hence by [K,KKS], $Z_F(s,\chi)$ and $Z_F(s,\chi\psi)$ completed with the appropriate $\Gamma$-factors have holomorphic continuations to $\mathbf{C}$, and functional equations relating $s$ to $2k-2-s$ (and $\chi$ resp. $\chi\psi$ to their conjugates) with root numbers $(\frac{\mathcal{G}(\chi)}{\sqrt{r}})^4$ resp. $(\frac{\mathcal{G}(\chi\psi)}{\sqrt{r}})^4$ hold; cf. the proof of Thm. 2.

Now assume that $\mathcal{P}$ is finite. Using the functional equations and observing that $\psi(p) = -1$ for $p \in \mathcal{P}$, one obtains in a similar way as in the proof of Thm. 2

$$(\frac{\mathcal{G}(\chi)}{\mathcal{G}(\chi\psi)})^4 = \prod_{p\in\mathcal{P}} \frac{Z_{F,p}(\overline{\chi}(p)p^{-s}) \cdot Z_{F,p}(-\chi(p)p^{s+2-2k})}{Z_{F,p}(-\overline{\chi}(p)p^{-s}) \cdot Z_{F,p}(\chi(p)p^{s+2-2k})} \quad (s \in \mathbf{C}),$$

i.e.

$$(\frac{\mathcal{G}(\chi)}{\mathcal{G}(\chi\psi)})^4 = \prod_{p\in\mathcal{P}} \frac{A(p^s)}{B(p^s)} \quad (s \in \mathbf{C}) \tag{11}$$

where

$$A_p(X) := X^4 Z_{F,p}(\overline{\chi}(p)\frac{1}{X}) \cdot Z_{F,p}(-\chi(p)p^{2-2k}X)$$

and

$$B_p(X) := X^4 Z_{F,p}(-\overline{\chi}(p)\frac{1}{X}) \cdot Z_{F,p}(\chi(p)p^{2-2k}X)$$

are polynomials of degree 8 with non-zero constant terms.

¿From (11) one infers in a standard way that $\mathcal{P}$ must be empty and $(\frac{\mathcal{G}(\chi)}{\mathcal{G}(\chi\psi)})^4 = 1$, hence by Lemma 4 we get a contradiction. Therefore $\mathcal{P}$ must be infinite. This proves Thm. 4.

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