RUPRECHT-KARLS-UNIVERSITÄT HEIDELBERG

Mathematisches Institut

Vorlesung Differentialgeometrie II
Heidelberg, 28.11.2014

Exercise sheet 6

Length spaces

To hand in by Friday, December 5, 2014, 12:00

Exercise 1. (10 points)
Consider the real line $\mathbb{R}$ and the map $d : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ with $(x, y) \mapsto d(x, y) := \sqrt{|y - x|}$.

(a) Show that d is a metric on $\mathbb{R}$.

(b) Show that the topology on $\mathbb{R}$ induced by d is the same as the usual topology induced by the Euclidean metric.

(c) Show that in the metric space $(\mathbb{R}, d)$ every non-constant curve has infinite length.

Exercise 2. (30 points)

(a) Let (X, d) be a length space. Show that X has approximate midpoints, that is, for all $x, y \in X$ and all $\varepsilon > 0$ there is a point $c \in X$ such that

$$d(x, c), d(c, y) \leq \frac{1}{2} d(x, y) + \varepsilon.$$

(b) Show that a complete metric space (X, d) with approximate midpoints is a length space.

(c) Let (X, d) be a geodesic space. Show that X has (exact) midpoints, that is, for all $x, y \in X$ there is a point $c \in X$ such that

$$d(x, c) = d(c, y) = \frac{1}{2} d(x, y).$$

(d) Show that a complete metric space (X, d) with (exact) midpoints is a geodesic space.

Hint for (b) and (d): You can use the following statement without proof. Let $A \subset [0, 1]$ be a dense subset, X a complete metric space and $f : A \longrightarrow X$ an L-Lipschitz function. Then f can be extended to a unique L-Lipschitz function on $[0, 1]$.

Exercise 3. (20 points) (“Hopf - Rinow“ theorem)
Let (X, d) be a metrically complete and locally compact length space.

(a) Show that X is proper, that is, every closed bounded subset of X is compact.

(b) Show that X is a geodesic metric space.

Hints:

(a) Show that for a fixed $z \in X$, the set $I_z := \{r \geq 0 \,|\overline{B_r(z)}$ is compact$\}$ is open and close in $\mathbb{R}_{\geq 0}$. For the closeness one can show that every sequence $(y_j)_{j \in \mathbb{N}} \subset \overline{B_R(z)}$ has a converging subsequence by additionally taking a decreasing sequence converging to 0 and using a diagonal argument.

(b) Use Exercise 2.