Seminar

Vector Bundles on Curves and p-adic Hodge Theory

Wintersemester 2014/15

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Contents

1 Introduction

Let us fix a finite extension $E/\mathbb{Q}_p$ with uniformizer π and residue field $\mathbb{F}_q$, and let F be an algebraically closed extension of $\mathbb{F}_q$ that is complete with respect to a non-trivial valuation v. In modern parlance one might call F perfectoid, since F is of characteristic p (not all perfectoid fields are algebraically closed, however).

Attached to this data, Fargues and Fontaine construct in their paper [2] a complete curve $X_{E,F}$ (now also refered to as the *fundamental curve of p-adic Hodge theory*), study its properties and classify vector bundles on it. They show that these vector bundles are closely related to p-adic representations of G_K, where K is any finite extension of $\mathbb{Q}_p$. Using these constructions, they are able to reprove/restate "Theorem A and B" of p-adic Hodge theory (namely, the classification of (crystalline/semistable/de Rham) p-adic representations of G_K via (φ, N)-modules and the fact that potentially semistable implies de Rham, respectively)

We remark that Fargues and Fontaine's fundamental curve has already been successfully used by Weinstein (see [5]) to give a new description of the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ as the étale fundamental group of an adic curve *defined over* $\mathbb{C}_p$. There is an upcoming workshop (Heidelberg/Münster) dedicated to understanding Weinstein's paper, and this seminar's goal is to give a good understanding of one of the basic ingredients.

The primary aim of our seminar is the construction of $X_{E,F}$ and the classification of vector bundles on it. In order to expand a bit on that, we quickly sketch the construction in section 2. For the interested reader we have put together some motivation for the construction in section 3. Section 4 consists of the talks of the seminar.

2 Construction of $X_{E,F}$

We start with E, F as before, and denote by $\mathcal{O}_E$ resp. $\mathcal{O}_F$ the ring of integers of E resp. F. Set

$$B_E^{b,+} := W_{\mathcal{O}_E}(\mathcal{O}_F)[\frac{1}{\pi}] = \left\{ \sum_{n \gg -\infty} [x_n]\pi^n : \ x_n \in \mathcal{O}_F \right\},$$

the $\mathcal{O}_E$-Witt vectors of the perfect ring $\mathcal{O}_F$ of characterstic p. For any $r \in \mathbb{R}_+$ there is, induced from the valuation v on $\mathcal{O}_F$, a valuation v_r on $B_E^{b,+}$. Let $B_{E,r}^+$ be the completion of $B_E^{b,+}$ with respect to this valuation and set $B_E^+ := \bigcap_{r \geq 0} B_{E,r}^+$.

For example, if $E = \mathbb{Q}_p$, $F = \mathrm{Frac}(R(\mathcal{O}_{\mathbb{C}_p}/p))$ ($R(-)$ denotes Fontaine's "épaississement universelle" construction, a universal construction that produces a perfect ring), then $B^+ = \widetilde{\mathbf{B}}^+_{\mathrm{rig}}$ in the notation of Berger.

The Frobenius on F induces by the definition of the Witt vectors a Frobenius φ on B_E^+. Let

$$P_{E,\pi} := \bigoplus_{d \geq 0} (B_E^+)^{\varphi = \pi^d}$$

as a graded algebra, where we denote by $P_{E,\pi,d} = (B_E^+)^{\varphi=\pi^d}$ the piece of degree d. Define $X_{E,F} := \mathrm{Proj}(P_{E,\pi})$.

In the first major theorem ([2], Theorem 10.2), some important results are the following:

Theorem 2.1. a) $X_{E,F}$ is a complete curve over E, and all its closed points are of degree 1.

b) If E'/E then $X_{E',F} = X_{E,F} \otimes_E E'$

c) One has a natural bijection $(P_{E,\pi,1} \setminus 0)/E^\times \to |X_{E,F}|$, where $|X_{E,F}|$ are the closed points of $X_{E,F}$.

d) $\mathrm{Pic}(X_{E,F}) \cong \mathbb{Z}$ via the degree map.

$X_{E,F}$ is not of finite type over E (hence not proper), so what does "complete" mean in this context? Fargues and Fontaine demand that a curve X over $\mathbb{Z}$ comes equipped with a degree map $\deg : |X| \to \mathbb{N}$. Any element $f \neq 0$ in the function field of X gives rise to a divisor $\mathrm{div}(f)$, and X is called *complete* if $\deg(\mathrm{div}(f)) = 0$.

Let now $d \in \mathbb{Z}$ and $P_{E,\pi}[d]$ be the graded algebra which has as underlying algebra $P_{E,\pi}$, but the grading is shifted by d. We consider $\mathcal{O}_{X_{E,F}}(d) = \widetilde{P_E[d]}$ as a sheaf of $\mathcal{O}_{X_{E,F}}$-modules. It is possible to generalize this definition to all $d \in \mathbb{Q}$.

The second major theorem ([2], Theorem 12.8) then contains the following result:

Theorem 2.2. The map

$$\{(\lambda_i)_{1 \leq i \leq n} \in \mathbb{Q}^n | \ \lambda_1 \geq \ldots \geq \lambda_n\} \longrightarrow \{\text{Isomorphism classes of vector bundles on } X_{E,F}\}$$
$$(\lambda_1, \ldots, \lambda_n) \longmapsto \bigoplus_{i=1}^n \mathcal{O}_{X_{E,F}}(\lambda_i)$$

is a bijection.

Let us remark that there exist more general constructions of $X_{E,F}$. For instance, F need not be algebraically closed. Statements in this setting are deduced by going to the algebraic closure und using descent theory on the original setting of [2] (see [3]). Also, E need not be of characteristic zero, but may be replaced by a general local field. As is mostly the case, in this equal characterstic situtation the constructions and proofs become considerably easier, which is why we will focus on the $\mathrm{char}(E) = 0$ case.

Another way to study this construction is to attach an analytic space (in the flavour of Berkovich or Huber) to $X_{E,F}$. However, the absence of a noetherian condition complicates things. Nevertheless, Fargues ([1]) was able to construct an adic space $X^{\mathrm{ad}}_{E,F}$ such that a GAGA formalism holds for coherent sheaves on $X_{E,F}$ and $X^{\mathrm{ad}}_{E,F}$.

Let us finally remark that it is this adification of $X_{E,F}$ that plays a decisive role in Weinstein's paper.

3 Motivation for the construction

3.1 Vector bundles on $X_{E,F}$ as B-pairs

One of the most direct ways to see why one should look at $X_{E,F}$ as defined above is the following, if one is interested in studying p-adic representations. For simplicity we assume that $E = \mathbb{Q}_p$. Let $t \in P_{E,\pi,1} \backslash 0$ and set $B_e = (B_E^+[1/t])^{\varphi=\mathrm{id}} = (P_{E,\pi}[1/t])_0$. Define B^+_{dR} as the t-adic completion of B_E^+ (for now we don't care about the dependence on t), which comes equipped with an action of $G_{\mathbb{Q}_p}$.

We recall that Berger has defined the category of *B-pairs* as follows: elements are pairs (W_e, W^+_{dR}), where W_e is a free B_e-module of finite type (equipped with a semi-linear φ-action), and W^+_{dR} is a B^+_{dR}-lattice of $B_{\mathrm{dR}} \otimes_{B_e} W_e$. Berger has shown that the category of p-adic representations of $G_{\mathbb{Q}_p}$ embeds into the category of B-pairs by the simple rule

$$V \mapsto (V \otimes_{\mathbb{Q}_p} B_e, V \otimes_{\mathbb{Q}_p} B^+_{\mathrm{dR}}),$$

where V is any such representation.

Now, vector bundles on a general curve X with an open affine subset $U = \mathrm{Spec}(B) \subset X$ such that $\mathrm{Pic}(U) = 0$ and $X \setminus U = \{\infty\}$ can be characterized by the following data: a free B-module M of finite type, a free $\widehat{\mathcal{O}}_{X,\infty}$-module N of finite type and an isomorphism $M \otimes_B \mathrm{Frac}(\widehat{\mathcal{O}}_{X,\infty}) \cong N \otimes_{\widehat{\mathcal{O}}_{X,\infty}} \mathrm{Frac}(\widehat{\mathcal{O}}_{X,\infty})$. If we apply this to the curve $X_{E,F}$ with $U = \mathrm{Spec}(B_e)$, we obtain from a vector bundle $\mathcal{F}$ a free B_e-module of finite type $\mathcal{F}|_U$, a free $\widehat{\mathcal{O}}_{X,\infty}$-module of finite type $\widehat{\mathcal{F}}_\infty$, and an isomorphism as before. Since one can easily show that $\widehat{\mathcal{O}}_{X,\infty} = B^+_{\mathrm{dR}}$ (where the closed point ∞ corresponds to t as chosen before) it follows that this is precisely the data of a B-pair.

3.2 $|X_{E,F}|$ as holomorphic functions on a rigid analytic space

A second point of view is the following: one would like to find a rigid analytic space Y such that the elements of the Fréchet algebra B_E^+ are the holomorphic functions on Y. From the classical point of view one should consider $\mathrm{MaxSpec}(B_E^+)$, although the ring B_E^+ is nothing like a Tate-algebra. Nevertheless, it is possible to attach an adic space $Y := \mathrm{Spa}(B_E^+)$ to B_E^+. We note that one has an action of φ on B_E^+, induced by the Frobenius on F. Let us denote by $|Y| = \mathrm{MaxSpec}(B_E^+) \subset \mathrm{Spa}(B_E^+)$ the subset of closed maximal ideals.

If we want to classify φ-modules on B_E^+, that is, free B_E^+-modules with a semi-linear action of φ, we should consider them as vector bundles on $Y/\varphi^{\mathbb{Z}}$. Hence, one is looking for a "proper" curve X that should be "equal" to $Y/\varphi^{\mathbb{Z}}$. Classically, for a projective curve X over a field, one finds a very ample line bundle $\mathcal{L}$ on X, so that

$$X = \mathrm{Proj}\left(\bigoplus_{n\geq 0} H^0(X, \mathcal{L}^{\otimes n})\right).$$

In our situation such a line bundle should come from a line bundle on Y that is φ-equivariant. If one puts $\mathcal{L} = B_E^+ \cdot e$ such that $\varphi(e) = p^{-1}e$, then

$$H^0(X, \mathcal{L}^{\otimes n}) = (B_E^+ \cdot e^{\otimes n})^{\varphi=\mathrm{id}} = (B_E^+)^{\varphi=p^n}.$$

Surprisingly, this turns out to be correct and explains the construction of X_E.

With this in hand one can show that $|Y|/\varphi^{\mathbb{Z}} = |X_{E,F}|$.

3.3 $X_{\mathbb{Q}_p,F}$ parametrizes all un-tilts of F

Another interesting point of view in light of the tilting process by Scholze can be made as follows. Recall that a nonarchimedean valued field L is called *perfectoid* if $|L^\times|$ is not discrete and if the Frobenius $x \mapsto x^p$ on $\mathcal{O}_L/p$ is surjective, where $\mathcal{O}_L$ denotes the ring of integers of L.

Now, if L is a non-archimedean field of characteristic 0, on may associate to it the following ring of characteristic p

$$\mathcal{O}_{L^\flat} := \varprojlim_{x\mapsto x^p} \mathcal{O}_L/p,$$

which is an integral domain. We denote $L^\flat := \mathrm{Frac}(\mathcal{O}_{L^\flat})$, and call it the *tilt* of L.

Suppose we are given a perfectoid field F of characteristic p. If L is a perfectoid field of characteristic 0 and F embeds into $L^\flat$ such that $L^\flat/F$ is finite, we call L an *un-tilt* of F.

How can we classify all un-tilts of F? Here is how the fundamental curve comes into play. Any un-tilt L of F is given by a map $F \to L^\flat$. This gives rise to a map $\# : \mathcal{O}_E \to \mathcal{O}_L$ via the $\#$-map from the tilting formalism, and hence to maps $W_{\mathbb{Q}_p}(\mathcal{O}_F) \to \mathcal{O}_L$ and $\theta_L : W_{\mathbb{Q}_p}(\mathcal{O}_F)[1/p] \to L$. The kernel of this last map is a maximal ideal. Unfortunately, it seems that $\mathrm{MaxSpec}(W_{\mathbb{Q}_p}(\mathcal{O}_F)[1/p])$ is too big, which is why one considers $W_{\mathbb{Q}_p}(\mathcal{O}_F)[1/p] \hookrightarrow B_E^+$. By continuity, the map θ_L extends to B_E^+, and its kernel is again a maximal ideal that, as one can show, corresponds precisely to one un-tilt L of F.

From the preceeding subsection, if we are now interested in φ-equivalence classes of un-tilts of F, we see that we may parametrize them by $|Y|/\varphi^{\mathbb{Z}} = |X_{\mathbb{Q}_p,F}|$.

4 List of talks

The numbers are in reference to the article [2]. [4] and [3] are overview articles, where the focus in the first article is on the relation with p-adic representations, and the second article also incorporates statements about the cases when E is not necessarily of characteristic 0 and when F is not necessarily algebraically closed. Unfortunately, it skips some essential proofs, which is why we will stick to the original article [2].

1. **Curves and Fiber bundles (1.1-2.3).** Define complete curves X over $\mathbb{Z}$, almost euclidean rings. Prove Theorem 1.13, skip 1.2.4, state the classification for fiber bundles on X in 2.1, show Lemma 2.5 and Proposition 2.6.

2. **Properties of Fiber bundles (3.1, 3.2.1, 4.1, 4.2.2).** Explain the Harder-Narasimhan formalism, define semi-stability, state Theorems 3.2, 3.3, 3.4 without proof. Give example 3.2.1. Define Riemannian spheres, sketch the proof of Theorem 4.2, state Proposition 4.3 without proof, treat section 4.2.2 as detailed as time permits.

3. **Generalized Riemannian spheres (4.2.3).** Define generalized Riemannian spheres, give example 4.21, prove Proposition 4.23, define pure vector bundles, give Proposition 4.25 and show the classification Theorem 4.26.

4. **Some Rings (5.1-5.2).** Define $W_{\mathcal{O}_E}(\mathcal{O}_F)$, define the standard operations F and V, treat the "torsion"-case by a Lubin-Tate law associated to E, define the rings $B_E^{b,+}$ and the valuations w_k as well as v_r, define the rings S_r and B^+ and state the elementary properties of these.

5. **Newton Polygons, Bivectors, the ring $\mathcal{R}$ (5.3, 5.4, 5.6).** Define the Legendre transform and the Newton polygon $Newt(b)$ of elements $b \in B^{b,+}$ and $b \in B^+$, show basic properties of $Newt(-)$, define bivectors, skip 5.5. Define the ring $\mathcal{R}_Q$ and the map θ, state the basic functorial properties and give the example 5.6.3.

6. **Elements of degree 1 (6.1, 6.2.1).** Define primitivity and the degree for elements in $W_{\mathcal{O}_E}(\mathcal{O}_F)$ (compare also Definition 6.41), define the set $\mathcal{A}$ and A associated to $a \in \mathcal{A}$, prove Lemma 6.2, sketch Proposition 6.5, explain the non-unique Weierstrass decomposition in this situation, show the identification $\mathcal{O}_F = \mathcal{R}(A)$, only sketch Proposition 6.19, deduce Corollary 6.21. Define the set Y and the metric d on Y.

7. **Properties of Y and elements of degree > 1 (6.2.2, 6.3, 6.4)**. Give the parametrization of Y by elements of $\mathfrak{m}_F$, show the homeomorphism in Proposition 6.33, explain how Y may be viewed as a subspace of a Berkovich space, show basic functorial properties in 6.3, prove Theorem 6.42 and state the Weierstrass decomposition in this case, show Theorems 6.45 and 6.46.

8. **Further study of elements of B_E^+ (6.5, 6.6, 6.7, 7.1, 7.2)**. Show Theorems 6.49 and 6.50 which decompose elements of B_E^+, explain how one can attach a ring $B_{dR,\mathfrak{m}}^+$ and a valuation $\mathrm{ord}_{\mathfrak{m}}$ to $W_{\mathcal{O}_E}(\mathcal{O}_F)$ for any $\mathfrak{m} \in Y$, define $\mathrm{Div}^+(Y)$ and $\mathrm{div}(f)$ for $f \in B_E^+$, show Corollary 6.54 and only state Theorem 6.55. Show Proposition 7.1, define $\overline{B}$, show Proposition 7.6.

9. **Fundamental exact sequence and main properties of the curve (9.1-9.4, 10.1).** Define the graded algebra $P_{E,\pi}$, the sets $\mathbb{M}_d$, $\mathbb{M}$ and the map Π. Define $\mathrm{Div}^+(Y/\varphi^{\mathbb{Z}})$ and show the Theorem 9.7. Explain the relation of Π with the Lubin-Tate logarithm, show the fundamental exact sequence 9.10, state Corollary 9.12 and Theorem 9.13. Prove the main theorem 10.2 and state Corollary 10.3.

10. **Vector bundles on the curve (12.1-12.5).** Define $\mathcal{O}_{X_{E,F}}(d)$, first for $d \in \mathbb{Z}$ (Definition 12.2), then for $\lambda \in \mathbb{Q}$ (Definition 12.4), show the statements 12.2-12.7. Prove Theorem 12.8 via Theorem 12.9, which in turn is proved via Banach-Colmez spaces, 12.5.1. Explain the proof as it is done in 12.5.2.

References

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