# Adiabatic limits and the vortex equation

S. Schmitt

Seminar

# Table of Contents

# Table of Contents

# Classic vortex Equations

Our central object of study are vortex equations on closed Riemann surfaces. In the following we will denote a closed Riemann surface with $\Sigma$ and $L \to \Sigma$ a Hermitian line bundle. The simplest vortex equation is the following:

**Classic vortex**

We call a pair of a unitary connection $A$ on $L$ and a section $\varphi$ of $L$, satisfying

$$\begin{cases} \overline{\partial}_A \varphi = 0, \\ i\Lambda F_A = 1 - |\varphi|^2, \end{cases} \tag{1}$$

where $\Lambda F_A$ is the Hodge dual of the curvature form, a (classical) vortex.

# A quick primer on necessary background

Note that the following will display the situation on vector bundles, but can be generalized to principal bundles.

- We call a triple $(E, X, \pi)$ where $\pi : E \to X$ is a fibre bundle with fibre given by a vector space $\mathbb{K}^r$, $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, a *vector bundle* and call $r$ the rank of the vector bundle. If $r = 1$, we call the bundle a line-bundle.
- A *connection* on a vector bundle is a choice of $\mathbb{K}$-linear differential operator $\nabla : \Gamma(E) \to \Gamma(T^*X \otimes E) = \Omega^1(E)$ such that $\forall f \in C^\infty(X)$ and sections $s \in \Gamma(E)$ we have $\nabla(fs) = df \otimes s + f\nabla s$.
- The *curvature* of $\nabla$ is given by the operator $F_\nabla \in \Omega^2(\mathrm{End}(E))$ with values in the endomorphism bundle, defined by $F_\nabla(v_1, v_2) = \nabla_{v_1}\nabla_{v_2} - \nabla_{v_2}\nabla_{v_1} - \nabla_{[v_1,v_2]}$. Every connection over a trivialising subset $U_\alpha$ differs from the trivial connection $d$ by some local connection one-form $A_\alpha \in \Omega^1(U_\alpha, \mathrm{End}(E))$ with the property that $\nabla = d + A_\alpha$ on $U_\alpha$. In terms of this local connection form, the curvature may be written as $F_A = dA_\alpha + A_\alpha \wedge A_\alpha$.

# A quick primer on necessary background

- A *Hermitian line bundle* is a complex line bundle with a hermitian metric.
- A *gauge transformation* is a diffeomorphism $\phi : E \to E$ commuting with the projection operator $\pi$ which is a linear isomorphism of vector spaces on each fibre. The gauge transformations of $E$ form a group under composition called the *gauge group*, typically denoted by $\mathcal{G}$.
- A gauge transformation $u$ of $E$ transforms a connection $\nabla$ into a connection $u \cdot \nabla$ by the conjugation $(u \cdot \nabla)_v(s) = u(\nabla_v(u^{-1}(s)))$. Under a local gauge transformation $g$ we have $\overline{A}_\alpha = gA_\alpha g^{-1} - (dg)^{-1}g$.
- The space of connections on a vector bundle is an infinite dimensional affine space $\mathcal{A}$ modelled on the vector space $\Omega^1(X, \mathrm{End}(E))$. Two connections $A, A' \in \mathcal{A}$ are said to be *gauge equivalent* if there exists a gauge transformation $u$ such that $A' = u \cdot A$.

# Classic vortex equations

## Classic vortex

We call a pair of a unitary connection $A$ on $L$ and a section $\varphi$ of $L$, satisfying

$$\begin{cases} \overline{\partial}_A \varphi = 0, \\ i\Lambda F_A = 1 - |\varphi|^2, \end{cases}$$

where $\Lambda F_A$ is the Hodge dual of the curvature form, a (classical) vortex.

The space of vortices takes a more familiar form if we look at the gauge equivalence classes of vortices, the equivalence classes take the form of the symmetric product $\mathrm{Sym}^d \Sigma$, where $d = \deg L$. The points in $\mathrm{Sym}^d \Sigma$ correspond to an effective divisor $D$ with degree $d$. Up to gauge equivalence there also exists a unique solution $(A, \varphi)$ such that $D$ is the zero divisor of $\varphi$.

# A first modification

We can modify our equation above by scaling the metric on $\Sigma$, say by $\varepsilon^{-1}$, this results in the modified equation:

## Modified vortex

$$\begin{cases} \overline{\partial}_A \varphi = 0, \\ \varepsilon^2 i \Lambda F_A = 1 - |\varphi|^2. \end{cases} \tag{2}$$

This new equation poses the question how the behaviour of solutions changes in the limiting case $\varepsilon \to 0$, i.e. what happens to our solutions if the volume of $\Sigma$ tends to infinity. An answer to that question was provided by Hong, Jost, and Struwe. We will reproduce this answer in this talk, albeit with a different approach which lends itself to proving a result of a more general form of the above mentioned equation.

# Framed vortex equations

To introduce the general equation, we will need to fix auxiliary unitary bundles $E_1, \dots, E_N$ over $\Sigma$ together with their respective connections $B_1, \dots, B_N$ and weights $k_1, \dots, k_N \in \mathbb{Z}^x = \mathbb{Z} \setminus \{0\}$. Now taking $\varepsilon > 0$ and $\tau \in \mathbb{R}$ along with a connection $A$ on $L$ and a section $\varphi = (\varphi^1, \dots, \varphi^N) \in \Gamma(\bigoplus_{j=1}^N E_j \otimes L^{\otimes k_j})$, gives us

## Framed vortex equations

$$\begin{cases} \overline{\partial}_{A\otimes B_j}\varphi^j = 0 & \text{for } j = 1, \dots, N, \\ \varepsilon^2 i\Lambda F_A + \sum_{j=1}^N k_j|\varphi|^2 + \tau = 0 \end{cases} \tag{3}$$

Notice that we recover equation (2), by setting $N = 1$, $k_1 = 1$ and $\tau = 1$. The moduli space of solutions to this generalized equation admits a holomorphic description as before. An important application of the theorem is to Seiberg-Witten theory, which we will introduce at the end of the talk. Finally note that solutions to (3) only exist if either the $k_i$ are of mixed signs or $k_i > 0 \forall i$ and $\tau < 0$.

# Framed vortex equations

We can now state the main theorem we are interested in:

## Main theorem

Let $(A_i, \varphi_i, \varepsilon_i)$ be a sequence of solutions to:

$$\begin{cases} \overline{\partial}_{A \otimes B_j} \varphi^j = 0 & \text{for } j = 1, \ldots, N, \\ \varepsilon^2 i \Lambda F_A + \sum_{j=1}^{N} k_j |\varphi|^2 + \tau = 0, \end{cases}$$

such that $\varepsilon_i \to 0$ and the sequence of norms $\|\varphi_i\|_{L^2}$ is bounded. Then there is a finite set of points $D \subset \Sigma$, such that after passing to a subsequence and applying gauge transformations, $(A_i, \varphi_i)$ converges in $C^\infty_{\text{loc}}$ on $\Sigma \setminus D$. The limit $(A, \varphi)$ satisfies the above equation with $\varepsilon = 0$ on $\Sigma \setminus D$.

# Table of Contents

# Overview

In this section of the talk we will introduce necessary notation, additional background knowledge and finally a series of analytical results needed to prove our result.

- An outlook on the proof

# Overview

In this section of the talk we will introduce necessary notation, additional background knowledge and finally a series of analytical results needed to prove our result.

- An outlook on the proof
- Gauge transformations

# Overview

In this section of the talk we will introduce necessary notation, additional background knowledge and finally a series of analytical results needed to prove our result.

- An outlook on the proof
- Gauge transformations
- A-priori estimates

# Overview

In this section of the talk we will introduce necessary notation, additional background knowledge and finally a series of analytical results needed to prove our result.

- An outlook on the proof
- Gauge transformations
- A-priori estimates
- Necessary lemmas

# Proof outline

- Establishing convergence in $\mathcal{G}^c$,
- Establishing convergence in real moduli spaces.

# Proof outline

- Convergence mod $\mathcal{G}^c$
  - We start by using the actions of $\mathcal{G}^c = C^\infty(\Sigma, \mathbb{C}^x)$ on $(A, \varphi)$.

# Proof outline

- Convergence mod $\mathcal{G}^c$
  - We start by using the actions of $\mathcal{G}^c = C^\infty(\Sigma, \mathbb{C}^x)$ on $(A, \varphi)$.
  - The moduli space of solutions to (3) is homeomorphic to the quotient of the set of solutions to the Cauchy-Riemann equation by $\mathcal{G}^c$. This was proved by J. A. Bryan and R. Wentworth. We show that this quotient is compact modulo the rescaling action of $\mathbb{C}^x$, by using elliptic estimates for Dolbeault operators.

# Proof outline

- Convergence mod $\mathcal{G}^c$
  - We start by using the actions of $\mathcal{G}^c = C^\infty(\Sigma, \mathbb{C}^x)$ on $(A, \varphi)$.
  - The moduli space of solutions to (3) is homeomorphic to the quotient of the set of solutions to the Cauchy-Riemann equation by $\mathcal{G}^c$. This was proved by J. A. Bryan and R. Wentworth. We show that this quotient is compact modulo the rescaling action of $\mathbb{C}^x$, by using elliptic estimates for Dolbeault operators.
  - This gives us complex gauge transformations $g_i = e^{f_i} u_i$ for $f_i \in C^\infty(\Sigma, \mathbb{R})$ and $u_i \in C^\infty(\Sigma, \mathrm{U}(1))$, such that the sequence $(A_i, \varphi_i)$ converges after rescaling it and applying $g_i$.

# Proof outline

- Real convergence
  - To establish convergence in the real moduli space as well, we need to control the $f_i$ that arose in the previous step.

# Proof outline

- Real convergence
  - To establish convergence in the real moduli space as well, we need to control the $f_i$ that arose in the previous step.
  - In the specific setting, this gives rise to the partial differential equation for $f_i$ of the form

$$\varepsilon^2 \Delta f + \sum_{j=1}^{n} A_j e^{\alpha_j f} - \sum_{j=1}^{m} B_j e^{-\beta_j f} + w = 0$$

for some functions $A_j\, B_j \geq 0$, $w$ and constants $\alpha_j, \beta_j > 0$.

# Proof outline

- Real convergence
  - To establish convergence in the real moduli space as well, we need to control the $f_i$ that arose in the previous step.
  - In the specific setting, this gives rise to the partial differential equation for $f_i$ of the form

$$\varepsilon^2 \Delta f + \sum_{j=1}^{n} A_j e^{\alpha_j f} - \sum_{j=1}^{m} B_j e^{-\beta_j f} + w = 0$$

    for some functions $A_j\, B_j \geq 0$, $w$ and constants $\alpha_j, \beta_j > 0$.
  - The bounds we establish will be independent of $\varepsilon \in (0, 1]$ and uniform on compact subsets of $\Sigma \setminus D$ where $D$ is the set of common zeroes of $A_j$ and $B_j$.

# Proof outline

- Real convergence
  - To establish convergence in the real moduli space as well, we need to control the $f_i$ that arose in the previous step.
  - In the specific setting, this gives rise to the partial differential equation for $f_i$ of the form

$$\varepsilon^2 \Delta f + \sum_{j=1}^{n} A_j e^{\alpha_j f} - \sum_{j=1}^{m} B_j e^{-\beta_j f} + w = 0$$

    for some functions $A_j\, B_j \geq 0$, $w$ and constants $\alpha_j, \beta_j > 0$.
  - The bounds we establish will be independent of $\varepsilon \in (0, 1]$ and uniform on compact subsets of $\Sigma \setminus D$ where $D$ is the set of common zeroes of $A_j$ and $B_j$.
  - This allows us to use the Arzelà–Ascoli theorem to guarantee the existence of a smoothly converging subsequence of $f_i$ on compact subsets of $\Sigma \setminus D$.

# Gauge transformations

Taking actions of the gouge group $\mathcal{G}$ of unitary automorphisms of $L$, we can identify $\mathcal{G}$ with $C^\infty(\Sigma, \mathrm{U}(1))$. Given a solution to (3), $(A, \varphi^1, \dots, \varphi^N)$, and a map $u : \Sigma \to \mathrm{U}(1)$, the action of $u$ on $(A, \varphi^1, \dots, \varphi^N)$ is given by

$$u(A, \varphi^1, \dots, \varphi^N) = (A - u^{-1}du, u^{k_1}\varphi^1, \dots, u^{k_N}\varphi^N)$$

. The set of these solutions is invariant under actions of $\mathcal{G}$.

# Gauge transformations

Even more so the Dolbelaut equation in (3), $\left(\overline{\partial}_{A\otimes B_j}\varphi^j = 0\right)$, is also invariant under the action of the complex gauge group $\mathcal{G}^c$. $\mathcal{G}^c$ consists of the complex automorphisms of $L$ and we identify it with $C^\infty(\Sigma, \mathbb{C}^x)$ where $\mathbb{C}^x$ is the complex numbers without 0, i.e. $\mathbb{C}^x = \mathbb{C} \setminus \{0\}$. Similar to before, for $g : \Sigma \to \mathbb{C}^x$ and $(A, \varphi^1, \ldots, \varphi^N)$ the action of $g$ is given by

$$g(A, \varphi^1, \ldots, \varphi^N) = (A + \overline{g}^{-1}\partial\overline{g} - g^{-1}\overline{\partial}g, g^{k_1}\varphi^1, \ldots, g^{k_N}\varphi^N).$$

Taking $s \in \Gamma(\Sigma, E_j \otimes L^{\otimes k_j})$ the associated Dolbeault operator transforms as

$$\overline{\partial}_{B_j, g(A)} s = g^{k_j}\overline{\partial}_{B_j A}(g^{-k_j} s).$$

# Gauge transformations

However the action of $\mathcal{G}^c$ does not preserve the second equation in (3) involving the curvature. Taking $f : \Sigma \to \mathbb{R}$, $u : \Sigma \to \mathrm{U}(1)$ and writing $g = e^f u$, then we obtain:

$$F_{g(A)} = F_A + 2\bar{\partial}\partial f$$

or taking the Hodge dual, along with $\Delta = \mathrm{d}\delta + \delta\mathrm{d}$, the Hodge-Laplacian acting on functions,

$$i\Lambda F_{g(A)} = i\Lambda F_A + \Delta f.$$

# A-priori estimates

The generalization we introduced also introduces new problems we have to deal with. There are two features that are in particularly difficult to deal with. The first one being that we have to introduce auxiliary cut-off functions to deal with possibility of our manifold having a boundary. The second major problem is having to deal with the degeneration happening if $\varepsilon = 0$.

# A-priori estimates

## Boundness theorem

Let X be a compact Riemannian manifold with (possibly empty) boundary $\partial X$, $\Omega \subset X$ an open subset, such that $\overline{\Omega} \subset X \setminus \partial X$. Let $\varepsilon_0$; $\alpha_1, \ldots, \alpha_n$; $\beta_1, \ldots, \beta_m$ be positive numbers and let $A_1, \ldots, A_n$; $B_1, \ldots, B_m$ and $w$ be smooth functions on $X$ such that $A_j, B_j \geq 0 \, \forall j$, and $A_1 + \cdots + A_n > 0, \quad B_1 + \cdots + B_m > 0$. Then there exist constants $M_0, M_1, M_2, \ldots$ depending only on the data listed above such that for any $\varepsilon \in [0, \varepsilon_0]$ and $f \in C^\infty(X)$ satisfying the equation

$$\varepsilon \Delta f + \sum_{j=1}^{n} A_j e^{\alpha_j f} - \sum_{j=1}^{m} B_j e^{-\beta_j f} + w = 0 \tag{4}$$

the following bound on $f$ holds:

$$\|f\|_{C^k(\Omega)} \leq M_k \quad \text{for } k = 0, 1, 2, \ldots$$

# A-priori estimates

## Remark

The bound $M_k$ depends on $A_j, B_j$ and $w$ as well was their derivatives. It will be important to consider sequences $\varepsilon_i \to 0$ and $f_i, A_1^i, \dots, A_n^i, B_1^i, \dots, B_m^i, w_i$ satisfying the PDE

$$\varepsilon_i \Delta f + \sum_{j=1}^{n} A_j^i e^{\alpha_j f_i} - \sum_{j=1}^{m} B_j^i e^{-\beta_j f_i} + w_i = 0.$$

Provided that $A_j^i, B_j^i$ and $w_i$ converge smoothly to $A_j, B_j$ and $w$ respectively where

$$A_1 + \cdots + A_n > 0, \quad B_1 + \cdots + B_n > 0,$$

the proof will hold and provide a $C^k$ estimate for large $i$ dependent on $k$

# Required lemmas

We also need the following fact:

**Lemma**

Let $L \to \Sigma$ be a Hermitian line bundle, $D \subset \Sigma$ a finite set of points, a unitary connection $A$ on $L|_{\Sigma \setminus D}$ and $\alpha \in \Gamma(\Sigma \setminus D, L)$. If $\overline{\partial}_A \alpha = 0$ and $|\alpha| = 1$ everywhere on $\Sigma \setminus D$, then

$$\nabla_A \alpha = 0 \ \text{and} \ F_A = 0.$$

Moreover for a small ball $B$ around a point $p \in D$ such that in a unitary local trivialisation $A = d + a$ for a one-form $a \in \Omega^1(B \setminus \{p\}, i\mathbb{R})$, and after identifying $\alpha$ with a smooth function $\alpha : B \setminus \{p\} \to S^1$. Then

$$\frac{i}{2\pi} \int_{\partial B} a = \deg(\alpha|_{\partial B}).$$

# Table of Contents

# Remarks

We are now in a position to work on the proof. We will present the proof for the classic vortex case, though the steps to prove the general result are the same. First let us restate the main theorem, on slide 10, in the case of the modified classical vortex equation, as mentioned before, we do this by setting: $N = 1$, $k_1 = 1$ and $\tau = 1$.

## Main theorem, classic case

Let $(A_i, \varphi_i, \varepsilon_i)$ be a solution to

$$\begin{cases} \overline{\partial}_A \varphi = 0, \\ \varepsilon^2 i \Lambda F_A = 1 - |\varphi|^2 \end{cases} \tag{5}$$

such that $\varepsilon_i \to 0$. Then there is a degree $d$ effective divisor D on $\Sigma$ such that after passing to a subsequence and applying gauge transformations $(A_i, \varphi_i)$ converges in $C^\infty_{\text{loc}}$ on $\Sigma \setminus D$ and $\frac{1}{2\pi}\Lambda F_A \to \delta_D := \sum_k m_k \delta_{x_k}$ as measures. The limit $(A, \varphi)$ satisfies $F_A = 0, |\alpha| = 1$ and $\nabla_A \alpha = 0$ on $\Sigma \setminus D$.

# The proof

We will prove the theorem in multiple steps, mirroring what we outlined before:

- Convergence modulo $\mathcal{G}^c$,
- Establishing $C^0$ estimates,
- Convergence outside $D$,
- The limiting configuration,
- Convergence of measures.

To start, notice that since $\varepsilon_i \to 0$, we may assume that none of the sections $\varphi_i$ are identically zero. We define $\mathcal{A}$ to be the space of unitary connections on $L$ and as before $\mathcal{G}^c$ to be the complex gauge group of $L$, that is the space of smooth maps from $\Sigma \to \mathbb{C}^x$.

# Convergence modulo $\mathcal{G}^c$

Our claim in this step of the proof is that we can find complex gauge transformations $g_i \in \mathcal{G}^c$ such that, after passing to a subsequence, $g_i(A_i, \varphi_i)$ converges in $C^\infty(\Sigma)$ to a pair $(A', \varphi')$. Where $\varphi'$ is not identically zero and satisfies $\overline{\partial}_{A'}\varphi' = 0$.

# Convergence modulo $\mathcal{G}^c$

To prove this we start by noting that $\mathcal{A}/\mathcal{G}^c$ is homeomorphic to the Jacobian torus $H^1(\Sigma, \mathbb{R})/H^1(\Sigma, \mathbb{Z})$ in the $C^\infty$ topology. The important consequence of this is that $\mathcal{A}/\mathcal{G}^c$ is compact. This gives us the existence of $g_i \in \mathcal{G}^c$ such that, after passing to a subsequence, we get convergence of $A_i' = g_i A_i$ in $C^\infty$ to a connection $A'$. Setting $\mu_i = \|g_i \varphi_i\|_{L^2}^{-1}$ and replacing $g_i$ with $\mu_i g_i$ we can normalize $\|g_i \varphi_i\|_{L^2} = 1 \ \forall i$. Since the constant gauge transformation $\mu_i$ acts trivially on the space of connections, $A_i' \to A'$ remains true. Finishing up our criterions for the choice of $g_i$, we assume them to be purely "imaginary".

# Convergence modulo $\mathcal{G}^c$

For a gauge transformation $u \in \mathrm{U}(1)$ and a real $f : \Sigma \to \mathbb{R}$ we can write any complex gauge transformation as $g = ue^f$. Enabling us to write $g_i = e^{f_i/2}$, with $f_i : \Sigma \to \mathbb{R}$ after incorporating $u \in \mathrm{U}(1)$ into our original sequence $(A_i, \varphi_i, \beta_i)$.
We also have that $\mathcal{G}^c$ preserves the Cauchy-Riemann equation, so for $\varphi_i' = g_i \varphi_i$ we have

$$\overline{\partial}_{A_i'} \varphi_i' = \overline{\partial}_{A_i'} g_i \varphi_i = 0.$$

For the $L^2$-norm this means

$$\|\overline{\partial}_{A_i'} \varphi_i'\|_{L^2} = \|(\overline{\partial}_{A'} - \overline{\partial}_{A_i'})\varphi_i'\|_{L^2} \leq \|A' - A_i'\|_{L^\infty} \|\varphi_i'\|_{L^2}.$$

# Convergence modulo $\mathcal{G}^c$

For the $L^2$-norm this means

$$\|\overline{\partial}_{A'_i}\varphi'_i\|_{L^2} = \|(\overline{\partial}_{A'} - \overline{\partial}_{A'_i})\varphi'_i\|_{L^2} \leq \|A' - A'_i\|_{L^\infty}\|\varphi'_i\|_{L^2}.$$

Since we already established $A'_i \to A'$, the right-hand side tends to zero and we obtain a bound on $\|\overline{\partial}_{A'_i}\varphi'_i\|_{L^2}$ independent of $i$. Using the elliptic estimate for the Dolbeault operators we can conclude that the sequence $\varphi'_i$ is bounded in $W^{1,2}$. Bootstrapping then gives us convergence in $C^k$ for any $k$. This lets us pass to another subsequence (which we will denote with the same symbols), that convergences in $C^\infty$ to $\varphi'$. The limit satisfies

$$\overline{\partial}_{A'}\varphi' = 0 \quad \text{and} \quad \|\varphi'\|_{L^2} = 1,$$

this finishes step 1.

# Establishing $C^0$ estimates

The next step is establishing $C^0$ estimates. We start with $D$, the set of zeroes of $\varphi'$. There are exactly $d = \deg(L)$ of them, counted with multiplicity. We need to establish uniform boundness of $f_i$ on compact subsets of $\Sigma \setminus D$. Since transformations in $\mathcal{G}^c$ do not preserve the curvature equation, we compute

$$\begin{aligned}
\varepsilon_i^2(2i\Lambda F_{A_i'}) &= \varepsilon_i^2(2i\Lambda F_{A_i'} + \Delta f_i) \\
&= 1 - |\varphi_i|^2 + \varepsilon_i^2 \Delta f_i \\
&= 1 - e^{-f_i}|\varphi_i'|^2 + \varepsilon_i^2 \Delta f_i,
\end{aligned}$$

# Establishing $C^0$ estimates

Now defining $q_i = |\varphi_i'|^2$ and $w_i = 1 - \varepsilon_i^2(2i\Lambda F_{A_i'})$, we obtain a partial differential equation for $f_i$ after rearranging:

$$\varepsilon_i^2 \Delta f_i = e^{-f_i}|\varphi_i'|^2 - 1 + \varepsilon_i^2(2i\Lambda F_{A_i'}) \tag{6}$$

$$= q_i e^{-f_i} - w_i, \tag{7}$$

Letting $u_i = e^{f_i}$ we then obtain

$$\begin{aligned} \varepsilon_i^2 \Delta u_i &= \varepsilon_i^2(-e^{f_i}|\nabla f_i|^2 + e^{f_i}\Delta f_i) \\ &\leq q_i - w_i u_i. \end{aligned}$$

# Establishing $C^0$ estimates

$$\varepsilon_i^2 \Delta u_i \leq q_i - w_i u_i.$$

The uniform convergence of $w_i \to 1$ then establishes $w_i \geq \frac{1}{2}$ when $i$ large enough. Because $\varphi_i'$ converges, the $q_i$ are bounded. Using the maximum principle we arrive at an upper bound for $u_i$ and since $u_i = e^{f_i}$, we also arrive at one for $f_i$. Computing further

$$\varepsilon_i^2 \Delta |\varphi_i|^2 + 2\varepsilon_i^2 |\partial_A \varphi_i|^2 = |\varphi_i|^2 (1 - |\varphi_i|^2),$$

and using the maximum principle again, we get $|\varphi_i|^2 \leq 1 \; \forall i$. We can now improve our estimate on $f_i$, we have that $|\varphi_i|^2 = e^{-f_i} |\varphi_i'|^2$ and $|\varphi_i'|^2 \to |\varphi'|^2$ uniformly, this implies a uniform lower bound for $f_i$ on compact subsets of $\Sigma \setminus D$.

# Convergence outside of $D$

The next step is to look at convergence outside of D. We pick up

$$\varepsilon_i^2 \Delta f_i = q_i e^{-f_i} - w_i,$$

(6), from the previous proof. Utilizing our main result of the last chapter, slide 20, as well as the remark following it, we can take the $C^0$ estimates we established in the last step and improve them to arrive at the stronger conclusion that $f_i$ is uniformly bounded along with its derivatives on all compact subsets of $\Sigma \setminus D$.

This allows us to choose a subsequence of $f_i$ which converges uniformly with all derivatives on compact subsets of $\Sigma \setminus D$ to a function $f \in C^\infty(\Sigma \setminus D, \mathbb{R})$. We will call the associated complex gauge transformation $g = e^{f/2}$ and also define $(A, \varphi) = (g^{-1}A', g^{-1}\varphi')$.

# Convergence outside of $D$

This pair is well-defined and in $C^\infty_{\mathrm{loc}}(\Sigma \setminus D)$ we have $(A_i, \varphi_i) \to (A, \varphi)$. We have

$$\begin{aligned}\varphi_i - \varphi &= g_i^{-1}\varphi_i' - g^{-1}\varphi' \\ &= g_i^{-1}\varphi_i' - g_i^{-1}\varphi' + g_i^{-1}\varphi' - g^{-1}\varphi' \\ &= g_i^{-1}(\varphi_i' - \varphi') + (g_i^{-1} - g^{-1})\varphi',\end{aligned}$$

using the convergence of $\varphi_i'$ and $g_i'$, then shows that for any compact $K \subset \Sigma \setminus D$ we can find constants $M_{l,K}$ with $l = 0, 1, \ldots$ such that

$$\|\varphi_i - \varphi\|_{C^l(K)} \leq M_{l,K}(\|\varphi_i' - \varphi'\|_{C^l(K)} + \|g_i^{-1} - g^{-1}\|_{C^l(K)}).$$

Since the right-hand side tends to zero, this shows that $\varphi_i$ converges to $\varphi$ in $C^l$ for any $l$ on $K$. A similar argument establishes convergence of connections.

# The situation at the limit

We pass to the limit in

$$\begin{cases} \overline{\partial}_A \varphi = 0, \\ \varepsilon^2 i \Lambda F_A = 1 - |\varphi|^2, \end{cases}$$

which shows us that $f : \Sigma \setminus D \to \mathbb{R}$ is given by

$$f = \log |\varphi'|^2,$$

since we also have $\varphi = e^{-f/2}\varphi'$, this is equivalent to $|\varphi| = 1$. Further we also have $\overline{\partial}_A \varphi = 0$, which means we can now use the lemma established in the preparation chapter, slide 22, to get $\nabla_A \varphi = 0$ and $F_A = 0$ on $\Sigma \setminus D$.

# Convergence in measure

The last thing we need to show is that

$$\frac{i}{2\pi}\Lambda F_{A_i} \to \sum_{j=1}^{d} \delta(x_j)$$

in measure. Equivalently we can show

$$\lim_{i\to\infty} \int_B \frac{i}{2\pi} F_{A_i} = k$$

for any small disc B around $x_j$, where k is the multiplicity of the section $\varphi'$ at $x_j$.

# Convergence in measure

Choosing local coordinates for $B$ together with a unitary trivialisation of $L$, gives us $A_i = d + a_i$ as a local representation of $A_i$ for $a_i \in \Omega^1(B, i\mathbb{R})$. The curvature then takes the form $F_{A_i} = da_i$. Using Stokes' theorem allows us to make the calculation,

$$\lim_{i\to\infty} \int_B \frac{i}{2\pi} F_{A_i} = \lim_{i\to\infty} \int_B \frac{i}{2\pi} da_i = \lim_{i\to\infty} \int_{\partial B} \frac{i}{2\pi} a_i = \int_{\partial B} \frac{i}{2\pi} a,$$

where $a \in \Omega^1(B \setminus \{x_j\}, i\mathbb{R})$ is the corresponding one-form to the singular connection $A = d + a$.

# Convergence in measure

Applying the same lemma we used in the last step again, gives us that

$$\frac{i}{2\pi}\int_{\partial B} a = \deg(\varphi|_{\partial B}).$$

Since $\varphi'$ and $\varphi$ differ by a non-zero function on $B \setminus \{x_j\}$, their degrees around $x_j$ are the same and equal to the multiplicity of $\varphi'$ at $x_j$. Finishing the proof.

# Table of Contents

# Seiberg-Witten theory

We can apply the result in the general case to achieve a result in three-dimensional Seiberg-Witten theory. To establish what we are talking about, we define the Seiberg-Witten equations with multiple spinors. To do so, let $Y$ be a closed Riemannian spin-three manifold. Define $S$ to be spinor bundle and $E, L$ vector bundles over $Y$ with structure groups $\mathrm{SU}(n)$ and $\mathrm{U}(1)$ respectively. Finally we equip $E$ with a connection $B$. This means for a connection $A$ on $L \to Y$ and $\Psi \in \Gamma(\mathrm{Hom}(E, S \otimes L))$ the Seiberg-Witten equations with multiple spinors are

$$\begin{cases} \not{D}_{A\otimes B}\Psi = 0 \\ F_A = \Psi\Psi^* - \frac{1}{2}|\Psi|^2. \end{cases} \tag{8}$$

where $\not{D}_{A\otimes B}$ is the Dirac operator twisted by $A$ and $B$, while the second equation is based on the identification $i\bigwedge^2 T^*Y \cong i\mathfrak{su}(S)$ given by Clifford multiplication.

# Seiberg-Witten background

A. Haydys and T. Walpuski proved a result regarding the limiting behaviour of solutions $(A_i, \Psi_i)$ such that $\|\Psi_i\|_{L^2} \to \infty$. They showed that there is a closed, nowhere dense subset $Z \in Y$ such that after passing to a subsequence and applying gauge transformations $A_i \to A$ in the Sobolev space $W^{1,2}_{\text{loc}}$ and $\Psi_i / \|\Psi_i\|_{L^2} \to \Psi$ weakly in $W^{1,2}_{\text{loc}}$ on $Y \setminus Z$, as well as that the limiting configuration $(A, \Psi)$ defined on $Y \setminus Z$ satisfies

$$\begin{cases} \not{D}_{A \otimes B} \Psi = 0 \\ 0 = \Psi\Psi^* - \frac{1}{2}|\Psi|^2. \end{cases} \tag{9}$$

Moreover, $Z$ is the zero locus of $\Psi$ and, if $\text{rank} E = 2$, $A$ is flat with holonmy contained in $\mathbb{Z}_2$. If $\text{rank} E > 2$, then $A$ induces a flat $\mathbb{Z}_2$ connection on a rank two subbundle of $E$ twisted by a line bundle.

# Seiberg-Witten theory - open questions

A number of open problems in Seiberg-Witten theory remain:

- The question of whether $Z$ is rectifiable or perhaps a smooth curve
- Improving the convergence statement for $(A_i, \Psi_i/\|\Psi_i\|_{L^2})$
- There are two ways of associating weights to the connected components of Z: one based on Taubes' frequency function and one developed by Haydys using topological methods. It is currently unknown whether these constructions are related.
- Haydys conjectured that, equipped with appropriate weights, $Z$ has the structure of a rectifiable current and that $\frac{1}{2\pi} F_{A_i}$ converges to Z as currents.

Application of our main theorem and the method used to prove it allow us to refine compactness and solve the above problems in the specific case that $Y = S^1 \times \Sigma$.

# Seiberg-Witten theory - some answers

In this context, our main theorem takes the following form:

## Main theorem Seiberg-Witten case

Let $Y = S^1 \times \Sigma$ equipped with a product metric. Then set $S$ the spinor bundle, $E, L$ vector bundles over $Y$ with structure groups $\mathrm{SU}(n)$ and $\mathrm{U}(1)$ respectively. Define $Z$ to be a closed, nowhere dense subset $Z \subset Y$. Equip $E$ with a connection pulled back from $\Sigma$, define a connection $A$ on $L \to Y$ along with $\Psi \in \Gamma(\mathrm{Hom}(E, S \otimes L))$ and let $(A_i, \Psi_i)$ be a sequence of solutions to

$$\begin{cases} \not{D}_{A \otimes B} \Psi = 0 \\ F_A = \Psi\Psi^* - \frac{1}{2}|\Psi|^2. \end{cases} \tag{10}$$

where $\not{D}_{A \otimes B}$ is the Dirac operator twisted by $A$ and $B$, while the second equation is based on the identification $i\Lambda^2 T^* Y \cong i\mathfrak{su}(S)$ given by Clifford multiplication, such that $\|\Psi_i\|_{L^2 \to \infty}$. Then

# Seiberg-Witten theory - theorem results

1. The singular set $Z$ is of the form $S^1 \times D$ for a degree $2d$ divisor $D = \sum_k m_k x_k$ with $d = \deg L$

# Seiberg-Witten theory - theorem results

1. The singular set $Z$ is of the form $S^1 \times D$ for a degree $2d$ divisor $D = \sum_k m_k x_k$ with $d = \deg L$
2. After passing to a subsequence and applying gauge transformations

$$A_i \to A \quad \text{and} \quad \frac{\Psi_i}{\|\Psi\|_{L^2}} \to \Psi$$

in $C^\infty_{\mathrm{loc}}$ on $Y \setminus Z$.

# Seiberg-Witten theory - theorem results

1. The singular set $Z$ is of the form $S^1 \times D$ for a degree $2d$ divisor $D = \sum_k m_k x_k$ with $d = \deg L$
2. After passing to a subsequence and applying gauge transformations

$$A_i \to A \quad \text{and} \quad \frac{\Psi_i}{\|\Psi\|_{L^2}} \to \Psi$$

   in $C^\infty_{\text{loc}}$ on $Y \setminus Z$.
3. $|\Psi|^4$ extends to a smooth function on Y whose zero set is $Z$ and for all $k$

$$|\Psi(x)| = O\left(\text{dist}(x, S^1 \times \{x_k\})^{|m_k|/2}\right)$$

# Seiberg-Witten theory - theorem results

1. The singular set $Z$ is of the form $S^1 \times D$ for a degree $2d$ divisor $D = \sum_k m_k x_k$ with $d = \deg L$
2. After passing to a subsequence and applying gauge transformations
$$A_i \to A \quad \text{and} \quad \frac{\Psi_i}{\|\Psi\|_{L^2}} \to \Psi$$
in $C^\infty_{\text{loc}}$ on $Y \setminus Z$.
3. $|\Psi|^4$ extends to a smooth function on Y whose zero set is $Z$ and for all $k$
$$|\Psi(x)| = O\left(\text{dist}(x, S^1 \times \{x_k\})^{|m_k|/2}\right)$$
4. If rank $E = 2$ then $\frac{i}{2\pi} F_{A_i} \to \frac{1}{2} Z$ as currents. If rank $E > 2$ then there is a rank two subbundle $F \subset E|_{Y \setminus Z}$ such that $\Psi \in \Gamma(Y \setminus Z, \text{Hom}(F, S \otimes L))$ and the previous statement holds if we replace $A$ and $A_i$ by the tensor product connections on $L \otimes (\det F)^{\frac{1}{2}}$. Here $F$ and $\det F$ are equipped with the unitary connections induced from $B$.

To apply our methods more clearly, we restate the previous discussion in the context of vortex equations.
For that, taking (3) and setting $N = 2$, $k_1 = 1, k_2 = -1$ along with choosing $E_1, E_2$ as Serre-dual to each other. Specifically, fixing a spin structure on $\Sigma$, and taking a $\mathrm{SU}(n)$-bundle $E$, we set $E_1 = E \otimes K^{1/2}$and $E_2 = E \otimes K^{1/2}$, where $K^{1/2}$ is said spin structure, which may be thought of as the square root of the canonical bundle of $\Sigma$.

# Seiberg-Witten equations

Naming our connection on $L$ once again $A$, the section we need to consider now becomes $\varphi = (\varphi^1, \varphi^2)$ where $\varphi^1 \in \Gamma(E \otimes K^{1/2} \otimes L)$ and similarly $\varphi^2 \in \Gamma(E^* \otimes K^{1/2} \otimes L^*)$, this gives us this "modified" version of (3):

$$\begin{cases} \overline{\partial}_{A \otimes B} \varphi^1 = 0, \\ \overline{\partial}_{A \otimes B} \varphi^2 = 0, \\ \varphi^1 \varphi^2 = 0, \\ \varepsilon^2 i \Lambda F_A + |\varphi^1|^2 - |\varphi^2|^2 = 0 \end{cases} \tag{11}$$

Notice that we have added an additional algebraic condition for $\varphi^1 \varphi^2 \in \Gamma(K)$. This is the image of $(\varphi^1, \varphi^2)$ under the pairing

$$\Gamma(E \otimes K^{1/2} \otimes L) \times \Gamma(E^* \otimes K^{1/2} \otimes L^*) \to \Gamma(K)$$

.

This means we can restate our theorem as follows:

**Main theorem, Seiberg-Witten case 2**

Let $(A_i, \varphi_i, \varepsilon_i)$ be a sequence of solutions to (11) with $\|\varphi_i\|_{L^2} = 1$ and $\varepsilon_i \to 0$. Then

1. There exists a degree $2d$ divisor $D = \sum_k m_k x_k$ and a configuration $(A, \varphi)$ defined on $\Sigma \setminus D$ and satisfying 11 with $\varepsilon = 0$,

1. There exists a degree $2d$ divisor $D = \sum_k m_k x_k$ and a configuration $(A, \varphi)$ defined on $\Sigma \setminus D$ and satisfying 11 with $\varepsilon = 0$,
2. $(A_i, \varphi_i) \to (A, \varphi)$ in $C^\infty_{\text{loc}}$ on $\Sigma \setminus D$,

1. There exists a degree $2d$ divisor $D = \sum_k m_k x_k$ and a configuration $(A, \varphi)$ defined on $\Sigma \setminus D$ and satisfying 11 with $\varepsilon = 0$,
2. $(A_i, \varphi_i) \to (A, \varphi)$ in $C^\infty_{\mathrm{loc}}$ on $\Sigma \setminus D$,
3. The function $|\varphi|^4$ extends to a smooth function on all of $\Sigma$ whose zero set consists of the points in $D$ and $\forall k$ we have

$$|\varphi(x)| = O\left(\mathrm{dist}(x, x_k)^{|m_k|/2}\right)$$

1. There exists a degree $2d$ divisor $D = \sum_k m_k x_k$ and a configuration $(A, \varphi)$ defined on $\Sigma \setminus D$ and satisfying 11 with $\varepsilon = 0$,
2. $(A_i, \varphi_i) \to (A, \varphi)$ in $C^\infty_{\text{loc}}$ on $\Sigma \setminus D$,
3. The function $|\varphi|^4$ extends to a smooth function on all of $\Sigma$ whose zero set consists of the points in $D$ and $\forall k$ we have

$$|\varphi(x)| = O\left(\text{dist}(x, x_k)^{|m_k|/2}\right)$$

4. If rank$E = 2$, then the limiting connection $A$ is flat, has holonomy contained in $\mathbb{Z}_2$, and $\frac{i}{2\pi}\Lambda F_{A_i} \to \frac{1}{2}\delta_D$ as measures. If rank$E > 2$, then there exists a rank two subbundle $F \subset E|_{\Sigma \setminus D}$ such that

$$\varphi^1 \in \Gamma(\Sigma \setminus D, F \otimes L \otimes K^{1/2}), \quad \varphi^2 \in \Gamma(\Sigma \setminus D, F^* \otimes L^* \otimes K^{1/2}),$$

and the previous statement holds if we replace $A$ and $A_i$ by the tensor product connections on $L \otimes (\det F)^{1/2}$. Here $F$ and $\det F$ are equipped with the unitary connections induced from $B$.