Reflection length in affine Coxeter groups

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Motivation: The conjecture

Conjecture A (Schwer). *Let $W = T \rtimes W_0$ be an affine Coxeter group with spherical Coxeter group W_0 over the same root system Φ. For $w \in W$ with normal form $w = t_\lambda u$ the reflection length can be written as*

$$\ell_R(w) = \frac{1}{2}\ell_R(t_\lambda) + \min_{v \in V_\lambda} \ell_R(vu) = \dim(\lambda) + \min_{v \in V_\lambda} \ell_R(vu).$$

Reflection length of translations: [MP11]
Reflection length in spherical Coxeter groups: [Car70]

Theorem B (Small rank). *Conjecture A is true in affine Coxeter groups of rank 1 and 2.*

Theorem C (Upper bound). *With the same notation as in Conjecture A holds*

$$\ell_R(w) \leq \dim(\lambda) + \min_{v \in V_\lambda} \ell_R(vu).$$

Outline

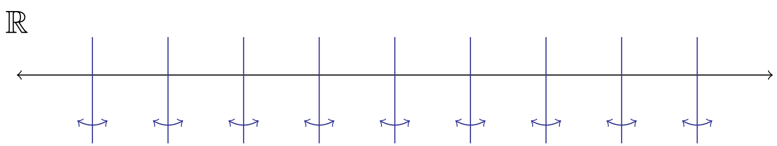
The conjecture in rank 1 Reflection length, normal forms

The conjecture in rank 2 $\dim(\lambda)$, the set V_λ

The conjecture in general

Rank 1: type affine A_1

The only affine Coxeter group of rank 1 is the **infinite dihedral group** D_∞.



$$D_\infty = \{$$

$$= \langle$$

Definition (Reflection length)**.** Let W be an affine Coxeter group with reflections R. The *reflection length* of $w \in W$ is

$$\ell_R(w) = \min\{k \mid w = r_1 r_2 \cdots r_k \text{ with all } r_i \in R \text{ reflections}\}.$$

Reflection lengths in D_∞:

$\mathbb{1}$	identity	$\ell_R(\mathbb{1})$	$=$
r	reflection	$\ell_R(r)$	$=$
t_λ	translation	$\ell_R(t_\lambda)$	$=$

Identify $D_\infty =$

Definition (Normal form)**.** Let $W = T \rtimes W_0$ be an affine Coxeter group. Then $w = t_\lambda u \in W$ is a *normal form* if $t_\lambda \in T$ and $u \in W_0$.

Proposition. *Let $D_\infty = T \rtimes \langle s \rangle$ as above. For $w \in D_\infty$ with normal form $w = t_\lambda u$ the reflection length fulfils*

$$\ell_R(w) = \frac{1}{2}\ell_R(t_\lambda) + \ell_R(v_\lambda u)$$

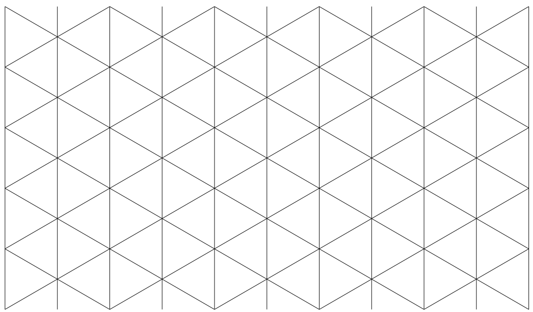
where

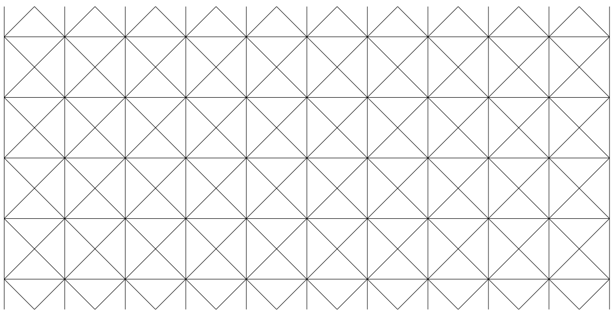
$$v_\lambda =$$

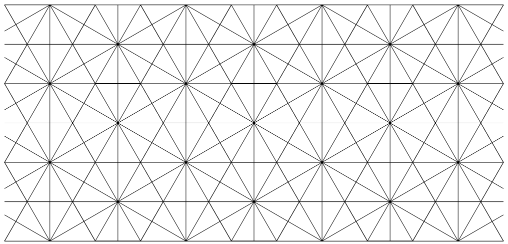
Proof.

Rank 2

There are three (irreducible) affine Coxeter groups of rank 2 [MT11, Prop. A.17]. These are their reflection hyperplanes in the Euclidean plane:

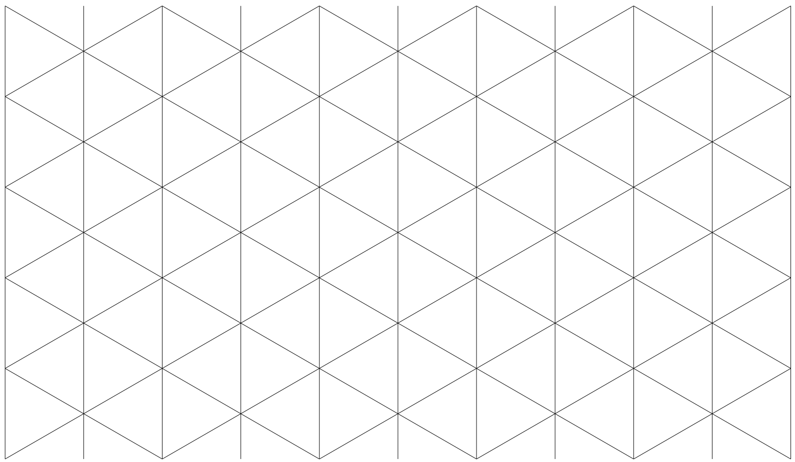




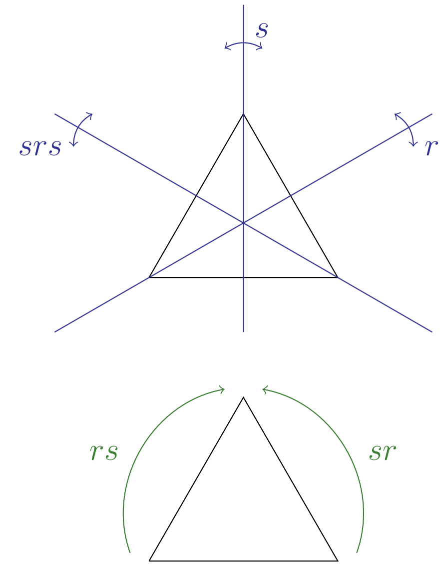


Type affine A_2

The affine Coxeter group W_A of type affine A_2 acting on the Euclidean plane:



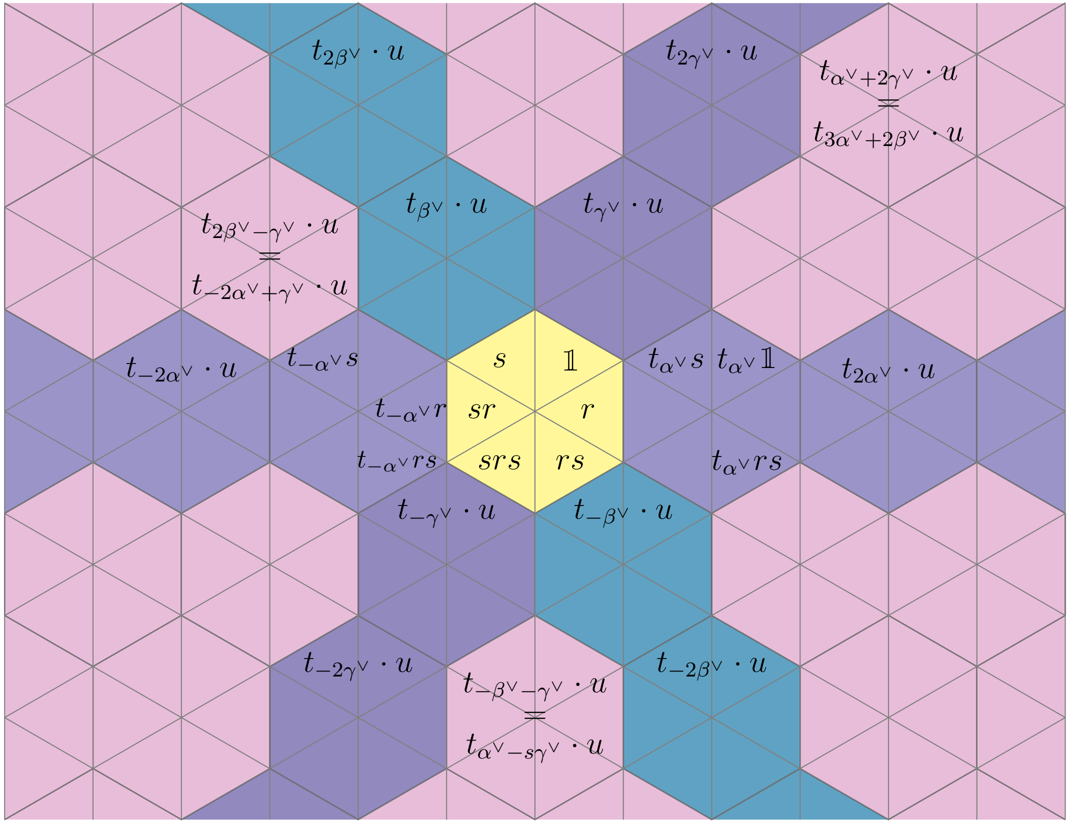
The spherical Coxeter group S_3 acting on an equilateral triangle:



$$S_3 = \{\mathbb{1},\ s,\ r,\ srs,\ sr,\ rs\}$$

$$= \langle$$

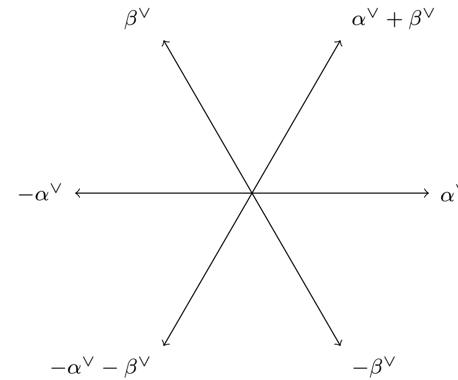
Type affine A_2



Identify W_A as semidirect product

$$W_A = T \rtimes S_3$$

where $T \cong \mathbb{Z}^2$.



$$\Phi_A^\vee = \{\pm\alpha^\vee, \pm\beta^\vee, \pm(\alpha^\vee + \beta^\vee)\}$$

$s_\alpha =$

$s_\beta =$

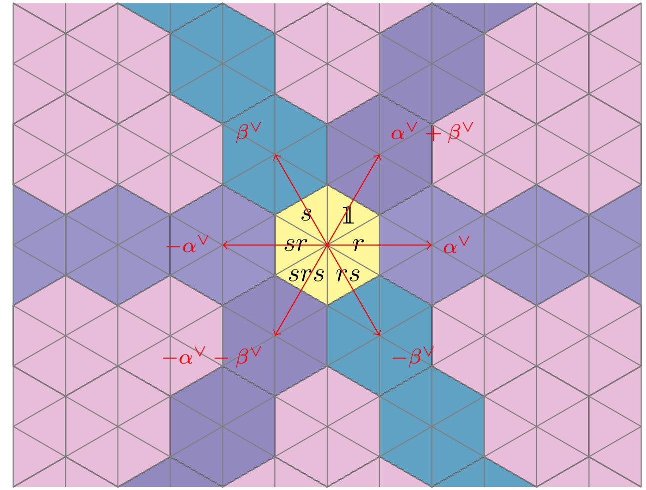
$s_{\alpha+\beta} =$

The conjecture in type affine A_2

Proposition. *Let $W_A = T \rtimes S_3$ as above. For a normal form $w = t_\lambda u \in W_A$ holds*

$$\ell_R(w) = \dim(\lambda) + \min_{v \in V_\lambda} \ell_R(vu).$$

What are $\dim(\lambda)$ and V_λ?

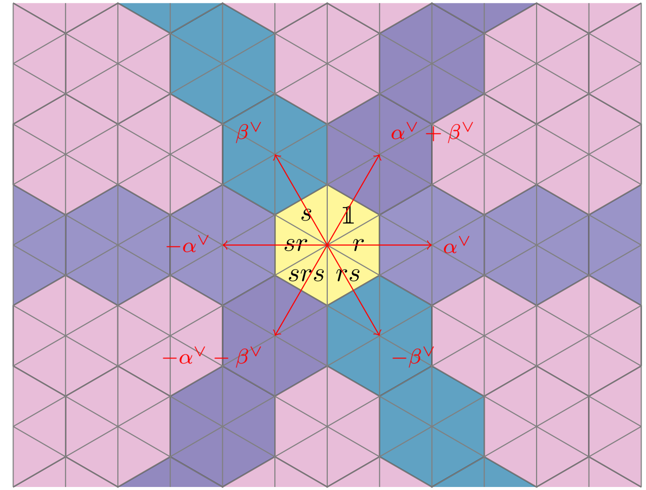


Dimension of a vector λ

Definition (Dimension of λ)**.** Let $\Phi^\vee$ be a coroot system. The *dimension* of $\lambda \in \mathbb{Z}\Phi^\vee$ is defined as

$$\dim(\lambda) = \min\{k \mid \lambda = \sum_{i=1}^{k} c_i \alpha_i^\vee \text{ with } c_i \in \mathbb{Z},\ \alpha_i^\vee \in \Phi^\vee\}.$$

In affine Coxeter groups it can be shown that $\dim(\lambda) = \frac{1}{2}\ell_R(t_\lambda)$ [MP11, Prop. 3.4].



The set V_λ

Definition (The set V_λ)**.** Let $\Phi^\vee$ be a coroot system and $\lambda \in \mathbb{Z}\Phi^\vee$ of dimension k. If

$$\lambda = \sum_{i=1}^{k} c_i \alpha_i^\vee$$

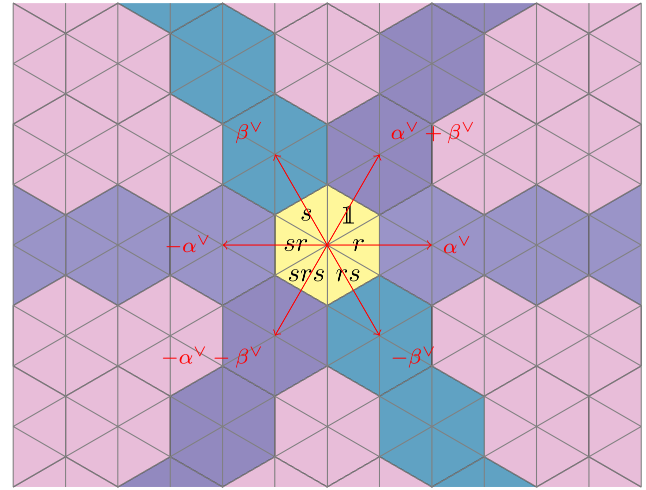
with coroots $\alpha_1^\vee, \dots, \alpha_k^\vee$ and all $c_i \in \mathbb{Z}_{\neq 0}$ then $s_{\alpha_1} \cdots s_{\alpha_k} \in V_\lambda$.

The set V_λ for λ of low dimension:

$\dim(\lambda) = 0:$

$\dim(\lambda) = 1:$

$\dim(\lambda) = 2:$



$$\Phi_A^\vee = \{\pm\alpha^\vee, \pm\beta^\vee, \pm(\alpha^\vee + \beta^\vee)\}$$

$$s_\alpha = s$$
$$s_\beta = r$$
$$s_{\alpha+\beta} = srs$$

The conjecture in type affine A_2

Proposition. *Let $W_A = T \rtimes S_3$ as above. For a normal form $w = t_\lambda u \in W_A$ holds*

$$\ell_R(w) = \dim(\lambda) + \min_{v \in V_\lambda} \ell_R(vu).$$

Proof. Case $\dim(\lambda) = 0$:

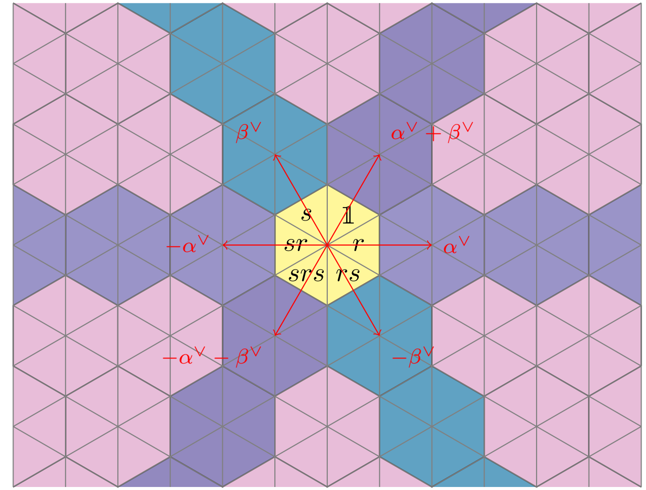
Case $\dim(\lambda) = 1$: $\lambda \in \mathbb{Z}\rho^\vee$

All coroots in $\Phi_A^\vee$ are conjugate [MT11, Cor. A.18].
ℓ_R is invariant under conjugacy [LMPS19, Rmk. 1.3].

$V_\lambda = \{$

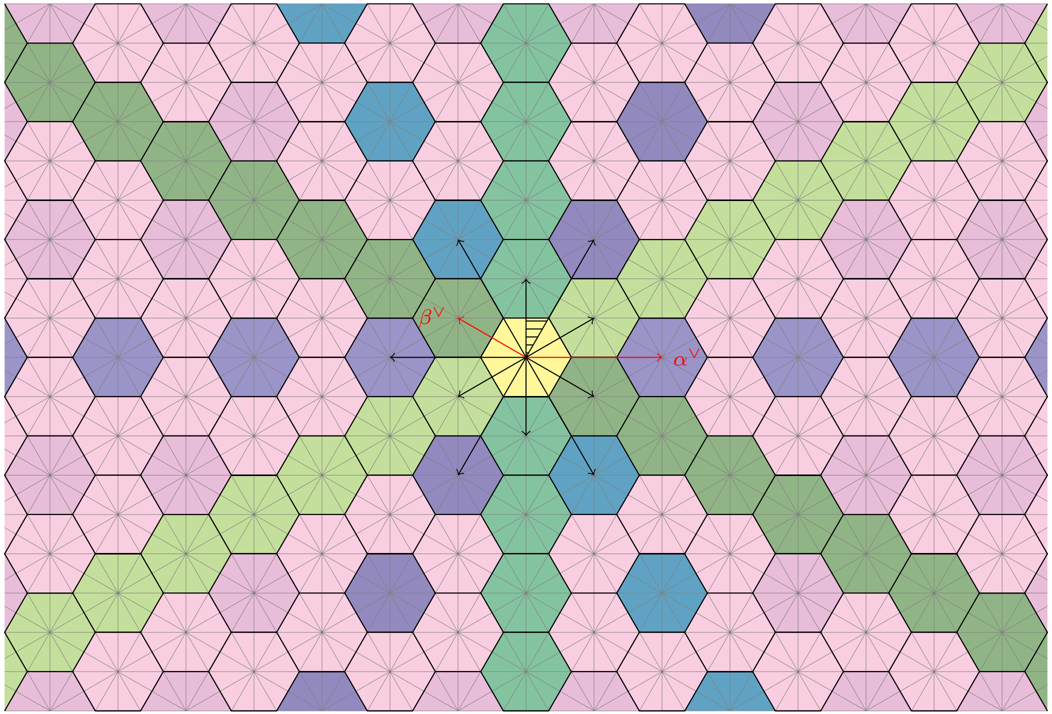
There are only 5 types of elements in rank two and those have the following reflection lengths [LMPS19, Table 1]:

identity	$\ell_R(w) = 0$
reflection	$\ell_R(w) = 1$
rotation	$\ell_R(w) = 2$
translation	$\ell_R(w) = 2$ or 4
glide reflection	$\ell_R(w) = 3$



Case $\dim(\lambda) = 2$: $V_\lambda = \{$

Type affine G_2



$\Phi_G^\vee$ is *generated* by $\alpha^\vee$ and $\beta^\vee$ or all short coroots [MT11, Prop. A.11 and Lem. B.20].

Identify W_G as semidirect product

$$W_G = T \rtimes D_6$$

where $T \cong \mathbb{Z}^2$ and

$$D_6 = \langle s, r \mid s^2, r^2, (sr)^6 \rangle.$$

Rotations in D_6:

Let $\lambda \in \Phi_G^\vee$ of dimension two. Then

$V_\lambda = \{$

The conjecture in general

Conjecture A (Schwer)**.** *Let* $W = T \rtimes W_0$ *be an affine Coxeter group with spherical Coxeter group* W_0 *over the same root system* Φ*. For* $w \in W$ *with normal form* $w = t_\lambda u$ *the reflection length can be written as*

$$\ell_R(w) = \dim(\lambda) + \min_{v \in V_\lambda} \ell_R(vu).$$

Root systems and spherical Coxeter groups

Definition. A subset $\Phi \subseteq V$ of a finite dimensional $\mathbb{R}$-vector space V is a *root system* if:

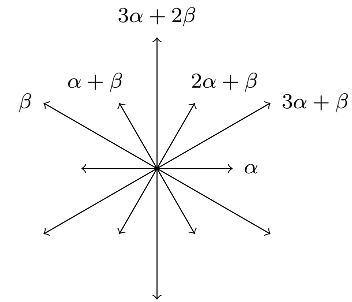
(R1) Φ is finite, $0 \notin \Phi$ and $\langle \Phi \rangle_{\mathbb{R}} = V$,

(R2) $\forall \alpha \in \Phi$: if $\lambda\alpha \in \Phi$ for $\lambda \in \mathbb{R}$, then $\lambda = \pm 1$,

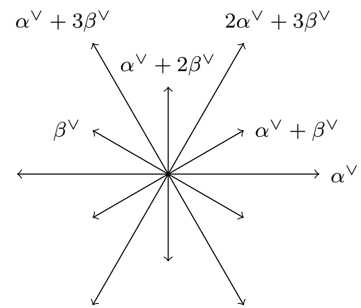
(R3) $\forall \alpha \in \Phi\ \exists$ a reflection $s_\alpha \in \mathrm{GL}(V)$ along α stabilising Φ,

(R4) (*crystallographic condition* or *integrality*)
$\forall \alpha, \beta \in \Phi$: $s_\alpha.\beta - \beta$ is an integral multiple of α

The group $W_0 = \langle s_\alpha \mid \alpha \in \Phi \rangle$ is called *spherical Coxeter group* and the set $\Phi^\vee = \{\frac{2}{(\alpha,\alpha)}\alpha \mid \alpha \in \Phi\}$ is the *coroot system* of Φ.



The root system Φ_G of type G_2.



The coroot system $\Phi_G^\vee$ of type G_2.

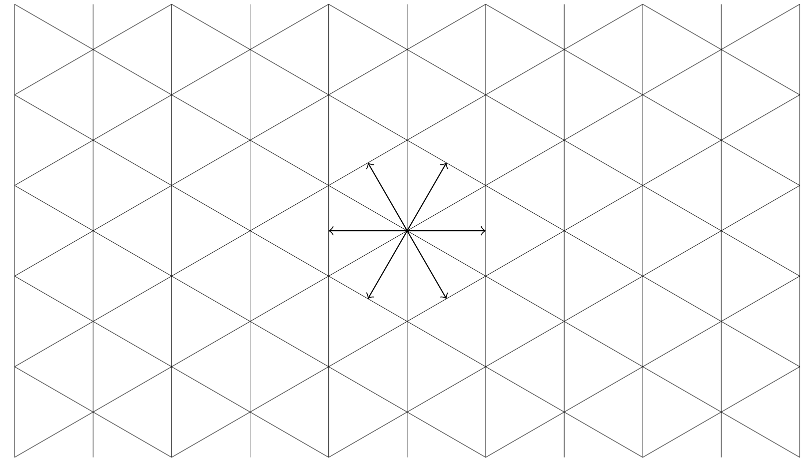
Affine Coxeter groups

Definition (Affine Coxeter groups)**.** Let V be a finite dimensional $\mathbb{R}$-vector space with root system Φ.
$\forall$ roots $\alpha \in \Phi$, $\forall$ integers $j \in \mathbb{Z}$ let

$H_{\alpha,j}$ the *(affine) hyperplane* of points $v \in V$ with $(v, \alpha) = j$

$r_{\alpha,j}$ the reflection about $H_{\alpha,j}$.

The set $R = \{r_{\alpha,j} \mid \alpha \in \Phi,\ j \in \mathbb{Z}\}$ generates the *affine Coxeter group* W.



The hyperplane arrangement of type A_2 with the root system Φ_A.

The upper bound of the conjecture

Theorem C (Upper bound)**.** *Let* $W = T \rtimes W_0$ *be an affine Coxeter group with spherical Coxeter group* W_0 *over the same root system* Φ*. For* $w \in W$ *with normal form* $w = t_\lambda u$ *the reflection length can be written as*

$$\ell_R(w) \leq \dim(\lambda) + \min_{v \in V_\lambda} \ell_R(vu).$$

Lemma (Rewriting reflection factorisations [MP11, Lem. 3.5])**.**
Let $w = r_1 \cdots r_l \in W$ *with all* r_i *reflections.*
For any $1 \leq i_1 < i_2 < \cdots < i_m \leq l$ *exist* $\tilde{w} \in W$ *of length* $l - m$*:*

$$w = \tilde{w} \cdot r_{i_1} r_{i_2} \cdots r_{i_m}.$$

Thank you for your attention!

References

[Car70] R. Carter. Conjugacy classes in the Weyl group. In *Seminar on Algebraic Groups and Related Finite Groups (The Institute for Advanced Study, Princeton, N.J., 1968/69)*, pages 297–318. Springer, Berlin, 1970.

[LMPS19] Joel Brewster Lewis, Jon McCammond, T. Kyle Petersen, and Petra Schwer. Computing reflection length in an affine Coxeter group. *Trans. Amer. Math. Soc.*, 371(6):4097–4127, 2019.

[MP11] Jon McCammond and T. Kyle Petersen. Bounding reflection length in an affine Coxeter group. *J. Algebraic Combin.*, 34(4):711–719, 2011.

[MT11] Gunter Malle and Donna Testerman. *Linear algebraic groups and finite groups of Lie type*, volume 133 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2011.