Character varieties and Lagrangian submanifolds

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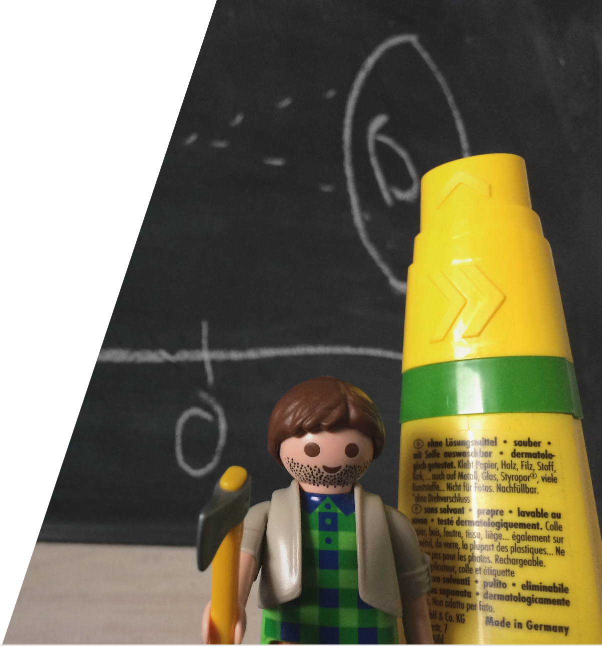

Representation varieties

Let $\Gamma = \langle \gamma_i | w_\lambda \rangle$ be a finitely generated group. Let G be an affine algebraic group over $\mathbb{C}$. The set of homomorphisms $\rho : \Gamma \to G$ can be embedded into G^N via

$$\mathrm{Hom}(\Gamma, G) \hookrightarrow G^N, \quad \rho \mapsto (\rho(\gamma_i))_i.$$

With the affine algebraic structure coming from G^N, $\mathrm{Hom}(\Gamma, G)$ is called *representation variety*.

Let G be a reductive algebraic group and let $\Gamma = \pi_1(S)$ be the fundamental group of a closed compact connected surface S. Then, the subset of irreducible representations $\mathrm{Hom}^i(\Gamma, G)$ consists of non-singular points only. In particular, it can be equipped with the structure of a complex manifold.

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Character varieties

Let G be a reductive affine algebraic group. G acts on $\mathrm{Hom}(\Gamma, G)$ via conjugation, i.e. $g.\rho(\gamma) = g\rho(\gamma)g^{-1}$. What is a "nice" orbit space?

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Character varieties

Definition

The categorical quotient $X_G(\Gamma) = \mathrm{Hom}(\Gamma, G) /\!/ G$ is called the *character variety* of Γ in G. It is an affine algebraic set.

Properties of the character variety:

- $X_G(\Gamma)$ represents closed orbits.
- $\mathrm{Hom}^i(\Gamma, G) /\!/ G = \mathrm{Hom}^i(\Gamma, G)/G$.
- A *good representation* ρ is an irreducible representation with stabilizer $G_\rho = C(G) \subseteq G$. The good representations ρ have (Zariski) tangent spaces $T_\rho X_G(\Gamma) = H^1(\Gamma, \mathfrak{g}_{\mathrm{Ad}\,\rho})$.
- The good character variety $X_G^g(S) = X_G^g(\pi_1(S))$ is non-singular.

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Goldman's symplectic form

From now on, let S be orientable. Suppose we have a non-degenerate, Ad-invariant, symmetric, $\mathbb{C}$-bilinear form $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$. It induces a bilinear form:

$$\begin{aligned} \omega_\rho^B : H^1(\pi_1(S), \mathfrak{g}_{\mathrm{Ad}\,\rho}) \times H^1(\pi_1(S), \mathfrak{g}_{\mathrm{Ad}\,\rho}) &\xrightarrow{\cup} H^2(\pi_1(S), \mathfrak{g}_{\mathrm{Ad}\,\rho} \otimes \mathfrak{g}_{\mathrm{Ad}\,\rho}) \\ &\xrightarrow{B} H^2(\pi_1(S), \mathbb{C}) \cong \mathbb{C}\,. \end{aligned}$$

ω_ρ^B is non-degenerate and anti-symmetric. It can be shown that ω^B is a symplectic form on the manifold $X_G^{\mathfrak{g}}(S)$. This construction is due to Bill Goldman [Gol84].

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The Lagrangian submanifold theorem

Let M be a compact connected 3-manifold, whose boundary is a compact connected orientable closed surface, i.e. $\partial M = S$. The embedding

$$S \hookrightarrow M$$

induces a group homomorphism

$$r : \pi_1(S) \longrightarrow \pi_1(M),$$

which induces a regular map

$$r^* : X_G(M) \longrightarrow X_G(S), \quad [\rho] \mapsto [\rho \circ r]$$

of the character varieties. **Idea:** The image of r^* is a Lagrangian submanifold.

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We have $r^* : X_G(M) \to X_G(S)$. Consider the "non-singular good image"

$$Y_G(M) = [r^* X_G(M) \cap X_G^g(S)]^{ns}.$$

Theorem (see [Sik09])

$Y_G(M) \subseteq X_G^g(S)$ is a disjoint union of isotropic submanifolds, i.e. $\omega^B|_{Y_G(M)} \equiv 0$.

A *Lagrangian submanifold* is an isotropic submanifold of dimension

$$\tfrac{1}{2} \dim_{\mathbb{R}} X_G^g(S).$$

It is possible to "characterize" Lagrangian components (components containing *reduced representations*).

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Anti-holomorphic involutions

Let S be a closed compact connected Riemann surface, and let $f : S \to S$ be an anti-holomorphic involution, i.e. $f^2 = \mathrm{id}$ and $z \mapsto f(\bar{z})$ is holomorphic. f induces a homomorphism

$$f_* : \pi_1(S, z_0) \to \pi_1(S, f(z_0)), \quad \gamma \mapsto f(\gamma).$$

Together with a path δ from z_0 to $f(z_0)$, we obtain an automorphism

$$f_{*,\delta} : \pi_1(S, z_0) \to \pi_1(S, z_0), \quad \gamma \mapsto \delta.f(\gamma).\delta^{-1}.$$

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Anti-holomorphic involutions

$\hat{f}$ preserves good representations. $\hat{f} : X_G^g(S) \to X_G^g(S)$ is independent of δ and an involution.

Theorem

The fixed point set $\mathcal{L}_G \subseteq X_G^g(S)$ of $\hat{f}$ is called the (A, B, A)-brane. It is a Lagrangian submanifold of $X_G^g(S)$.

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We have found two isotropic (and possibly Lagrangian) submanifolds of $X_G^g(S)$, namely $Y_G(M)$ and $\mathcal{L}_G$. Are they related to each other?

Yes, if we choose the right M.

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Task

We will construct a 3-manifold M with $\partial M = S$, for which $Y_G(M) \subseteq \mathcal{L}_G$.

Consider the 3-manifold $\Sigma = S \times [-1, 1]$ together with the smooth involution

$$\sigma : \Sigma \to \Sigma, \quad (z, t) \mapsto (f(z), -t).$$

Via σ, $\mathbb{Z}/2\,\mathbb{Z}$ acts on Σ. Denote the orbit space by $M = \Sigma/\sigma$.

Denote the fixed point set of f by $F \subseteq S$. On $\Sigma \backslash F$, σ acts freely, so that $M \backslash F = (\Sigma \backslash F)/\sigma$ is a smooth 3-manifold with boundary $\partial M = S$. It remains to construct a smooth neighborhood around $F \subset M$ compatible with the smooth structure of $M \backslash F$.

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Comparing Lagrangian submanifolds

Construction of a smooth chart around $F \subset M$:

- F consists of disjoint copies of the circle S^1.
- Every circle $S^1 \subseteq \Sigma$ has a neighborhood $S^1 \subset V \cong S^1 \times B^2_\varepsilon(0)$ with
$$\sigma(x, y, t) = (x, -y, -t).$$
- In M, we have $[x, y, t] = [x, -y, -t]$. Define $w = y + it \in \mathbb{C}$, we have
$$[x, w] = [x, -w] \text{ in } M.$$
- Let $\pi : \Sigma \to M$ be the projection. We have a commutative diagram
$$\begin{tikzcd} V \arrow[rr, "(x,w)\mapsto(x,w^2)"] \arrow[dr, "\pi:(x,w)\mapsto[x,w]"'] & & \mathbb{R}^3 \\ & \pi(V). \arrow[ur, "\phi:[x,w]\mapsto(x,w^2)"'] & \end{tikzcd}$$

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We choose $\phi : \pi(V) \to \mathbb{R}^3$ to be the chart around $S^1 \subset M$. It is compatible with the smooth structure of $M \backslash F$. Hence, M is a smooth manifold.

Theorem (see [BS14])

For M constructed as above, $Y_G(M)$ is contained in $\mathcal{L}_G$.

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